NON-COMMUTATIVE PHASE-SPACE EFFECTS IN FERMIONIC STRING THEORY*

Mohamed Adib Abdelmoumene†, Nadir Belaloui

University Mentouri Constantine 1, Physics Department
Laboratoire de Physique Mathématique et Physique subatomique (LPMPS)
Constantine 1 BP, 325 Route de Ain El Bey, Constantine, Algérie, 25017

Received 26 August 2025, accepted 9 March 2026,
published online 7 July 2026

We study free open fermionic strings on a non-commutative phase-space. Modified super-Virasoro algebras in both the Ramond and Neveu–Schwarz sectors acquire non-commutativity anomalies, and this non-commtativity breaks also the Lorentz symmetry and give a non-diagonal mass operator. Redefining the Fock space diagonalizes the mass operator. Extra constraints on non-commutativity parameters cancel the anomalies, restore the standard spectrum, and make the GSO projection possible.

DOI:10.5506/APhysPolB.57.7-A6

1. Introduction

The non-commutativity was first studied by Connes [1], who considered it as a relation between many connections in physics, and later in string theory [2, 3].

One can also obtain non-commutativity when considering a string interacting with an antisymmetric B-field or a Neveu–Schwarz (N–S) B-field [4–11]. The main result is the fact that the equations of motion do not depend on the B-field while the Noether currents, the boundaries conditions, and in particular, the momentum do. A great number of works [4–13] have investigated the covariant quantization of this theory and showed that the coordinates extremities of the string are non-commutative and that the physical states are subject to the Virasoro conditions, which depend on the B-field. Throughout these works, the background antisymmetric field $B_{\mu\nu}$ is understood to have support only along the Neumann directions of the D_p-brane, B_{ij} with $i, j = 0, \dots, p$ labeling directions along the brane worldvolume, since only these components contribute to the open-string boundary variation and modify the effective worldvolume dynamics, while mixed or

* Funded by SCOAP[3] under Creative Commons License, CC-BY 4.0.
† Corresponding author: abdelmoumene.medadib@umc.edu.dz

Dirichlet components do not. As a result, the effect of the B-field is fully captured by an effective field strength F_{ij}, and the induced non-commutative structure arises solely on the D-brane worldvolume.

The B-field dependence of the momentum makes the definition of the light-cone gauge difficult. This difficulty is resolved for the closed bosonic string [14] when it wraps a compactified dimension along which the only non-zero component of the B-field is present. The periodicity conditions allow then for the use of the light-cone gauge.

Another way consists of considering a string propagating in a non-commutative worldsheet. Our motivation for investigating non-commutativity on the worldsheet stems from the fact that when the latter propagates in a non-commutative spacetime, it inherits this non-commutative structure and consequently becomes a two-dimensional non-commutative space [15–17]. The commutation relations between the modes and the Virasoro algebra are modified. These modifications will affect the mass spectrum and the Lorentz invariance.

It is important to emphasize that such a theory is inherently non-conformal and lacks Lorentz invariance. Nevertheless, it is worthwhile to investigate this framework in order to explore possible ways of addressing these issues by taking advantage of the existence of two independent non-commutativity parameters $\theta_{(m)}$ and $\gamma_{(m)}$, and examining potential relations between them. This is the principal motivation for introducing non-commutativity in the full phase-space, rather than only between coordinate variables, to explore whether the Virasoro (and Lorentz) algebras can be restored or not, by imposing a consistent relation between the two deformation parameters, $\theta_{(m)}$ and $\gamma_{(m)}$. Our results show that the Virasoro algebra can indeed be recovered in a non-commutative phase-space when the deformation is restricted to the center-of-mass variables. On the other hand, in order to restore the Lorentz algebra completely, one must return to an ordinary (commutative) phase-space.

In contrast to the Seiberg–Witten framework [6], where non-commutativity arises in the target-space coordinates of open strings propagating in a background antisymmetric $B_{\mu\nu}$ field, the deformation considered here is canonical rather than geometric: it affects the equal-time commutation relations of the string coordinates and their conjugate momenta, while leaving the target-space geometry itself commutative. As a result, the theory does not define a non-commutative quantum field theory on spacetime, and the well-known consistency and regularization issues discussed in non-commutative gauge theories — such as UV/IR mixing, unitarity violation, or renormalization ambiguities — do not arise within the present framework. Moreover, since the super-Virasoro generators depend explicitly on both the coordinate and momentum variables, a deformation restricted solely to the coordinate sector would generically obstruct the closure of the algebra. A simultaneous deformation of coordinates and momenta seems therefore to be the minimal extension required to preserve the algebraic consistency perturbatively.

In this paper, we investigate a fermionic open string theory that moves in a non-commutative phase-space. In Section 2, we describe how non-commutativity in the phase-space is obtained from non-commutativity in the worldsheet. In Section 3, the commutation relations for the string coordinates are postulated and the oscillator algebra is obtained. In Section 4, we calculate the modified super-Virasoro algebra for both the Ramond and Neveu–Schwarz sectors. In Section 5, we deduce the modified Lorentz algebra. In Section 6, mass spectrum and the GSO projection are discussed. Finally, we summarize our results.

2. Non-commutative phase-space from a non-commutative worldsheet

In the standard formulation of string theory, the worldsheet is a two-dimensional manifold parameterized by the coordinates $\sigma^a = (\tau, \sigma)$. However, in a non-commutative framework, these coordinates satisfy the fundamental commutation relation [16, 17] given by

$$\left[\sigma^a, \sigma^b\right] = i\theta^{ab}\,, \tag{1}$$

where θ^{ab} is an antisymmetric constant tensor representing the non-commutativity of the worldsheet. As a result, a deformation of the usual product of functions, introducing the Moyal star product [16, 17] is as follows:

$$(f * g)(\sigma, \tau) = f(\sigma, \tau)\, \mathrm{e}^{2i\theta^{ab}\overleftarrow{\partial_a}\,\overrightarrow{\partial_b}}\, g(\sigma, \tau)\,. \tag{2}$$

Expanding the exponential, we find

$$\begin{aligned} f(\sigma, \tau) * g(\sigma, \tau) \ &= \ f(\sigma, \tau) g(\sigma, \tau) + \frac{i}{2}\theta^{ab}\partial_a f \partial_b g \\ &\quad -\frac{1}{8}\theta^{ab}\theta^{cd}\left(\partial_a\partial_c f\right)\left(\partial_b\partial_d g\right) + \mathcal{O}\left(\theta^3\right)\,. \end{aligned} \tag{3}$$

For all calculations in the manuscript, we restrict ourselves to linear order in θ, as higher orders produce higher-derivative (non-local) corrections which are beyond the present semiclassical treatment. This truncation is consistent with standard treatments of non-commutative field theory [16].

The action of the superstring in this framework is modified as well

$$S_* = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\left(\partial_a X^\mu * \partial^a X_\mu - i\bar{\psi}^\mu * \rho^a \partial_a \psi_\mu\right)\,, \tag{4}$$

where the star product replaces the usual multiplication in all field interactions, and

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{5}$$

$$\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{6}$$

that verify the Clifford algebra given by

$$\left\{\rho^\alpha, \rho^\beta\right\} = -2\eta^{\alpha\beta}. \tag{7}$$

We will show how non-commutative structure propagates to the spacetime coordinates.

The non-commutativity of the worldsheet directly induces non-commutativity in the spacetime coordinates [16]. By using the star product, the commutator between spacetime coordinates takes the form

$$\begin{aligned}\left[X^\mu(\sigma,\tau), X^\nu\left(\sigma',\tau\right)\right]_* &= \theta(2\alpha')^{3/2}\sum_{n\neq 0}\left[(p^\mu\alpha_n^\nu - p^\nu\alpha_n^\mu)]\mathrm{e}^{-in\tau}\sin(n\sigma)\right] \\ &\quad -2\alpha'\sum_{m\neq 0}\sum_{n\neq 0}\left(\frac{1}{mn}\left(\alpha_n^\mu\alpha_m^\nu - \alpha_n^\nu\alpha_m^\mu\right)\mathrm{e}^{-i(m+n)\tau}\right. \\ &\quad \left.\times\cos\left(m\left(\sigma+\frac{1}{2}n\theta\right)\right)\cos\left(n\left(\sigma'-\frac{1}{2}m\theta\right)\right)\right).\end{aligned} \tag{8}$$

This result shows that the non-commutativity of the worldsheet introduces a deformed algebra for spacetime coordinates.

Since the momentum density is defined as

$$P^\mu(\sigma,\tau) = \frac{1}{2\pi\alpha'}\dot{X}^\mu \tag{9}$$

and inherits again the non-commutative structure as X^μ, the corresponding commutator in the presence of a worldsheet non-commutativity is found to be

$$\begin{aligned}\left[P^\mu(\sigma,\tau), P^\nu\left(\sigma',\tau\right)\right]_* &= \frac{1}{2\alpha'\pi^2}\sum_{m\neq 0}\sum_{n\neq 0}\left(\left(\alpha_n^\mu\alpha_m^\nu - \alpha_n^\nu\alpha_m^\mu\right)\mathrm{e}^{-i(m+n)\tau}\right. \\ &\quad \left.\times\cos\left(m\left(\sigma+\frac{1}{2}n\theta\right)\right)\cos\left(n\left(\sigma'-\frac{1}{2}m\theta\right)\right)\right).\end{aligned} \tag{10}$$

Thus, both position and momentum variables obey the non-commutative relations. This will leads to a fundamental deformation of the phase-space, where both position and momentum no longer satisfy the usual Poisson structure but instead obey a deformed algebra.

3. Non-commutative phase-space and oscillator algebra

Let us now consider a non-commutative phase-space described by the following non-commutative commutation relations [18, 19]:

$$\begin{aligned}
\left[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\right] &= i\eta^{\mu\nu}\delta\left(\sigma-\sigma'\right) , \\
\left[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\right] &= i\theta^{\mu\nu}\left(\sigma-\sigma'\right) , \\
\left[P^\mu(\tau,\sigma), P^\nu(\tau,\sigma')\right] &= i\gamma^{\mu\nu}\left(\sigma-\sigma'\right) , \\
\left\{\psi^\mu(\tau,\sigma), \psi^\nu(\tau,\sigma')\right\} &= \eta^{\mu\nu}\delta\left(\sigma-\sigma'\right) ,
\end{aligned} \tag{11}$$

where $P^\mu(\tau,\sigma) = \frac{1}{2\pi\alpha'}\partial_\tau X^\mu(\tau,\sigma)$, $\theta^{\mu\nu}$ represents the non-commutativity parameters of the space part, and $\gamma^{\mu\nu}$ those of the momentum part of the phase-space.

For the sake of simplicity, this non-commutativity is only postulated by assuming the existence of a non-commutative worldsheet as the origin of (11), without worrying about the latter. Consequently, since the equation of motion remains unchanged (see Appendix B), the solutions and the Virasoro operators also remain unaffected.

The equations of motions are

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = \partial_+\psi_-^\mu = \partial_-\psi_+^\mu = 0\,, \tag{12}$$

where $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$, while ψ_- and ψ_+ are the right- and left-moving components of ψ^μ.

One can write the Fourier expansions for the variables $\theta^{\mu\nu}(\sigma-\sigma')$, $\gamma^{\mu\nu}(\sigma-\sigma')$ [18], and $X^\mu(\tau,\sigma)$, $\psi^\mu(\tau,\sigma)$ [20–22]

$$\theta^{\mu\nu}\left(\sigma-\sigma'\right) = \sum_{n=-\infty}^{+\infty} \theta_n^{\mu\nu}\, \mathrm{e}^{in(\sigma-\sigma')}\,, \tag{13}$$

$$\gamma^{\mu\nu}\left(\sigma-\sigma'\right) = \sum_{n=-\infty}^{+\infty} \gamma_n^{\mu\nu}\, \mathrm{e}^{in(\sigma-\sigma')}\,, \tag{14}$$

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\cos(n\sigma)\,\mathrm{e}^{-in\tau}\,, \tag{15}$$

$$\begin{aligned} \text{N–S sector} : \psi^{\mu}(\tau,\sigma) &= \frac{1}{\sqrt{2}} \sum_{r\in Z+\frac{1}{2}} b_r^{\mu}\, \mathrm{e}^{-ir(\tau-\sigma)}\,, \\ \text{R sector} : \psi^{\mu}(\tau,\sigma) &= \frac{1}{\sqrt{2}} \sum_{n\in Z} d_n^{\mu}\, \mathrm{e}^{-in(\tau-\sigma)}\,. \end{aligned} \tag{16}$$

By using equations (13)–(16), we can verify that equations (11) are equivalent to the following commutation relations of the oscillator algebra [18]:

$$\begin{aligned} [p^{\mu},p^{\nu}] &= i\pi^2{\gamma_0}^{\mu\nu}\,, \\ [x^{\mu},p^{\nu}] &= i\eta^{\mu\nu} - 2i\pi^2\alpha'\tau\gamma_0^{\mu\nu}\,, \\ [x^{\mu},x^{\nu}] &= i\theta_0^{\mu\nu} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\nu}\,, \end{aligned} \tag{17}$$

$$[\alpha_m^{\mu},\alpha_n^{\nu}] = \left(m\eta^{\mu\nu} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\mu\nu} + i\frac{n^2}{2\alpha'}\theta_n^{\mu\nu} \right)\delta_{n+m,0}\,, \tag{18}$$

$$\begin{cases} \{d_m^{\mu},d_n^{\nu}\} &= \eta^{\mu\nu}\delta_{m+n,0}\,, \\ \{b_r^{\mu},b_s^{\nu}\} &= \eta^{\mu\nu}\delta_{r+s,0}\,. \end{cases} \tag{19}$$

4. Modified super-Virasoro algebra

We define the Virasoro generators in a quantized system as [20–23]

— For the Ramond sector:

$$L_m = L_m^{\alpha} + L_m^{d} = \begin{cases} L_m^{\alpha} = \frac{1}{2}\sum\limits_{n\in Z} :\alpha_{-n}\alpha_{m+n}: \\ L_m^{d} = \frac{1}{2}\sum\limits_{n\in Z} \left(n+\frac{1}{2}m\right) :d_{-n}d_{m+n}: \end{cases} \tag{20}$$

$$F_m = \sum_{n\in Z} \alpha_{-n}d_{m+n} \tag{21}$$

which represent the fermionic sector.

— For the Neveu–Schwarz sector:

$$L_m = L_m^{\alpha} + L_m^{b} = \begin{cases} L_m^{\alpha} = \frac{1}{2}\sum\limits_{n\in Z} :\alpha_{-n}\alpha_{m+n}: \\ L_m^{b} = \frac{1}{2}\sum\limits_{r\in Z+\frac{1}{2}} \left(r+\frac{1}{2}m\right) :b_{-r}b_{m+r}: \end{cases} \tag{22}$$

$$G_r = \sum_{n\in Z} \alpha_{-n}b_{r+n} \tag{23}$$

which represent the bosonic sector.

Due to the modifications in the oscillator algebra (18), one can deduce the modified super-Virasoro algebras for both sectors

$$\left[L_m^{(\alpha)}, L_n^{(\alpha)}\right] = (m-n)L_{n+m}^{(\alpha)} + \frac{d}{12}m\left(m^2-1\right)\delta_{m+n,0} + R_{mn}\,, \tag{24}$$

where n represents the integer modes of the bosonic oscillators, where $n \in \mathbb{Z}$, and R_{mn} represents the anomaly part due to the non-commutativity, defined as

$$\begin{aligned} R_{mn} &= -\frac{1}{2}\sum_{p=-\infty}^{+\infty}\Big[2i\alpha'\pi^2\left(\gamma_{p-n}^{\nu\mu} + \gamma_{m-p}^{\mu\nu}\right) \\ &\quad + \frac{i}{2\alpha'}\left((p-n)^2\theta_{p-n}^{\nu\mu} + (m-p)^2\theta_{m-p}^{\mu\nu}\right)\Big]\alpha_p^\mu\alpha_{m+n-p}^\nu\,, \end{aligned} \tag{25}$$

which is not the same result given by Mousavi in [18].

The super-algebra then, is given by

— For the Neveu–Schwarz sector:

$$[L_m, L_n] = (m-n)L_{n+m} + \frac{D}{8}m\left(m^2-1\right)\delta_{m+n,0} + R_{mn}\,, \tag{26}$$

$$[L_m, G_r] = \left(\frac{1}{2}m - r\right)G_{m+r} + V_{mr}\,, \tag{27}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s} + B_{rs}\,, \tag{28}$$

with the new anomaly terms B_{rs} and V_{mr} given by

$$\begin{aligned} B_{rs} &= -\frac{1}{2}\sum_{q=-\infty}^{+\infty}\Big[2i\alpha'\pi^2\left(\gamma_{q-s}^{\nu\mu} + \gamma_{r-q}^{\mu\nu}\right) \\ &\quad + \frac{i}{2\alpha'}\left((q-s)^2\theta_{q-s}^{\nu\mu} + (r-q)^2\theta_{r-q}^{\mu\nu}\right)\Big]b_q^\mu b_{r+s-q}^\nu\,, \end{aligned} \tag{29}$$

$$\begin{aligned} V_{mr} &= -\frac{1}{2}\sum_{q=-\infty}^{+\infty}\Big[2i\alpha'\pi^2\left(\gamma_{q-r}^{\nu\mu} + \gamma_{m-q}^{\mu\nu}\right) \\ &\quad + \frac{i}{2\alpha'}\left((q-r)^2\theta_{q-r}^{\nu\mu} + (m-q)^2\theta_{m-q}^{\mu\nu}\right)\Big]\alpha_q^\mu b_{r+m-q}^\nu\,, \end{aligned} \tag{30}$$

where $n \in \mathbb{Z}$ and $r \in \mathbb{Z}+\frac{1}{2}$. In B_{rs}, q sums over all half-integer modes, while in V_{mr}, q sums over all integer modes.

— For the Ramond sector:

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{n+m} + \frac{D}{8}m^3\delta_{m+n,0} + R_{mn}\,, && (31)\\
[L_m, F_n] &= \left(\frac{1}{2}m - n\right)F_{m+n} + W_{mn}\,, && (32)\\
\{F_r, F_s\} &= 2L_{r+s} + \frac{D}{2}r^2\delta_{r+s} + D_{rs}\,, && (33)
\end{aligned}
$$

again with the new anomaly terms D_{rs} and W_{mn} given by

$$
\begin{aligned}
D_{rs} &= -\frac{1}{2}\sum_{q=-\infty}^{+\infty}\Bigg[2i\alpha'\pi^2\left(\gamma^{\nu\mu}_{q-s} + \gamma^{\mu\nu}_{r-q}\right)\\
&\quad + \frac{i}{2\alpha'}\left((q-s)^2\theta^{\nu\mu}_{q-s} + (r-q)^2\theta^{\mu\nu}_{r-q}\right)\Bigg]d^\mu_q d^\nu_{r+s-q}\,, && (34)\\
W_{mn} &= -\frac{1}{2}\sum_{q=-\infty}^{+\infty}\Bigg[2i\alpha'\pi^2\left(\gamma^{\nu\mu}_{q-n} + \gamma^{\mu\nu}_{m-q}\right)\\
&\quad + \frac{i}{2\alpha'}\left((q-n)^2\theta^{\nu\mu}_{q-n} + (m-q)^2\theta^{\mu\nu}_{m-q}\right)\Bigg]\alpha^\mu_q d^\nu_{n+m-q}\,, && (35)
\end{aligned}
$$

where $n, r \in \mathbb{Z}$ and q sums over all integer modes.

5. Modified Lorentz algebra

The angular momentum $M^{\mu\nu}$ is given by [20–23]

$$
M^{\mu\nu} = \begin{cases}
x^\mu p^\nu - x^\nu p^\mu - i\sum\limits_{n=1}^{\infty}\frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right)\\
-\frac{i}{4}\sum\limits_{r=-\infty}^{+\infty}\left(b^\mu_{-r}b^\nu_r - b^\nu_{-r}b^\mu_r\right) \to \text{N–S sector}\,,\\[2ex]
x^\mu p^\nu - x^\nu p^\mu - i\sum\limits_{n=1}^{\infty}\frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right)\\
-\frac{i}{4}\sum\limits_{m=-\infty}^{+\infty}\left(d^\mu_{-m}d^\nu_m - d^\nu_{-m}d^\mu_m\right) \to \text{R sector}\,,
\end{cases} \qquad (36)
$$

By using equations (17)–(19), a direct calculation (see Appendix A) gives the following modified Lorentz algebra:

$$
\left[M^{\mu\nu}, M^{\rho\lambda}\right] = -i\eta^{\nu\rho}M^{\mu\lambda} + i\eta^{\mu\lambda}M^{\rho\nu} + i\eta^{\nu\lambda}M^{\mu\rho} - i\eta^{\mu\rho}M^{\lambda\nu} + T^{\mu\nu\rho\lambda}\,, \quad (37)
$$

$$
[p^\mu, M^{\nu\rho}] = i\eta^{\rho\mu}p^\nu - i\eta^{\nu\mu}p^\rho + K^{\nu\mu\rho}\,, \qquad (38)
$$

$$
[p^\mu, p^\nu] = i\pi^2\gamma^{\mu\nu}_0\,, \qquad (39)
$$

where $T^{\mu\nu\rho\lambda}$ and $K^{\nu\mu\rho}$ represent the anomalies due to the non-commutativity and are given by

$$
\begin{aligned}
T^{\mu\nu\rho\lambda} \;=\; & i\pi^2\left(\gamma_0^{\nu\lambda}x^\mu x^\rho + \gamma_0^{\nu\rho}x^\mu x^\lambda + +\gamma_0^{\mu\lambda}x^\nu x^\rho + \gamma_0^{\mu\rho}x^\nu x^\lambda\right) \\
& +2i\pi^2\alpha'\tau\left(\gamma_0^{\nu\rho}x^\mu p^\lambda - \gamma_0^{\mu\lambda}x^\rho p^\nu + \gamma_0^{\nu\lambda}x^\mu p^\rho - \gamma_0^{\mu\rho}x^\lambda p^\nu + \gamma_0^{\mu\rho}x^\nu p^\lambda\right. \\
& \left.-\gamma_0^{\nu\lambda}x^\rho p^\mu + \gamma_0^{\mu\lambda}x^\nu p^\rho - \gamma_0^{\nu\rho}x^\lambda p^\mu\right) \\
& +\left(i\theta_0^{\mu\rho} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\rho}\right)p^\lambda p^\nu + \left(i\theta_0^{\mu\lambda} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\lambda}\right)p^\rho p^\nu \\
& +\left(i\theta_0^{\nu\rho} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\nu\rho}\right)p^\lambda p^\mu + \left(i\theta_0^{\nu\lambda} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\nu\lambda}\right)p^\rho p^\mu \\
& +\left(i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\nu\rho} + i\frac{n^2}{2\alpha'}\theta_n^{\nu\rho}\right)\left(\alpha_{-n}^\mu\alpha_n^\lambda + \alpha_{-n}^\lambda\alpha_n^\mu\right) \\
& +\left(i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\mu\lambda} + i\frac{n^2}{2\alpha'}\theta_n^{\mu\lambda}\right)\left(\alpha_{-n}^\rho\alpha_n^\lambda + \alpha_{-n}^\lambda\alpha_n^\rho\right) \\
& +\left(i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\nu\lambda} + i\frac{n^2}{2\alpha'}\theta_n^{\nu\lambda}\right)\left(\alpha_{-n}^\rho\alpha_n^\mu + \alpha_{-n}^\mu\alpha_n^\rho\right) \\
& +\left(i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\mu\rho} + i\frac{n^2}{2\alpha'}\theta_n^{\mu\rho}\right)\left(\alpha_{-n}^\lambda\alpha_n^\nu + \alpha_{-n}^\nu\alpha_n^\lambda\right), \qquad (40)
\end{aligned}
$$

$$
K^{\nu\mu\rho} \;=\; 2i\pi^2\alpha'\tau\gamma_0^{\nu\mu}p^\rho - 2i\pi^2\alpha'\tau\gamma_0^{\rho\mu}p^\nu + i\pi^2\gamma_0^{\mu\rho}p^\nu - i\pi^2\gamma_0^{\mu\nu}p^\rho\,. \qquad (41)
$$

6. Mass spectrum and GSO projection

The calculation of the mass spectrum required working in the light-cone coordinates. Equation (18) will take the form

$$
\left[\alpha_m^i, \alpha_n^j\right] = \left(m\eta^{ij} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{ij} + i\frac{n^2}{2\alpha'}\theta_n^{ij}\right)\delta_{n+m,0}\,, \qquad (42)
$$

where $i, j = 2\ldots D-1$ represent the transversal indices, while the anti-commutation relations between the fermionic modes remain unchanged

$$
\left\{d_m^i, d_n^j\right\} \;=\; \eta^{ij}\delta_{m+n,0}\,, \qquad (43)
$$

$$
\left\{b_r^i, b_s^j\right\} \;=\; \eta^{ij}\delta_{r+s,0}\,. \qquad (44)
$$

The mass operator for both sectors are given by

— For the Ramond sector:

$$
M_{\mathrm{R}}^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i + \sum_{r=1}^{\infty}r d_{-r}^i d_r^i\right). \qquad (45)
$$

— For the Neveu–Schwarz sector:

$$M_{\rm NS}^2=\frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^i+\sum_{r=\frac{1}{2}}^{\infty}rb_{-r}^ib_r^i-\frac{1}{2}\right). \tag{46}$$

We need to diagonalize the antisymmetric matrices θ_m^{ij} and γ_m^{ij} by introducing the unitary matrix U_m such that

$$\left(U_m^{-1}i\theta_mU_m\right)^{ij}=D_m^{ij}=\mu_i^{(m)}\delta^{ij}\,, \tag{47}$$

and,

$$\left(U_m^{-1}i\gamma_mU_m\right)^{ij}=T_m^{ij}=\nu_i^{(m)}\delta^{ij}\,, \tag{48}$$

with $[\theta_m,\gamma_m]=0$, and $\mu_i^{(m)}$ and $\nu_i^{(m)}$ being the eigenvalues of $i{\theta_m}^{ij}$ and $i{\gamma_m}^{ij}$ respectively.

This latter diagonalization, given by equations (47) and (48), can be obtained through a redefinition of the Fock space [24] in order to get a diagonal mass in the new basis. The redefinition takes the form

$$\prod_{i=2}^{D-1}\prod_{m=1}^{\infty}\left(\alpha_{-m}^i\right)^{\lambda_{m,i}}\prod_{j=2}^{D-1}\begin{pmatrix}\prod\limits_{r=\frac{1}{2},\frac{3}{2}\ldots}\left(b_{-r}^j\right)^{\rho_{r,j}}\\ \text{or}\\ \prod\limits_{n=1,2\ldots}\left(d_{-n}^j\right)^{\rho_{n,j}}\end{pmatrix}\left|p^+,\vec{p}^T\right\rangle$$

$$\to\prod_{i=2}^{D-1}\prod_{m=1}^{\infty}\left\{\left(U_m^{-1}\,\alpha_{-m}\right)^i\right\}^{\lambda_{m,i}}\prod_{j=2}^{D-1}\begin{pmatrix}\prod\limits_{r=\frac{1}{2},\frac{3}{2}\ldots}\left(b_{-r}^j\right)^{\rho_{r,j}}\\ \text{or}\\ \prod\limits_{n=1,2\ldots}\left(d_{-n}^j\right)^{\rho_{n,j}}\end{pmatrix}\left|p^+,\vec{p}^T\right\rangle, \tag{49}$$

where the non-negative integer $\lambda_{m,i}$ shows [20] how many times the creation operator α_{-m}^i appears, and $\rho_{n,k}$ takes either zero or one.

In order to get an equivalent of a GSO projection, one can follow the usual way to get the following steps summarized in Table 1 and Table 2. The results of the GSO projection for the two sectors are grouped in Table 3.

Table 1. The mass spectrum in terms of redefined modes for the Neveu–Schwarz sector.

Level	Neveu–Schwarz sector	
	State	Mass
0	$\lvert 0\rangle$	$-\frac{1}{2\alpha'}$
1	$b^i_{-\frac{1}{2}}\lvert 0\rangle$	0
2	$b^i_{-\frac{1}{2}} b^j_{-\frac{1}{2}}\lvert 0\rangle$	$\frac{1}{2\alpha'}$
	$U_1^{-1}\alpha^i_{-1}\lvert 0\rangle$	$\frac{1}{\alpha'}\left(\frac{1}{2}-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\,\nu_j^{(1)}\right)\right)$
3	$b^i_{-\frac{1}{2}} b^j_{-\frac{1}{2}} b^k_{-\frac{1}{2}}\lvert 0\rangle$	$\frac{1}{\alpha'}$
	$b^i_{-\frac{3}{2}}\lvert 0\rangle$	$\frac{1}{\alpha'}$
	$U_1^{-1}\alpha^i_{-1} b^j_{-\frac{1}{2}}\lvert 0\rangle$	$\frac{1}{\alpha'}\left(1-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\,\nu_j^{(1)}\right)\right)$
4	$b^i_{-\frac{1}{2}} b^j_{-\frac{1}{2}} b^k_{-\frac{1}{2}} b^l_{-\frac{1}{2}}\lvert 0\rangle$	$\frac{3}{2\alpha'}$
	$b^i_{-\frac{3}{2}} b^j_{-\frac{1}{2}}\lvert 0\rangle$	$\frac{3}{2\alpha'}$
	$U_2^{-1}\alpha^i_{-2}\lvert 0\rangle$	$\frac{1}{\alpha'}\left(\frac{3}{2}-\frac{1}{2\alpha'}\left(4\mu_j^{(2)}+(2\pi\alpha')^2\,\nu_j^{(2)}\right)\right)$
	$U_1^{-1}\alpha^j_{-1}U_1^{-1}\alpha^k_{-1}\lvert 0\rangle$	$\frac{1}{\alpha'}\left(\frac{3}{2}-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\,\nu_j^{(1)}+\mu_k^{(1)}+(2\pi\alpha')^2\,\nu_k^{(1)}\right)\right)$
	$U_1^{-1}\alpha^j_{-1} b^k_{-\frac{1}{2}} b^l_{-\frac{1}{2}}\lvert 0\rangle$	$\frac{1}{\alpha'}\left(\frac{3}{2}-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\,\nu_j^{(1)}\right)\right)$
5	$b^i_{-\frac{1}{2}} b^j_{-\frac{1}{2}} b^k_{-\frac{1}{2}} b^l_{-\frac{1}{2}} b^m_{-\frac{1}{2}}\lvert 0\rangle$	$\frac{2}{\alpha'}$
	$b^i_{-\frac{5}{2}}\lvert 0\rangle$	$\frac{2}{\alpha'}$
	$b^i_{-\frac{1}{2}} b^j_{-\frac{1}{2}} b^k_{-\frac{1}{2}}$	$\frac{2}{\alpha'}$
	$U_2^{-1}\alpha^j_{-2} b^k_{-\frac{1}{2}}\lvert 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(4\mu_j^{(2)}+(2\pi\alpha')^2\,\nu_j^{(2)}\right)\right)$
	$U_1^{-1}\alpha^j_{-1}U_1^{-1}\alpha^k_{-1} b^l_{-\frac{1}{2}}\lvert 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\,\nu_j^{(1)}+\mu_k^{(1)}+(2\pi\alpha')^2\,\nu_k^{(1)}\right)\right)$
	$U_1^{-1}\alpha^j_{-1} b^k_{-\frac{1}{2}} b^l_{-\frac{1}{2}} b^m_{-\frac{1}{2}}\lvert 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\,\nu_j^{(1)}\right)\right)$
	$U_1^{-1}\alpha^j_{-1} b^k_{-\frac{3}{2}}\lvert 0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\,\nu_j^{(1)}\right)\right)$

Table 2. The mass spectrum in terms of redefined modes for the Ramond sector.

Level	Ramond sector	
	State	Mass
0	$\|0\rangle$	0
1	$d^j_{-1}\|0\rangle$	$\frac{1}{\alpha'}$
	$U_1^{-1}\alpha^j_{-1}\|0\rangle$	$\frac{1}{\alpha'}\left(1-\left(\frac{1}{2\alpha'}\mu^{(1)}_j+\frac{(2\pi\alpha')^2}{2\alpha'}\nu^{(1)}_j\right)\right)$
2	$d^j_{-2}\|0\rangle$	$\frac{2}{\alpha'}$
	$d^j_{-1}d^k_{-1}\|0\rangle$	$\frac{2}{\alpha'}$
	$U_2^{-1}\alpha^j_{-2}\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(4\mu^{(2)}_j+(2\pi\alpha')^2\nu^{(2)}_j\right)\right)$
	$\left(U_1^{-1}\alpha^j_{-1}\right)\left(U_1^{-1}\alpha^k_{-1}\right)\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu^{(1)}_j+(2\pi\alpha')^2\nu^{(1)}_j +\mu^{(1)}_k+(2\pi\alpha')^2\nu^{(1)}_k\right)\right)$
	$\left(U_1^{-1}\alpha^j_{-1}\right)d^k_{-1}\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu^{(1)}_j+(2\pi\alpha')^2\nu^{(1)}_j\right)\right)$

Then, one can impose

$$\nu^{(1)}_i = \frac{-1}{(2\pi\alpha')^2}\mu^{(1)}_i \tag{50}$$

to restore the value of the mass for the first excited state (for example), and in general

$$\nu^{(m)}_i = \frac{-m^2}{(2\pi\alpha')^2}\mu^{(m)}_i \tag{51}$$

being an equivalent to

$$T^{ij}_{(m)} = \frac{-m^2}{(2\pi\alpha')^2}D^{ij}_{(m)} \tag{52}$$

which can be use to restore mass values of the other levels, where $m > 0$ represent the number of state level.

In the present construction, the non-commutative deformation is implemented through a redefinition of the bosonic oscillators, while the fermionic modes remain unchanged. Consequently, the fermionic Fock space and its canonical anticommutation relations are preserved. The supersymmetric pairing underlying the GSO projection remains therefore intact within the considered perturbative regime.

Table 3. The GSO projection for the Neveu–Schwarz and Ramond sectors.

Level	Neveu–Schwarz sector	
	State	Mass
1	$b^i_{-\frac{1}{2}}\|0\rangle$	0
3	$b^i_{-\frac{1}{2}}b^j_{-\frac{1}{2}}b^k_{-\frac{1}{2}}\|0\rangle$	$\frac{1}{\alpha'}$
	$b^i_{-\frac{3}{2}}\|0\rangle$	$\frac{1}{\alpha'}$
	$U_1^{-1}\alpha^i_{-1}b^j_{-\frac{1}{2}}\|0\rangle$	$\frac{1}{\alpha'}\left(1+\left(\frac{1}{2\alpha'}\mu_j^{(1)}+\frac{(2\pi\alpha')^2}{2\alpha'}\nu_j^{(1)}\right)\right)$
5	$b^i_{-\frac{1}{2}}b^j_{-\frac{1}{2}}b^k_{-\frac{1}{2}}b^l_{-\frac{1}{2}}b^m_{-\frac{1}{2}}\|0\rangle$	$\frac{2}{\alpha'}$
	$b^i_{-\frac{5}{2}}\|0\rangle$	$\frac{2}{\alpha'}$
	$b^i_{-\frac{1}{2}}b^j_{-\frac{1}{2}}b^k_{-\frac{1}{2}}$	$\frac{2}{\alpha'}$
	$U_2^{-1}\alpha^j_{-2}b^k_{-\frac{1}{2}}\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(4\mu_j^{(2)}+(2\pi\alpha')^2\nu_j^{(2)}\right)\right)$
	$U_1^{-1}\alpha^j_{-1}U_1^{-1}\alpha^k_{-1}b^l_{-\frac{1}{2}}\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}+\mu_k^{(1)}+(2\pi\alpha')^2\nu_k^{(1)}\right)\right)$
	$U_1^{-1}\alpha^j_{-1}b^k_{-\frac{1}{2}}b^l_{-\frac{1}{2}}b^m_{-\frac{1}{2}}\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$
	$U_1^{-1}\alpha^j_{-1}b^k_{-\frac{3}{2}}\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$

Level	Ramond sector	
	State	Mass
1	$\|0\rangle$	0
3	$d^i_{-1}\|0\rangle$	$\frac{1}{\alpha'}$
	$U_1^{-1}\alpha^i_{-1}\|0\rangle$	$\frac{1}{\alpha'}\left(1+\left(\frac{1}{2\alpha'}\mu_j^{(1)}+\frac{(2\pi\alpha')^2}{2\alpha'}\nu_j^{(1)}\right)\right)$
5	$d^j_{-2}\|0\rangle$	$\frac{2}{\alpha'}$
	$d^j_{-1}d^k_{-1}\|0\rangle$	$\frac{2}{\alpha'}$
	$U_2^{-1}\alpha^j_{-2}\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(4\mu_j^{(2)}+(2\pi\alpha')^2\nu_j^{(2)}\right)\right)$
	$\left(U_1^{-1}\alpha^j_{-1}\right)\left(U_1^{-1}\alpha^k_{-1}\right)\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}+\mu_k^{(1)}+(2\pi\alpha')^2\nu_k^{(1)}\right)\right)$
	$\left(U_1^{-1}\alpha^j_{-1}\right)d^k_{-1}\|0\rangle$	$\frac{1}{\alpha'}\left(2-\frac{1}{2\alpha'}\left(\mu_j^{(1)}+(2\pi\alpha')^2\nu_j^{(1)}\right)\right)$

Under the imposed condition on the deformation parameters (51), the resulting mass spectrum coincides with the commutative one at first order of θ (see Table 4). Beyond this order, higher-order corrections are expected to modify the spectrum, and a full analysis of these effects lies outside the scope of the present work.

Table 4. The first levels of the mass spectrum after the GSO projection and the application of equation (51) for the Neveu–Schwarz and Ramond sectors.

Level	Neveu–Schwarz sector		Ramond sector	
	State	Mass	State	Mass
1	$b^i_{-\frac{1}{2}}\|0\rangle$	0	$\|0\rangle$	0
3	$b^i_{-\frac{1}{2}}b^j_{-\frac{1}{2}}b^k_{-\frac{1}{2}}\|0\rangle$	$\frac{1}{\alpha'}$	$d^i_{-1}\|0\rangle$	$\frac{1}{\alpha'}$
	$b^i_{-\frac{3}{2}}\|0\rangle$	$\frac{1}{\alpha'}$		
	$U_1^{-1}\alpha^i_{-1}b^j_{-\frac{1}{2}}\|0\rangle$	$\frac{1}{\alpha'}$	$U_1^{-1}\alpha^i_{-1}\|0\rangle$	$\frac{1}{\alpha'}$
5	$b^i_{-\frac{1}{2}}b^j_{-\frac{1}{2}}b^k_{-\frac{1}{2}}b^l_{-\frac{1}{2}}b^m_{-\frac{1}{2}}\|0\rangle$	$\frac{2}{\alpha'}$	$d^j_{-2}\|0\rangle$	$\frac{2}{\alpha'}$
	$b^i_{-\frac{5}{2}}\|0\rangle$	$\frac{2}{\alpha'}$	$d^j_{-1}d^k_{-1}\|0\rangle$	$\frac{2}{\alpha'}$
	$b^i_{-\frac{1}{2}}b^j_{-\frac{1}{2}}b^k_{-\frac{1}{2}}$	$\frac{2}{\alpha'}$		
	$U_2^{-1}\alpha^j_{-2}b^k_{-\frac{1}{2}}\|0\rangle$	$\frac{2}{\alpha'}$	$U_2^{-1}\alpha^j_{-2}\|0\rangle$	$\frac{2}{\alpha'}$
	$U_1^{-1}\alpha^j_{-1}U_1^{-1}\alpha^k_{-1}b^l_{-\frac{1}{2}}\|0\rangle$	$\frac{2}{\alpha'}$	$\left(U_1^{-1}\alpha^j_{-1}\right)\left(U_1^{-1}\alpha^k_{-1}\right)\|0\rangle$	$\frac{2}{\alpha'}$
	$U_1^{-1}\alpha^j_{-1}b^k_{-\frac{1}{2}}b^l_{-\frac{1}{2}}b^m_{-\frac{1}{2}}\|0\rangle$	$\frac{2}{\alpha'}$		
	$U_1^{-1}\alpha^j_{-1}b^k_{-\frac{3}{2}}\|0\rangle$	$\frac{2}{\alpha'}$	$\left(U_1^{-1}\alpha^j_{-1}\right)d^k_{-1}\|0\rangle$	$\frac{2}{\alpha'}$

By applying $(U_m U_m^{-1})$ on the both sides of equations (47) and (48), one can show that equation (52) can be expressed with respect to θ^{ij}_m and γ^{ij}_m as

$$\gamma^{ij}_{(m)} = \frac{-m^2}{(2\pi\alpha')^2}\theta^{ij}_{(m)} \,. \tag{53}$$

From this result, we can fix our starting model (11) by imposing to $\theta^{\mu\nu}$ and $\gamma^{\mu\nu}$ the following relation:

$$\gamma^{\mu\nu}_{(m)} = \frac{-m^2}{(2\pi\alpha')^2}\theta^{\mu\nu}_{(m)} \,, \tag{54}$$

where $m \neq 0$ and $\mu, \nu = 0, 1 \ldots, D-1$.

With the condition (54), one can easily verify that all the anomaly terms (25), (29), (30), (34), and (35) of the modified Virasoro algebra are eliminated due to non-commutativity. This result is a direct consequence of the fact that we considered non-commutativity between coordinates and momenta rather than only between coordinates.

On the other hand, the Lorentz algebra's anomaly term (40) is simplified to

$$\begin{aligned}
T^{\mu\nu\rho\lambda} &= i\pi^2\left(\gamma_0^{\nu\lambda}x^\mu x^\rho+\gamma_0^{\nu\rho}x^\mu x^\lambda+\gamma_0^{\mu\lambda}x^\nu x^\rho+\gamma_0^{\mu\rho}x^\nu x^\lambda\right)\\
&\quad+2i\pi^2\alpha'\tau\left(\gamma_0^{\nu\rho}x^\mu p^\lambda-\gamma_0^{\mu\lambda}x^\rho p^\nu+\gamma_0^{\nu\lambda}x^\mu p^\rho-\gamma_0^{\mu\rho}x^\lambda p^\nu\right.\\
&\quad\left.+\gamma_0^{\mu\rho}x^\nu p^\lambda-\gamma_0^{\nu\lambda}x^\rho p^\mu+\gamma_0^{\mu\lambda}x^\nu p^\rho-\gamma_0^{\nu\rho}x^\lambda p^\mu\right)\\
&\quad+\left(i\theta_0^{\mu\rho}-4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\rho}\right)p^\lambda p^\nu+\left(i\theta_0^{\mu\lambda}-4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\lambda}\right)p^\rho p^\nu\\
&\quad+\left(i\theta_0^{\nu\rho}-4i\pi^2\alpha'^2\tau^2\gamma_0^{\nu\rho}\right)p^\lambda p^\mu+\left(i\theta_0^{\nu\lambda}-4i\pi^2\alpha'^2\tau^2\gamma_0^{\nu\lambda}\right)p^\rho p^\mu\,, \qquad (55)\\
K^{\nu\mu\rho} &= 2i\pi^2\alpha'\tau\gamma_0^{\nu\mu}p^\rho-2i\pi^2\alpha'\tau\gamma_0^{\rho\mu}p^\nu+i\pi^2\gamma_0^{\mu\rho}p^\nu-i\pi^2\gamma_0^{\mu\nu}p^\rho\,. \qquad (56)
\end{aligned}$$

Although the imposed condition on the deformation parameters (54) eliminates all anomaly terms in the modified super-Virasoro algebra, the Lorentz algebra still contains residual anomaly terms (55) and (56). Therefore, as it is the case in several studies [6, 11, 25, 26], in the present work, Lorentz symmetry cannot be restored pertubatively at a first order.

7. Summary and results

In this work, we have explored the effects of non-commutative phase-space on free open fermionic string theory. By introducing non-commutativity in both coordinates and momenta, we derived the modified super-Virasoro algebra for both the Ramond and Neveu–Schwarz sectors. This modification introduced additional anomaly terms, which could affect the consistency of the theory, including conformal invariance and the structure of the mass spectrum.

To resolve these issues, we imposed a specific relation between the non-commutativity parameters of space $\theta^{\mu\nu}_{(m)}$ and momentum $\gamma^{\mu\nu}_{(m)}$ (54). Physically, this relation shows that the deformations of coordinates and momenta are not independent but rather are part of a single consistent phase-space structure. It links the spatial and momentum non-commutativity in such a way that conformal symmetry is preserved (see Appendix C). In other words, the relation (54) defines the exact balance required for non-commutative effects to exist without breaking the Virasoro symmetry. This condition led also to the cancellation of all Virasoro anomalies, allowing the algebra to recover its standard form while maintaining the presence of non-commutativity at the fundamental level. Additionally, a redefinition of the Fock space was necessary to diagonalize the mass operator and preserve the usual mass spectrum.

Notice that it is the simultaneous presence of the non-commutative parameters θ and γ that allows us to impose the necessary restrictions, demonstrating that non-commutative deformations can be incorporated into string theory preserving the Virasoro algebra, provided that specific constraints are applied. This is the main motivation that led us to consider not a non-commutative spacetime, but a non-commutative phase-space.

Notice that in the previous study [25, 27], non-commutativity in the target space was found to produce partial supersymmetry breaking. In the present approach, the deformation acts on the phase-space and not directly on the target-space geometry. As a result, no explicit supersymmetry breaking is observed at first order. We restrict our conclusions to this perturbative regime.

Also, the analysis presented in this work is restricted to first order in the deformation parameters. Beyond this order, higher powers of the deformation parameters generate increasingly non-local contributions to the Virasoro generators. These terms might as well reintroduce new anomalies in the Virasoro and Lorentz algebra. A systematic investigation of these effects is left for future work.

This work is supported by the Algerian Ministry of High Education and Research under the PRFU project No. B00L02UN250120220011.

Appendix A

Calculation of $[M^{\mu\nu}, M^{\rho\lambda}]$

Using equations (17)–(19), one can find that

$$\begin{aligned}\left[x^\mu p^\nu, x^\rho p^\lambda\right] &= i\pi^2\gamma_0^{\nu\lambda}x^\mu x^\rho + \left(2i\pi^2\alpha'\tau\gamma_0^{\nu\rho} - i\eta^{\nu\rho}\right)x^\mu p^\lambda \\ &\quad + \left(i\eta^{\mu\lambda} - 2i\pi^2\alpha'\tau\gamma_0^{\mu\lambda}\right)x^\rho p^\nu + \left(i\theta_0^{\mu\rho} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\rho}\right)p^\lambda p^\nu\,,\end{aligned} \tag{A.1}$$

$$\begin{aligned}\left[x^\mu p^\nu, x^\rho p^\lambda\right] &= i\pi^2\gamma_0^{\nu\lambda}x^\mu x^\rho + \left(2i\pi^2\alpha'\tau\gamma_0^{\nu\rho} - i\eta^{\nu\rho}\right)x^\mu p^\lambda \\ &\quad + \left(i\eta^{\mu\lambda} - 2i\pi^2\alpha'\tau\gamma_0^{\mu\lambda}\right)x^\rho p^\nu + \left(i\theta_0^{\mu\rho} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\rho}\right)p^\lambda p^\nu\,,\end{aligned} \tag{A.2}$$

$$\left[x^\nu p^\mu, x^\lambda p^\rho\right] = i\pi^2\gamma_0^{\mu\rho}x^\nu x^\lambda + \left(2i\pi^2\alpha'\tau\gamma_0^{\mu\lambda} - i\eta^{\mu\lambda}\right)x^\nu p^\rho + \left(i\eta^{\nu\rho} - 2i\pi^2\alpha'\tau\gamma_0^{\nu\rho}\right)x^\lambda p^\mu + \left(i\theta_0^{\nu\lambda} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\nu\lambda}\right)p^\rho p^\mu\,, \tag{A.3}$$

$$\left[x^\nu p^\mu, x^\rho p^\lambda\right] = i\pi^2\gamma_0^{\mu\lambda}x^\nu x^\rho + \left(2i\pi^2\alpha'\tau\gamma_0^{\mu\rho} - i\eta^{\mu\rho}\right)x^\nu p^\lambda + \left(i\eta^{\nu\lambda} - 2i\pi^2\alpha'\tau\gamma_0^{\nu\lambda}\right)x^\rho p^\mu + \left(i\theta_0^{\nu\rho} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\nu\rho}\right)p^\lambda p^\mu\,. \tag{A.4}$$

For the mode part, we find that

$$\left[\alpha^\mu_{-n}\alpha^\nu_n, \alpha^\rho_{-m}\alpha^\lambda_m\right] = \left(n\eta^{\nu\rho} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\nu\rho} + i\frac{n^2}{2\alpha'}\theta_n^{\nu\rho}\right)\alpha^\mu_{-n}\alpha^\lambda_n + \left(-n\eta^{\mu\lambda} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\mu\lambda} + i\frac{n^2}{2\alpha'}\theta_n^{\mu\lambda}\right)\alpha^\rho_{-n}\alpha^\nu_n\,, \tag{A.5}$$

$$\left[\alpha^\mu_{-n}\alpha^\nu_n, \alpha^\lambda_{-m}\alpha^\rho_m\right] = \left(n\eta^{\nu\lambda} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\nu\lambda} + i\frac{n^2}{2\alpha'}\theta_n^{\nu\lambda}\right)\alpha^\mu_{-n}\alpha^\rho_n + \left(-n\eta^{\mu\rho} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\mu\rho} + i\frac{n^2}{2\alpha'}\theta_n^{\mu\rho}\right)\alpha^\lambda_{-n}\alpha^\nu_n\,, \tag{A.6}$$

$$\left[\alpha^\nu_{-n}\alpha^\mu_n, \alpha^\lambda_{-m}\alpha^\rho_m\right] = \left(n\eta^{\mu\lambda} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\mu\lambda} + i\frac{n^2}{2\alpha'}\theta_n^{\mu\lambda}\right)\alpha^\nu_{-n}\alpha^\rho_n + \left(-n\eta^{\nu\rho} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\nu\rho} + i\frac{n^2}{2\alpha'}\theta_n^{\nu\rho}\right)\alpha^\lambda_{-n}\alpha^\mu_n\,, \tag{A.7}$$

$$\left[\alpha^\nu_{-n}\alpha^\mu_n, \alpha^\rho_{-m}\alpha^\lambda_m\right] = \left(n\eta^{\mu\rho} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\mu\rho} + i\frac{n^2}{2\alpha'}\theta_n^{\mu\rho}\right)\alpha^\nu_{-n}\alpha^\lambda_n + \left(-n\eta^{\nu\lambda} + i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\nu\lambda} + i\frac{n^2}{2\alpha'}\theta_n^{\nu\lambda}\right)\alpha^\rho_{-n}\alpha^\mu_n\,, \tag{A.8}$$

$$\left[b^\mu_{-r}b^\nu_r, b^\rho_{-s}b^\lambda_s\right] = 2b^\mu_{-r}b^\rho_{-s}b^\nu_r b^\lambda_s + 2b^\mu_{-r}b^\nu_r b^\rho_{-s}b^\lambda_s + 2b^\rho_{-s}b^\mu_{-r}b^\lambda_s b^\nu_r + 2b^\mu_{-r}b^\rho_{-s}b^\lambda_s b^\nu_r - \eta^{\nu\lambda}b^\mu_{-r}b^\rho_r - \eta^{\nu\rho}b^\mu_{-r}b^\lambda_r - \eta^{\mu\lambda}b^\rho_{-r}b^\nu_r - \eta^{\mu\rho}b^\lambda_{-r}b^\nu_r\,, \tag{A.9}$$

$$\left[b^\nu_{-r}b^\mu_r, b^\rho_{-s}b^\lambda_s\right] = 2b^\nu_{-r}b^\rho_{-s}b^\mu_r b^\lambda_s + 2b^\nu_{-r}b^\mu_r b^\rho_{-s}b^\lambda_s + 2b^\rho_{-s}b^\nu_{-r}b^\lambda_s b^\mu_r + 2b^\nu_{-r}b^\rho_{-s}b^\lambda_s b^\mu_r - \eta^{\mu\lambda}b^\nu_{-r}b^\rho_r - \eta^{\mu\rho}b^\nu_{-r}b^\lambda_r - \eta^{\nu\lambda}b^\rho_{-r}b^\mu_r - \eta^{\nu\rho}b^\lambda_{-r}b^\mu_r\,, \tag{A.10}$$

$$\begin{aligned}\left[b^{\mu}_{-r}b^{\nu}_{r}, b^{\rho}_{-s}b^{\lambda}_{s}\right] &= 2b^{\mu}_{-r}b^{\lambda}_{-s}b^{\nu}_{r}b^{\rho}_{s} + 2b^{\mu}_{-r}b^{\nu}_{r}b^{\lambda}_{-s}b^{\rho}_{s} + 2b^{\lambda}_{-s}b^{\mu}_{-r}b^{\rho}_{s}b^{\nu}_{r} + 2b^{\mu}_{-r}b^{\lambda}_{-s}b^{\rho}_{s}b^{\nu}_{r} \\ &\quad -\eta^{\nu\rho}b^{\mu}_{-r}b^{\lambda}_{r} - \eta^{\nu\lambda}b^{\mu}_{-r}b^{\rho}_{r} - \eta^{\mu\rho}b^{\lambda}_{-r}b^{\nu}_{r} - \eta^{\mu\lambda}b^{\rho}_{-r}b^{\nu}_{r}\,, \qquad \text{(A.11)}\end{aligned}$$

$$\begin{aligned}\left[b^{\nu}_{-r}b^{\mu}_{r}, b^{\lambda}_{-s}b^{\rho}_{s}\right] &= 2b^{\nu}_{-r}b^{\lambda}_{-s}b^{\mu}_{r}b^{\rho}_{s} + 2b^{\nu}_{-r}b^{\mu}_{r}b^{\lambda}_{-s}b^{\rho}_{s} + 2b^{\lambda}_{-s}b^{\nu}_{-r}b^{\rho}_{s}b^{\mu}_{r} + 2b^{\nu}_{-r}b^{\lambda}_{-s}b^{\rho}_{s}b^{\mu}_{r} \\ &\quad -\eta^{\mu\rho}b^{\nu}_{-r}b^{\lambda}_{r} - \eta^{\mu\lambda}b^{\nu}_{-r}b^{\rho}_{r} - \eta^{\nu\rho}b^{\lambda}_{-r}b^{\mu}_{r} - \eta^{\nu\lambda}b^{\rho}_{-r}b^{\mu}_{r}\,. \qquad \text{(A.12)}\end{aligned}$$

Now, we use equation (36) to calculate $[M^{\mu\nu}, M^{\rho\lambda}]$

$$\begin{aligned}\left[M^{\mu\nu}, M^{\rho\lambda}\right] &= -i\eta^{\nu\rho}M^{\mu\lambda} + i\eta^{\mu\lambda}M^{\rho\nu} + i\eta^{\nu\lambda}M^{\mu\rho} - i\eta^{\mu\rho}M^{\lambda\nu} \\
&\quad +i\pi^2\left(\gamma_0^{\nu\lambda}x^{\mu}x^{\rho} + \gamma_0^{\nu\rho}x^{\mu}x^{\lambda} + \gamma_0^{\mu\lambda}x^{\nu}x^{\rho} + \gamma_0^{\mu\rho}x^{\nu}x^{\lambda}\right) \\
&\quad +2i\pi^2\alpha'\tau\Big(\gamma_0^{\nu\rho}x^{\mu}p^{\lambda} - \gamma_0^{\mu\lambda}x^{\rho}p^{\nu} + \gamma_0^{\nu\lambda}x^{\mu}p^{\rho} - \gamma_0^{\mu\rho}x^{\lambda}p^{\nu} \\
&\quad +\gamma_0^{\mu\rho}x^{\nu}p^{\lambda} - \gamma_0^{\nu\lambda}x^{\rho}p^{\mu} + \gamma_0^{\mu\lambda}x^{\nu}p^{\rho} - \gamma_0^{\nu\rho}x^{\lambda}p^{\mu}\Big) \\
&\quad +\left(i\theta_0^{\mu\rho} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\rho}\right)p^{\lambda}p^{\nu} + \left(i\theta_0^{\mu\lambda} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\mu\lambda}\right)p^{\rho}p^{\nu} \\
&\quad +\left(i\theta_0^{\nu\rho} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\nu\rho}\right)p^{\lambda}p^{\mu} + \left(i\theta_0^{\nu\lambda} - 4i\pi^2\alpha'^2\tau^2\gamma_0^{\nu\lambda}\right)p^{\rho}p^{\mu} \\
&\quad +\left(i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\nu\rho} + i\frac{n^2}{2\alpha'}\theta_n^{\nu\rho}\right)\left(\alpha^{\mu}_{-n}\alpha^{\lambda}_{n} + \alpha^{\lambda}_{-n}\alpha^{\mu}_{n}\right) \\
&\quad +\left(i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\mu\lambda} + i\frac{n^2}{2\alpha'}\theta_n^{\mu\lambda}\right)\left(\alpha^{\rho}_{-n}\alpha^{\lambda}_{n} + \alpha^{\lambda}_{-n}\alpha^{\rho}_{n}\right) \\
&\quad +\left(i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\nu\lambda} + i\frac{n^2}{2\alpha'}\theta_n^{\nu\lambda}\right)\left(\alpha^{\rho}_{-n}\alpha^{\mu}_{n} + \alpha^{\mu}_{-n}\alpha^{\rho}_{n}\right) \\
&\quad +\left(i\frac{(2\pi\alpha')^2}{2\alpha'}\gamma_n^{\mu\rho} + i\frac{n^2}{2\alpha'}\theta_n^{\mu\rho}\right)\left(\alpha^{\lambda}_{-n}\alpha^{\nu}_{n} + \alpha^{\nu}_{-n}\alpha^{\lambda}_{n}\right)\,. \qquad \text{(A.13)}\end{aligned}$$

Appendix B

Derivation of the equations of motion from the deformed action

Starting from the deformed worldsheet action

$$S_* = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\left(\partial_a X^{\mu} * \partial^a X_{\mu} - i\,\bar{\psi}^{\mu} * \rho^a\partial_a\psi_{\mu}\right)\,, \qquad \text{(B.1)}$$

we compute the functional variations with respect to the X^{μ} and $\bar{\psi}^{\mu}$ fields.

The non-commutative structure of the worldsheet is defined by

$$\left[\sigma^a, \sigma^b\right] = i\,\theta^{ab}\,, \qquad \theta^{ab} = -\theta^{ba}\,, \qquad \text{(B.2)}$$

and the Moyal (star) product of two fields A and B is given by

$$A * B = A\, \mathrm{e}^{\frac{i}{2}\theta^{cd}\overleftarrow{\partial_c}\overrightarrow{\partial_d}} B = AB + \frac{i}{2}\theta^{cd}(\partial_c A)(\partial_d B) + \mathcal{O}\left(\theta^2\right) . \tag{B.3}$$

We retain only first-order terms in θ^{ab} throughout.

Applying (B.3) to the bosonic and fermionic parts of (B.1) gives

— For bosonic term:

$$\partial_a X^\mu * \partial^a X_\mu = \left(\partial_a X^\mu\right)\left(\partial^a X_\mu\right) + \frac{i}{2}\theta^{cd}\left(\partial_c\partial_a X^\mu\right)\left(\partial_d\partial^a X_\mu\right) . \tag{B.4}$$

— For fermionic term:

$$\bar{\psi}^\mu * \rho^a\partial_a\psi_\mu = \bar{\psi}^\mu\rho^a\partial_a\psi_\mu + \frac{i}{2}\theta^{cd}\left(\partial_c\bar{\psi}^\mu\right)\rho^a\left(\partial_d\partial_a\psi_\mu\right) . \tag{B.5}$$

Substituting (B.4) and (B.5) into (B.1) gives the expanded action

$$\begin{aligned} S_* \ &= \ -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\left[\left(\partial_a X^\mu\right)\left(\partial^a X_\mu\right) + \frac{i}{2}\theta^{cd}\left(\partial_c\partial_a X^\mu\right)\left(\partial_d\partial^a X_\mu\right)\right. \\ &\quad \left. -i\bar{\psi}^\mu\rho^a\partial_a\psi_\mu + \frac{1}{2}\theta^{cd}\left(\partial_c\bar{\psi}^\mu\right)\rho^a\left(\partial_d\partial_a\psi_\mu\right)\right] . \end{aligned} \tag{B.6}$$

We separate the commutative and first-order terms

$$S_*^{(X)} = S_0^{(X)} + S_1^{(X)} , \tag{B.7}$$

where

$$\begin{aligned} S_0^{(X)} \ &= \ -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\left(\partial_a X^\mu\right)\left(\partial^a X_\mu\right) , \\ S_1^{(X)} \ &= \ -\frac{i}{8\pi\alpha'}\int \mathrm{d}^2\sigma\theta^{cd}\left(\partial_c\partial_a X^\mu\right)\left(\partial_d\partial^a X_\mu\right) . \end{aligned} \tag{B.8}$$

From the ordinary part, one get the ordinary solution, so that

$$\delta S_0^{(X)} = +\frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\ \left(\partial_a\partial^a X_\mu\right)\delta X^\mu \tag{B.9}$$

and $S_1^{(X)}$ do not contribute in the equations of motion, indeed

$$\delta S_1^{(X)} = -\frac{i}{8\pi\alpha'}\int \mathrm{d}^2\sigma\theta^{cd}\left[\left(\partial_c\partial_a\delta X^\mu\right)\left(\partial_d\partial^a X_\mu\right) + \left(\partial_c\partial_a X^\mu\right)\left(\partial_d\partial^a\delta X_\mu\right)\right] . \tag{B.10}$$

Since both terms are equal, we write

$$\delta S_1^{(X)} = -\frac{i}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\theta^{cd}\left(\partial_c\partial_a X_\mu\right)\left(\partial_d\partial^a\delta X^\mu\right) . \tag{B.11}$$

Integrating by parts, first with respect to ∂_d and then with respect to ∂_a, while neglecting boundary terms, gives

$$\delta S_1^{(X)} = -\frac{i}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\theta^{cd}\left(\partial_a\partial_d\partial_c\partial^a X_\mu\right)\delta X^\mu . \tag{B.12}$$

Now, by combining (B.9) and (B.12), we get

$$\delta S_*^{(X)} = \frac{1}{2\pi\alpha'}\int \mathrm{d}^2\sigma\left[\partial_a\partial^a X_\mu - \frac{i}{2}\theta^{cd}\partial_a\partial_d\partial_c\partial^a X_\mu\right]\delta X^\mu . \tag{B.13}$$

Since θ^{cd} is antisymmetric whereas $\partial_d\partial_c$ is symmetric, the second term vanishes. The Euler–Lagrange equation reduces therefore to

$$\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0 . \tag{B.14}$$

Now, varying with respect to $\bar\psi^\mu$, from (B.6), we obtain the fermionic Lagrangian density

$$\mathcal{L}_\psi = \frac{i}{4\pi\alpha'}\left[\bar\psi^\mu\rho^a\partial_a\psi_\mu - \frac{i}{2}\theta^{cd}\left(\partial_c\bar\psi^\mu\right)\rho^a\left(\partial_d\partial_a\psi_\mu\right)\right] . \tag{B.15}$$

The same calculations lead to the standard Dirac-type equation

$$\rho^a\partial_a\psi^\mu = 0 . \tag{B.16}$$

Appendix C

Conformal invariance verification

We verify explicitly that the Virasoro generators defined in equations (20)–(23) continue to generate conformal transformations of the fields X^μ and ψ^μ in the presence of the deformed commutation relations (17)–(19), provided the non-commutative structure tensors satisfy the consistency condition

$$\gamma_{(m)}^{\mu\nu} = -\frac{m^2}{\left(2\pi\alpha'\right)^2}\,\theta_{(m)}^{\mu\nu} \tag{C.1}$$

— Bosonic field:

First, we calculate the commutator between L_m and α_n^μ. From (22) we have

$$L_m = \frac{1}{2}\sum_k :\alpha^\rho_{m-k}\alpha_{k\,\rho}: + \frac{1}{4}\sum_r (2r-m) :b^\rho_{m-r} b_{r\,\rho}:, \tag{C.2}$$

and using the deformed oscillator algebra (18), we get

$$[\alpha_m^\mu, \alpha_n^\nu] = \left(m\eta^{\mu\nu} + i\frac{(2\pi\alpha')^2}{2\alpha'}\,\gamma_n^{\mu\nu} + i\frac{n^2}{2\alpha'}\,\theta_n^{\mu\nu} \right)\delta_{m+n,0}\,. \tag{C.3}$$

Computing the commutator, we obtain

$$[L_m, \alpha_n^\mu] = \frac{1}{2}\sum_k \left([\alpha^\rho_{m-k}, \alpha_n^\mu]\,\alpha_{k\,\rho} + \alpha^\rho_{m-k}\,[\alpha_{k\,\rho}, \alpha_n^\mu] \right)\,. \tag{C.4}$$

Substituting (C.3) and simplifying using the delta functions gives

$$[L_m, \alpha_n^\mu] = -\,n\,\alpha^\mu_{m+n} + i\frac{(2\pi\alpha')^2}{4\alpha'}\,\gamma^\mu_{n\,\rho}\alpha^\rho_{m+n} + i\frac{n^2}{4\alpha'}\,\theta^\mu_{n\,\rho}\alpha^\rho_{m+n}\,, \tag{C.5}$$

where the first term reproduces the standard conformal transformation; the remaining terms are the non-commutative corrections.

Second, we calculate the action of L_m on $X^\mu(\sigma,\tau)$, from (15) we get

$$X^\mu(\sigma,\tau) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\,\mathrm{e}^{-in\tau}\cos n\sigma\,. \tag{C.6}$$

By applying (C.5) we obtain

$$\begin{aligned}
[L_m, X^\mu] &= i\sqrt{2}\alpha'\sum_{n\neq 0}\frac{1}{n}\,[L_m, \alpha_n^\mu]\;\mathrm{e}^{-in\tau}\cos n\sigma \\
&= -i\sqrt{2}\alpha'\sum_{n\neq 0}\alpha^\mu_{m+n}\,\mathrm{e}^{-in\tau}\cos n\sigma \\
&\quad + \frac{i\sqrt{2\alpha'}}{4\alpha'}\sum_{n\neq 0}\frac{1}{n}\left[(2\pi\alpha')^2\,\gamma^\mu_{n\rho} + n^2\theta^\mu_{n\rho} \right]\alpha^\rho_{m+n}\,\mathrm{e}^{-in\tau}\cos n\sigma\,.
\end{aligned} \tag{C.7}$$

By shifting the summation index $n \to n-m$ in the first term of (C.7) and using trigonometric identities, we can identify the derivative structure of $\partial_+ X^\mu = (\partial_\tau + \partial_\sigma)X^\mu$. This yields:

$$[L_m, X^\mu] = i\,\mathrm{e}^{im(\tau+\sigma)}\partial_+ X^\mu + \frac{i}{4\alpha'}\left[(2\pi\alpha')^2\,\gamma^\mu_{n\rho} + n^2\theta^\mu_{n\rho} \right]\mathrm{e}^{im(\tau+\sigma)}\partial_+ X^\rho\,. \tag{C.8}$$

Applying the structural relation between $\gamma_{(m)}$ and $\theta_{(m)}$ (53), and substituting into (C.8) yields

$$[L_m, X^\mu(\sigma,\tau)] = i\,\mathrm{e}^{im(\tau+\sigma)}\,\partial_+ X^\mu(\sigma,\tau)\,. \tag{C.9}$$

Hence, the Virasoro generators act on X^μ exactly as in the commutative theory.

— Fermionic field:

The field ψ^μ is not affected by non-commutativity, and the reasoning follows the same logic as in the ordinary case. Using (19) and the fermionic part of L_m, we get

$$[L_m, b_s^\mu] = \left(\frac{m}{2} - s\right) b_{m+s}^\mu\,, \tag{C.10}$$

and therefore, in coordinate space, we have

$$[L_m, \psi^\mu(\sigma,\tau)] = i\,\mathrm{e}^{im(\tau+\sigma)} \left(\partial_+ + \frac{im}{2}\right) \psi^\mu(\sigma,\tau)\,. \tag{C.11}$$

Both the bosonic and fermionic sectors preserve their standard conformal transformation laws, demonstrating that conformal invariance holds exactly in the deformed theory.

REFERENCES

[1] A. Connes, «A short survey of noncommutative geometry», *J. Math. Phys.* **41**, 3832 (2000).

[2] E. Witten, «Bound states of strings and p-branes», *Nucl. Phys. B* **460**, 335 (1996).

[3] M. Sheikh-Jabbari, «More on mixed boundary conditions and D-branes bound states», *Phys. Lett. B* **425**, 48 (1998).

[4] M. Sheikh-Jabbari, «Super Yang–Mills theory on noncommutative torus from open strings interactions», *Phys. Lett. B* **450**, 119 (1999).

[5] M. Sheikh-Jabbari, «Open strings in a B-field background as electric dipoles», *Phys. Lett. B* **455**, 129 (1999).

[6] N. Seiberg, E. Witten, «String theory and noncommutative geometry», *J. High Energy Phys.* **1999**, 032 (1999).

[7] C.-S. Chu, P.-M. Ho, «Non-commutative open string and D-brane», *Nucl. Phys. B* **550**, 151 (1999).

[8] J. Jing, Z.-W. Long, «Open string in the constant B-field background», *Phys. Rev. D* **72**, 126002 (2005).

[9] T. Kawano, T. Takahashi, «Open–Closed String Field Theory in a Background B Field», *Prog. Theor. Phys.* **104**, 1267 (2000).

[10] T. Kawano, T. Takahashi, «Open String Field Theory on Noncommutative Space», *Prog. Theor. Phys.* **104**, 459 (2000).

[11] M.R. Douglas, N.A. Nekrasov, «Noncommutative field theory», *Rev. Mod. Phys.* **73**, 977 (2001).

[12] A. Fayyazuddin, M. Zabzine, «Note on bosonic open strings in a constant B field», *Phys. Rev. D* **62**, 046004 (2000).

[13] F. Ardalan, H. Arfaei, M.M. Sheikh-Jabbari, «Noncommutative geometry from strings and branes», *J. High Energy Phys.* **1999**, 016 (1999).

[14] G. Grignani, M. Orselli, G.W. Semenoff, «Matrix strings in a B-field», *J. High Energy Phys.* **2001**, 004 (2001).

[15] D. Kamani, «Noncommutativity parameter as a field on the string worldsheet», *Phys. Lett. B* **548**, 231 (2002).

[16] D. Kamani, «Some Consequences of Noncommutative Worldsheet of Superstring», *Mod. Phys. Lett. A* **19**, 375 (2004).

[17] D. Kamani, «Non-commutative superstring worldsheet», *Eur. Phys. J. C* **26**, 285 (2002).

[18] S.S. Shahid-Zadeh Mousavi, «Generalization of Some Algebras in the Bosonic String Theory», *Int. J. Theor. Phys.* **48**, 2068 (2009).

[19] M.A. Seridi, N. Belaloui, «Paraquantum strings in noncommutative space-time», *Int. J. Mod. Phys. A* **30**, 1550175 (2015).

[20] B. Zwiebach, «A First Course in String Theory», 2$^{\text{nd}}$ edition, *Cambridge University Press*, 2009.

[21] J. Polchinski, «String Theory. Vol. 2: Superstring Theory and Beyond», Cambridge Monographs on Mathematical Physics, *Cambridge University Press*, 2007.

[22] K. Becker, M. Becker, J.H. Schwarz, «String Theory and M-Theory: A Modern Introduction», *Cambridge University Press*, 2006.

[23] F. Ferrari, «Superconformal String Theory», Proseminar in *Conformal Field Theory & String Theory*, ETH Zürich, May 2013.

[24] X.-J. Wang, «Strings in Noncommutative Spacetime», `arXiv:hep-th/0503111`.

[25] R.J. Szabo, «Quantum field theory on noncommutative spaces», *Phys. Rep.* **378**, 207 (2003).

[26] S.M. Carroll *et al.*, «Noncommutative Field Theory and Lorentz Violation», *Phys. Rev. Lett.* **87**, 141601 (2001).

[27] M. Chaichian, A. Demichev, P. Prešnajder, A. Tureanu, «Space-time noncommutativity, discreteness of time and unitarity», *Eur. Phys. J. C* **20**, 767 (2001).