THE SCALING POMERON*

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In honor of Andrzej Białas on his 90th anniversary

We examine the Regge theoretical properties for the scaling observed in pp elastic scattering differential cross sections at the LHC. A positive signature amplitude (*i.e.* the Pomeron) with scaling properties has been derived. It is found to describe the dip–bump region of momentum transfer at LHC energies in agreement with data. We derive the analytic continuation in the whole plane of the t-channel partial waves of index l_t specific to the Regge formalism. The analytic form of the partial-wave amplitude exhibits a specific scaling property without singularities, except for a series of poles in the l_t real axis at fractional values.

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1. Scaling proton–proton scattering amplitudes

Recently [1], scaling properties of proton–proton elastic scattering at LHC energies have been revealed, based on the elastic differential cross sections measured by the TOTEM Collaboration [2]. Consider the pp differential cross section

$$\frac{\mathrm{d}\sigma^{pp}}{\mathrm{d}t}(s,t) = \frac{1}{\left(s/\mathrm{TeV}^2\right)^2}\,\left|\mathcal{A}^{pp}(s,t)\right|^2 , \tag{1}$$

where the usual Mandelstam variables are s — the positive center-of-mass energy squared and t — the negative of the momentum transfer squared. $\mathcal{A}^{pp}(s,t)$ is the proton–proton elastic spin-averaged scattering amplitude. For convenience, formula (1) is normalized in a dimensionless way such that the amplitude is dimensioned to an energy squared.

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The scaling properties empirically found in [1] are such that a conveniently scaled cross section appears, within error bars, to depend only on one variable $t^{**}(s,t) = (s/\mathrm{TeV}^2)^{.065}|t|^{.72}$. One finds

$$\left(s/\mathrm{TeV}^2\right)^{-.305} \frac{\mathrm{d}\sigma^{pp}}{\mathrm{d}t}(s,t) = f\left(t^{**}\right) , \tag{2}$$

where t is measured in $-\mathrm{GeV}^2$. The empirical scaling property, Eq. (2), means that a model proton–proton elastic amplitude $\mathcal{A}^{pp}(s,t)$ in Eq. (1) may be written [1] using one scaling variable

$$\tau \equiv t \times \left(s/\mathrm{TeV}^2\right)^{\gamma/2} , \tag{3}$$

and two scaling exponents, $\alpha = 0.35$ and $\gamma = .13/.72 = 0.1806$, namely,

$$\mathcal{A}^{pp}(s,t) = \left(s/\mathrm{TeV}^2\right)^{\alpha/2} \; F(\tau) . \tag{4}$$

Indeed, in Ref. [1], one finds a concrete realization of the amplitude $\mathcal{A}^{pp}$ verifying the scaling Eq. (2) when performing a fit of the cross sections measured by the TOTEM Collaboration [2] in the dip–bump region of transverse momentum, namely,

$$\begin{aligned} \mathcal{A}^{pp} &= \mathrm{e}^{i\theta}\left(\mathcal{A}_1 - \mathrm{e}^{i\varphi}\mathcal{A}_2\right) , \\ \mathcal{A}_{j=1,2} &= i\, N_j^0 \left(s/\mathrm{TeV}^2\right)^{\frac{\alpha}{2}+1} \mathrm{e}^{B_j^0 \tau} , \end{aligned} \tag{5}$$

where the parameters N_j^0, B_j^0, $j = 1,2$, and the complex phase φ were phenomenologically adjusted [1] by the fit. Then the overall phase θ was fixed by the known ratio of the real over imaginary part of the amplitude in the very forward direction.

Note that a QCD interpretation of the scaling property, *cf.* Eq. (2), has been recently proposed [3].

2. Scaling amplitudes with positive signature (Pomeron)

The problem we want to study is how to describe a scaling amplitude such as Eqs. (5) in the S-matrix formalism of the Regge theory [4] without *a priori* referring to any specific phenomenological Regge model. As is well known, this is related to analyticity properties of t-channel amplitudes. In this framework, the amplitude $\mathcal{A}^{pp}$ can be written as a sum of two components

$$\mathcal{A}_\eta^{pp}(s,t) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{s^l + \eta\,(-s)^l}{\sin \pi l}\; a_\eta(l,t)\,\mathrm{d}l , \tag{6}$$

where $\eta = \pm$ is called the signature and $a_\eta(l,t)$ are the analytic continuations of the partial-wave amplitudes of both $\pm$ signatures in the crossed two-body t-channel. The contour $\mathcal{C}$ is curled around the poles, due to the positive, even or odd, integer zeros of $\sin \pi l$, depending on η, since the poles of opposite $(\mp)$ parity are canceled by the numerator. Indeed, the integrand of (6) can also be written as

$$\begin{aligned} \frac{s^l + \eta\,(-s)^l}{\sin \pi l} &= \frac{\left(\mathrm{e}^{-i\frac{\pi}{2}} s\right)^l}{\sin(\pi l/2)} \qquad \text{if } \eta = +1\,, \\ &= i\frac{\left(\mathrm{e}^{-i\frac{\pi}{2}} s\right)^l}{\cos(\pi l/2)} \qquad \text{if } \eta = -1\,. \end{aligned} \tag{7}$$

As shown by Eq. (7), the S-matrix Regge formalism [4] imposes a thorough connection between the s dependence and the accompanying complex phases of the amplitudes through the substitution everywhere

$$s \to \sigma = \mathrm{e}^{-i\pi/2} s\,. \tag{8}$$

The expression, Eq. (6), can be rewritten as

$$\begin{aligned} \mathcal{A}^{pp}_{+}(s,t) &= \frac{1}{2i\pi}\int_{\mathcal{C}} \mathrm{d}l\ \sigma^l\ \hat{a}_+(l,t) \qquad \text{if } \eta = +1\,, \\ \mathcal{A}^{pp}_{-}(s,t) &= \frac{1}{2i\pi}\int_{\mathcal{C}} \mathrm{d}l\ i\sigma^l\ \hat{a}_-(l,t) \qquad \text{if } \eta = -1\,, \end{aligned} \tag{9}$$

with the redefinitions

$$\hat{a}_+(l,t) = a_+(l,t)/\sin(\pi l/2) \quad \text{and} \quad \hat{a}_-(l,t) = a_-(l,t)/\cos(\pi l/2)\,.$$

There are important consequences of Eqs. (9) since the unique energy dependence of the amplitudes $\mathcal{A}^{pp}_\eta$ is in terms of σ. Hence, there exists a precise energy-phase relationship, Eq. (8), predicted by the Regge formalism independently of any phenomenological model. Moreover, for a negative signature amplitude, there appears a phase shift of $\pi/2$ w.r.t. the positive signature amplitude, *cf.* the i factor in Eq. (7) for $\eta = -1$. However, the partial-wave amplitudes, being *a priori* complex functions, may have their own phases.

In the following, we will focus on the positive sign signature which corresponds to the so-called Pomeron t-channel exchange. We shall examine whether a scaling Pomeron amplitude may describe the data as well as the good fit obtained with the amplitudes, Eqs. (5).

Using the complex variable σ, Eq. (8), and the substitution, Eq. (8), acting on the amplitudes, Eqs. (5), we are led to write the scaling Pomeron amplitudes as

$$\begin{aligned} \mathcal{A}^{pp} &= \tilde{\mathcal{A}}_1 - \mathrm{e}^{i\tilde{\varphi}}\tilde{\mathcal{A}}_2\,, \\ \tilde{\mathcal{A}}_{i=1,2} &= \tilde{N}_i^0 \left(\frac{\sigma}{\mathrm{TeV}^2}\right)^{\alpha/2+1} \mathrm{e}^{\tilde{B}_i^0\tau}\,, \end{aligned} \tag{10}$$

where the tilded amplitudes and parameters are to be determined by a best fit to TOTEM data. One indeed finds a comparably good fit with such parameterizations [5], see Fig. 1.

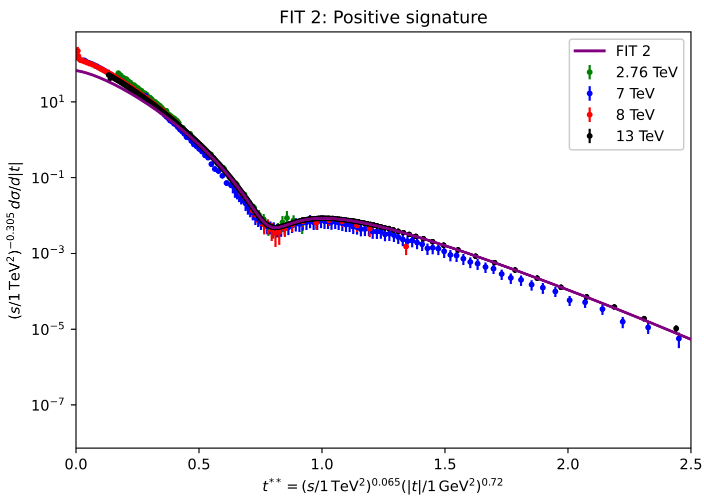

Fig. 1. Scaling Pomeron *versus* TOTEM data (from [5]). Both the fit using Eq. (10) and TOTEM data are displayed on the scaling plot $\log(s^{-\alpha}\mathrm{d}\sigma/\mathrm{d}t)$ *versus* t^{**}.

3. Regge formalism: Determination of the t-channel partial waves

As discussed in Section 2, the integrand of the Regge integral $a(l,t)$ in the first of Eqs. (9) represents the analytic continuation of the partial-wave amplitude in the complex l-plane. Let us determine such an analytic continuation for the scaling amplitudes, Eqs. (5).

As a starting point, we make the scaling ansatz. The scaling amplitudes expand as follows:

$$\tilde{\mathcal{A}}_{i=1,2} = -\tilde{N}_i^0 \left(\frac{\sigma}{\mathrm{TeV}^2}\right)^{\alpha/2+1} \mathrm{e}^{\tilde{B}_i^0 t \left(\frac{\sigma}{\mathrm{TeV}^2}\right)^{\gamma/2}}$$
$$= -\tilde{N}_i^0 \sum_{n=0}^{\infty} \frac{\left(\tilde{B}_i^0 t\right)^n}{n!} \left(\frac{\sigma}{\mathrm{TeV}^2}\right)^{1+\alpha/2+n\gamma/2} . \quad (11)$$

The summation, Eq. (11), can be realized through a sum of residues of poles of the following integrals in the complex l-plane:

$$\tilde{\mathcal{A}}_{i=1,2} = 2i\pi\tilde{N}_i^0 \sum_{n=0}^{\infty} \frac{\left(\tilde{B}_i^0 t\right)^n}{n!} \int_{\mathcal{C}} \frac{\sigma^l \mathrm{d}l}{l-1-\frac{\alpha+n\gamma}{2}} , \quad (12)$$

where the contour $\mathcal{C}$ goes around all poles of the integrand. By denoting

$$x \equiv \tilde{B}_i^0 t , \qquad \lambda \equiv \frac{2}{\gamma}\left(l-1-\frac{\alpha}{2}\right) , \quad (13)$$

the sum in Eq. (12) can be cast into the two equivalent forms (see Appendix A)

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}\frac{1}{\lambda-n} = -\mathrm{e}^x \sum_{k=0}^{\infty} \frac{(-x)^k \varGamma(-\lambda)}{\varGamma(-\lambda+k+1)} . \quad (14)$$

Interestingly, the second expression of Eq. (14) is known [6] to be related to the upper incomplete Gamma function $G(u,y)$,

$$\mathrm{e}^x \sum_{k=0}^{\infty} \frac{(-x)^k \varGamma(-\lambda)}{\varGamma(-\lambda+k+1)} = \varGamma(-\lambda)\ G^*(-\lambda,-x) , \quad (15)$$

where $G^*(u,y)$ is the holomorphic factor of $G(u,y)$, see Appendix A.

In Eq. (14), only $\varGamma(-\lambda)$ exhibits single poles for every positive integer λ. The scaling ansatz has induced that no other singularity is present with respect to both complex variables. Hence, in their scaling form, Eq. (10), the amplitudes have only single-pole singularities at positive integer values of λ. Singularities at infinity in the complex plane may exist but do not seem to be necessary to our analysis. There are no other Regge singularities in the complex l-plane, for instance, standard Regge poles or cuts. The scaling form has thus a specific singularity structure. Note, however, that scaling is only seen at moderate but nonzero momentum transfer, so the full amplitude will probably generate the singularities. The problem remains open to see the connection with the scaling properties.

Since λ is a positive integer, it is interesting to replace l by its value in terms of λ in the complex integral, Eq. (12). Using identity (14), one finally obtains

$$\begin{aligned}\tilde{\mathcal{A}}_{1,2} &= \frac{\tilde{N}_i^0}{2i\pi}\,\sigma^{1+\alpha/2} \\ &\quad\times \int\limits_{\mathcal{C}} \sigma^{\gamma\lambda/2}\,\mathrm{d}\lambda\; \Gamma(-\lambda) G^*\left(-\lambda, -\tilde{B}_i^0 t\right) .\end{aligned} \tag{16}$$

Equation (16) provides the complete Regge form of the pp elastic amplitude in its scaling form, Eq. (10).

It is interesting to note that the second line of Eq. (16) compares well with the generic Regge formula, Eq. (6), if we perform the replacements

$$l \to 2\lambda\,, \qquad \sigma \to \sigma^{\gamma/2}\,, \qquad a(l,t) \to \gamma^*\left(-\lambda, -\tilde{B}_i^0 t\right) . \tag{17}$$

3.1. Highlights

In the present paper, we essentially derive two results inspired by the use of scaling amplitudes [1] to describe pp elastic data at the LHC.

— Using the Regge formalism, we define and derive a scaling amplitude with a positive signature, namely the Pomeron. It successfully describes the TOTEM data [2].

— Looking for a t-channel point of view on the scaling amplitudes, we find the following analytical result, namely an everywhere analytic continuation of partial waves of t-channel index $l_t = \lambda$ in the whole λ complex plane up to a rescaling of the complexified variable $\sigma \to \sigma^{\gamma/2}$. This means a rescaling of the "momentum transfer value in the t-channel" (*i.e.* the energy value in the s-channel). This is realized thanks to the analytic functions $G^*(-\lambda, -\tilde{B}_i^0 t)$, see Appendix A.

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Appendix A

Proof of relation (14)

In order to prove Eq. (14), let us multiply each of the two sums by a factor $-x^{-\lambda}$. We then show that both functions have equal derivative with respect to x and equal values at $x=0$, for all complex values of λ.

For the left-hand side of Eq. (14), the derivative gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{-x^{-\lambda}\sum_{n=0}^{\infty}\frac{x^n}{n!}\,\frac{1}{\lambda-n}\right\}=\sum_{n=0}^{\infty}\frac{x^{n-\lambda-1}}{n!}=x^{-\lambda-1}\,\mathrm{e}^x\,. \tag{A.1}$$

For the right-hand side of Eq. (14), the derivative gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^{-\lambda}\,\mathrm{e}^x\mathit{\Gamma}(-\lambda)\sum_{k=0}^{\infty}\frac{(-x)^k}{\mathit{\Gamma}(-\lambda+k+1)}\right\}$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\left\{(-1)^{\lambda}G(-\lambda,-x)\right\}=x^{-\lambda-1}\,\mathrm{e}^x\,, \tag{A.2}$$

where $G(u,y)=\int_0^y t^{u-1}\mathrm{e}^{-t}$ is the lower incomplete gamma function. Note that the function

$$G^*(u,y)\equiv\mathrm{e}^y\mathit{\Gamma}(u)\sum_{k=0}^{\infty}\frac{y^k}{\mathit{\Gamma}(u+k+1)} \tag{A.3}$$

is known [6] to be holomorphic in the complex $\mathcal{C}\otimes\mathcal{C}$ space of its u,y variables.

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