A STUDY OF RETARDATION MODELS FOR PHENOMENOLOGICAL INTERACTIONS IN HADRONIC PHYSICS*

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A study of retardation models for the hadronic quark interaction is performed starting from a coordinate-space calculation that elaborates a classical electrodynamics procedure. The possibility of constructing a corresponding quantum operator is critically analyzed also performing some numerical matrix element calculations. A comparison of the developed model with the Feynman diagram interaction at tree-level is studied, showing a substantial physical correspondence of the two models. Possible applications and generalizations are discussed.

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1. Introduction

The study of hadronic systems in terms of interacting constituent quarks still represents an interesting field of investigation, due to the non-perturbative character of QCD and the difficulties of the lattice calculations.

Among the different issues, we study in the present paper the problem of *retardation* in the quark strong interaction. Some of the content of this work also appears in Ref. [1], where the didactic aspects are emphasized. In the present work, we focus our attention on *heavy* $q\bar{q}$ systems, such as charmonium and bottomonium. The dynamics of these hadronic systems, composed of two equal mass particles, can be reproduced, in their center of mass (CM) reference frame, by means of a relativistic wave equation in which the constituent quark interaction is represented by a specific coordinate-space potential.

In this respect, we cite here a series of works developed by the author in the framework of a three-dimensional reduction of a two-body Dirac equation [2–8]. The references to other relativistic models for hadronic systems can be found particularly in Ref. [3].

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It has been possible to show that a good reproduction of the charmonium data can be obtained by taking a *vector* interaction term, that is reminiscent of the underlying QCD dynamics, plus a scalar (or mass) term that reproduces the correct spin–orbit splittings and can be related to the scalar resonances of the hadronic spectrum [4, 6]. However, the main role in the interaction is played by the vector term, which can be represented by a Coulombic potential, *i.e.* given by the interchange of a massless particle, regularized by a chromo-electric charge distribution of the interacting quarks [2].

The interaction vector potential is usually derived in the framework of a three-dimensional relativistic wave equation, which is obtained by *reducing* the four-dimensional Bethe–Salpeter equation [9, 10]. This treatment, which was originally developed for the study of *electromagnetically* bound systems, allows us to add the retardation corrections by means of a cumbersome procedure [11] that relies on the smallness of the coupling constant and, for this reason, is not straightforwardly applicable in the case of strongly interacting particles.

Retardation corrections together with other terms of relativistic corrections have also been added perturbatively to a standard Schrödinger effective Hamiltonian [12, 13]. However, this technique is now superseded by fully relativistic calculations.

In the standard *instantaneous* or *unretarded* approach, the bound system is studied by means of a relativistic three-dimensional equation in which the interaction is represented by a potential operator that depends only on the interquark distance r, completely ignoring the retardation effects of the interaction. As it will be recalled in Subsection 1.1, this modelization (studied in more detail in Ref. [5]) does not violate relativity *per se*. Phenomenologically, it is able to reproduce accurately the charmonium data.

Nevertheless, it is worthwhile to investigate the possible effects related to retardation, taking into account that retardation is generally considered a specific characteristic of particle interactions in field theory and could give non-negligible contributions to strongly interacting systems.

This last point represents the main motivation of the present work. In other words, we show that retardation (or *relative time effect*), which is usually lost when reducing the four-dimensional Bethe–Salpeter equation [9, 10] to a three-dimensional wave equation without relative time, can be absorbed into the momentum dependence of the interaction operator, directly affecting the physical strength of the interaction. To this aim, we try to study different modelizations of the retardation effects that can be used in hadronic constituent quark interactions.

In the first part of the work, the use of coordinate space will be emphasized in order to understand in a more intuitive way the contribution of the different terms. To this aim, we try to construct a retarded interaction *operator* by using a procedure that relies on classical electrodynamics, more precisely, on the well-known Liénard–Wiechert (LW) construction, also assuming constant momenta of the interacting particles.

Then, a series expansion of this operator is performed in order to calculate the operator matrix elements numerically.

Finally, we compare our model with the tree-level Feynman diagram calculation, showing a substantial physical agreement between the two approaches. In this respect, the present work helps to understand the physical meaning of retardation in coordinate space and, in this way, motivates further calculations that can be developed directly in momentum space, starting from the tree-level Feynman diagram interaction.

The problem addressed in this article presents a high level of complexity, mainly related to the treatment of time in microscopic systems and to the formulation of a relativistic quantum model that could represent an *alternative* to standard quantum field theories. No definitive conclusions will be drawn.

In more detail, the contents of the paper are organized as follows. In Subsection 1.1, we recall the construction of a relativistic interaction model *without* retardation, commenting on some related developments. In Section 2, the classical derivation, based on the LW construction, is explained. Then, in Section 3, the corresponding quantum operator is constructed in different forms and some of its properties are also studied by means of test numerical calculations. In Subsection 3.1, some critical comments about the applicability of the model, in particular for the case of light quarks, will be given. In Section 4, we study the relationship between our model and the interaction given by the Feynman diagram at tree-level. Finally, in Subsection 4.1, a discussion about the results of this comparison, in view of possible applications and generalizations, will be performed.

Throughout the work, we shall use the standard natural units, that is $\hbar = c = 1$.

1.1. Relativistic interaction without retardation

In this subsection, we recall synthetically that retardation is not strictly necessary to formulate a relativistic interaction model. This point has been discussed in Ref. [5] considering the relativistic (integral) wave equation of that model, written in the momentum space. Physically, an unretarded potential would correspond to make the hypothesis that the interaction is mediated by a static field, in contrast with the standard mechanism of a bosonic particle interchange.

Here, we note that in the coordinate space, one can always define the CM interparticle distance r in a relativistic invariant form, by using a standard trick [14]. Let us consider, in a generic reference frame, the 4-vector distance between two events

$$r^\mu = \left(r^0, \boldsymbol{r}\right) , \tag{1}$$

where the time component r^0 can take *any* form and $\boldsymbol{r}$ is the interparticle 3-vector distance. We introduce the *projected* 4-vector distance

$$\rho^\mu = r^\mu - \left(\frac{r_\nu P^\nu}{M}\right)\frac{P^\mu}{M} , \tag{2}$$

where P^μ represents the total 4-momentum of the bound system and M its mass. In the CM, ρ^μ takes the form

$$\rho^\mu_{\rm CM} = (0, \boldsymbol{r}_{\rm CM}) . \tag{3}$$

In this way, in the CM, we get rid of the time component r^0 and the interparticle invariant distance can be written as

$$r = \sqrt{-\rho^\mu \rho_\mu} , \tag{4}$$

where, in the l.h.s., the label 'CM' has not been written explicitly. Note that the last equation defines the variable that is used in the instantaneous potential functions.

As a side remark, we mention the possibility of introducing an interaction given by the sum of an unretarded contribution plus a retarded term, as it is done in the Coulomb gauge in quantum electrodynamics [15]. We shall not develop this point in the present paper.

2. The standard LW construction

The problem of retardation has been studied in the context of classical electrodynamics [16, 17], highlighting that the finite speed of light, that originates retardation, does not allow us to write down an *exact* Hamiltonian for two (or more) interacting particles. The retardation effects are taken into account as terms of relativistic corrections in an expansion of the Hamiltonian in powers of p/m. In this work, we try to follow a different method, writing an (initially) exact expression for the retarded interaction not including the acceleration terms that, classically, are related to the radiation emission. This last effect, in a quantum model, should be treated apart, in a different way.

Along the derivation, we bear in mind that the expression for the interaction obtained by means of the classical calculation must be transformed into a quantum operator for our study of hadronic systems. This last point will be fully developed in Section 3.

We start our calculation deriving the expression of the *retarded distance* between two interacting particles. Then, we shall use this quantity in the LW retarded interaction.

We recall that, in quantum mechanics, the *standard distance* between two particles is represented by a Hermitian operator given by the difference of the position operators of the two particles, at the same time. On the other hand, we want to derive the *retarded distance* operator whose expression depends (as it will be shown) on the *standard distance* and on the *momenta* of the two particles. We perform the following derivation according to a classical procedure [16, 17], with the aim of obtaining, finally, a quantum operator in which the relative time variable does not explicitly appear. In a given reference frame, particle 1, at time t' and from the spatial point $\boldsymbol{r}'_1 = \boldsymbol{r}_1(t')$, "emits" the vector field. The emission event is defined as $r_1'^{\mu} = (t', \boldsymbol{r}'_1)$. The field is "observed" (*i.e.* "received") later, at time $t > t'$, in the spatial point $\boldsymbol{r}_{\rm o}$; in this way, we can define the observation event $r_{\rm o}^{\mu} = (t, \boldsymbol{r}_{\rm o})$. At time t, the position of particle 1 is $\boldsymbol{r}_1 = \boldsymbol{r}_1(t)$, which defines the event $r_1^{\mu} = (t, \boldsymbol{r}_1)$. We can introduce the *simultaneous* 4-vector distance

$$r^{\mu} = r_{\rm o}^{\mu} - r_1^{\mu} = (0, \boldsymbol{r})\,, \tag{5}$$

where $\boldsymbol{r}$ represents the *standard vector distance* between the observation point and particle 1, at the same time t. Incidentally, note that, in the classical theory, $\boldsymbol{r}$ depends on time t through $\boldsymbol{r}_1$. For the following developments, recall that at time t, the velocity of particle 1 is $\boldsymbol{v}_1$.

Given that the speed of light ($c = 1$) is *finite*, the observed field depends on the position and velocity of particle 1 at the emission time t'. The emission time and position are usually called *retarded* time and position. Consequently, we can introduce the *retarded, light-like*, 4-vector distance

$$r'^{\mu} = r_{\rm o}^{\mu} - r_1'^{\mu}\,. \tag{6}$$

Considering the time difference $\Delta t = t - t' > 0$ and taking into account that the field propagates at the speed of light $c = 1$, we have

$$r'^{\,0} = \Delta t = |\boldsymbol{r}'| = r' \tag{7}$$

that expresses the light-like character of r'^{μ}, and

$$\boldsymbol{r}' = \boldsymbol{r}_{\rm o} - \boldsymbol{r}'_1\,. \tag{8}$$

We make the *additional* hypothesis that in the interval between t' and t, particle 1 moves with the constant velocity $\boldsymbol{v}_1$. This hypothesis amounts to disregarding radiation emission in the classical theory. By using Δt of Eq. (7), we have for the particle's displacement

$$\boldsymbol{r}_1 - \boldsymbol{r}'_1 = \boldsymbol{v}_1 \Delta t = \boldsymbol{v}_1 r' \tag{9}$$

that gives, for the spatial components of the retarded distance, the following expression:

$$\boldsymbol{r}' = \boldsymbol{v}_1 r' + \boldsymbol{r}\,. \tag{10}$$

This equation simply means that the retarded vector distance $\boldsymbol{r}'$ is given by the sum of the particle's displacement in the time interval Δt (first term of the r.h.s.) plus the simultaneous spatial vector distance (second term of the r.h.s.). The time component associated to $\boldsymbol{r}'$ is straightforwardly given by Eq. (7). We can now solve Eq. (10) analytically to find the expression of r' as a function of $\boldsymbol{r}$ and $\boldsymbol{v}_1$. The result is

$$r' = \gamma_1^2 \left[\sqrt{(\boldsymbol{r}\cdot\boldsymbol{v}_1)^2 + \frac{r^2}{\gamma_1^2}} + \boldsymbol{r}\cdot\boldsymbol{v}_1 \right], \tag{11}$$

where $\gamma_1 = 1/\sqrt{1-\boldsymbol{v}_1^2}$ is the standard Lorentz factor of particle 1.

The result of Eq. (11) can be now used to determine the 4-vector LW potential observed at $\boldsymbol{r}_{\rm o}^{\mu}$, that is $A^{\mu}(t, \boldsymbol{r}_{\rm o})$. The general expression for this quantity, given for the electromagnetic field in Refs. [16, 17], has the following form:

$$A^{\mu}(t, \boldsymbol{r}_{\rm o}) = q_1 u_1^{\mu} \frac{1}{d_1}\,, \tag{12}$$

where q_1 represents the generalized strong charge of particle 1, u_1^{μ} is the standard particle 4-velocity and, finally, the denominator d_1 is given by

$$d_1 = u_1^{\nu} r'_{\nu}\,. \tag{13}$$

Note that d_1 of Eq. (13) depends on the *retarded* 4-vector distance r'^{μ}. On the other hand, due to our hypothesis of constant velocity, it is not necessary to specify that u_1^{ν} is taken at time t'.

We now elaborate on Eq. (13) in order to express d_1 in terms of $\boldsymbol{r}$ and of the 3-momentum $\boldsymbol{p}_1$. First, by using the standard 4-velocity definition, d_1 can be written in the form

$$d_1 = \gamma_1 \left(r' - \boldsymbol{v}_1 \cdot \boldsymbol{r}' \right)\,. \tag{14}$$

Then, we multiply Eq. (10) by $\boldsymbol{v}_1$ and replace the expression obtained for $\boldsymbol{v}_1 \cdot \boldsymbol{r}'$ in Eq. (14). We have

$$d_1 = \frac{r'}{\gamma_1} - \boldsymbol{r} \cdot \boldsymbol{u}_1 \,. \tag{15}$$

We also replace the expression of r' given by Eq. (11) in Eq. (15) and, finally, recalling that $\boldsymbol{u}_1 = \boldsymbol{p}_1/m$, we obtain the searched result

$$d_1 = \sqrt{\boldsymbol{r}^2 + \left(\frac{\boldsymbol{r} \cdot \boldsymbol{p}_1}{m}\right)^2} \,. \tag{16}$$

The complete expression of the field takes the form

$$A^\mu(t, \boldsymbol{r}_\mathrm{o}) = q_1 u_1^\mu \frac{1}{\sqrt{\boldsymbol{r}^2 + \left(\frac{\boldsymbol{r} \cdot \boldsymbol{p}_1}{m}\right)^2}} \,. \tag{17}$$

In order to construct an invariant *interaction* between two particles, we assume that the interacting particle 2, with generalized charge q_2, is placed at the observation point, that is $r_\mathrm{o}^\mu = r_2^\mu$. In this way, instead of Eq. (5), we have *from now on* the following definition:

$$r^\mu = r_2^\mu - r_1^\mu = (0, \boldsymbol{r}) = (0, \boldsymbol{r}_2 - \boldsymbol{r}_1) \,. \tag{18}$$

Furthermore, we make the hypothesis that particle 2 moves with the constant velocity $\boldsymbol{v}_2$, being u_2^μ the corresponding 4-velocity. In this way, the invariant interaction at time t takes the standard form

$$W_{12} = q_2 u_2^\mu A_\mu(t, \boldsymbol{r}) \,, \tag{19}$$

where $A_\mu(t, \boldsymbol{r})$ is given by Eq. (17).

We develop the last expression in the following way. First, in order to construct the quantum model in Section 3, we can formally drop the dependence on the (generic) time t, because, in the Schrödinger representation, operators do not depend on time. Furthermore, we specialyze the interaction to a $q\bar{q}$ *system*. In this case, the product of the two strong charges is given by

$$q_1 q_2 = -\frac{4}{3}\alpha_\mathrm{s} \,. \tag{20}$$

where $-4/3$ represents the quark color factor and α_s is the strong effective coupling constant. We obtain

$$W_{12} = -\frac{4}{3}\alpha_\mathrm{s} u_1^\mu u_2^\nu g_{\mu\nu} \frac{1}{\sqrt{\boldsymbol{r}^2 + \left(\frac{\boldsymbol{r} \cdot \boldsymbol{p}_1}{m}\right)^2}} \,. \tag{21}$$

Now we have to consider the contribution W_{21} obtained from Eq. (21) by interchanging the two particles. Then, we sum the two contributions multiplying by a factor of 1/2, primarily to avoid double counting. The result for the total retarded interaction is

$$W_{\rm ret}(\boldsymbol{r},\boldsymbol{p}_1,\boldsymbol{p}_2) = -\frac{4}{3}\alpha_{\rm s} u_1^\mu u_2^\nu g_{\mu\nu}\,\frac{1}{2}\left[\frac{1}{\sqrt{\boldsymbol{r}^2+\left(\frac{\boldsymbol{r}\cdot\boldsymbol{p}_1}{m}\right)^2}}+\frac{1}{\sqrt{\boldsymbol{r}^2+\left(\frac{\boldsymbol{r}\cdot\boldsymbol{p}_2}{m}\right)^2}}\right]. \tag{22}$$

In this way, we have obtained an expression that is not only symmetrized with respect to the two particles, but also reconstructs a time-symmetric interaction within the stationary hadronic bound state. This procedure, which eliminates the problem of the classical chronological order (where one particle is considered the "source" and the other the "observer"), seems highly consistent with the unobservability of the relative time between the two particles.

For practical calculations of bound-state properties, we introduce, in the CM, the relative momentum $\boldsymbol{p} = \boldsymbol{p}_2 = -\boldsymbol{p}_1$. In this reference frame, the interaction can be written in the following form:

$$W_{\rm ret}(\boldsymbol{r},\boldsymbol{p}) = -\frac{4}{3}\alpha_{\rm s} u_1^\mu u_2^\nu g_{\mu\nu}\, V_{\rm ret}(\boldsymbol{r},\boldsymbol{p})\,, \tag{23}$$

where we have introduced the *spatial function*

$$V_{\rm ret}(\boldsymbol{r},\boldsymbol{p}) = \frac{1}{\sqrt{\boldsymbol{r}^2+\left(\frac{\boldsymbol{r}\cdot\boldsymbol{p}}{m}\right)^2}}\,. \tag{24}$$

In the last equation, the denominator of $V_{\rm ret}(\boldsymbol{r},\boldsymbol{p})$ represents the *invariant retarded distance* of the two particles. In Eq. (23), the 4-velocity product, according to the *classical theory*, is given by

$$u_1^\mu u_2^\nu g_{\mu\nu} = 1 + \frac{2\boldsymbol{p}^2}{m^2}\,, \tag{25}$$

but, in the quantum model discussed in the next section, the 4-velocities will be replaced by the Dirac matrices of the two particles.

3. The quantum operator for bound-state calculations

In this section, starting from the retarded interaction of Eqs. (23) and (24), we construct a quantum operator to be used in a reduced two-body Dirac equation. To this aim, we replace the 4-velocities u_i^μ of Eq. (23)

with the Dirac matrices of the two particles. For clarity, we write here the expression that will be elaborated and studied in the remainder of the work

$$W_{\rm ret}(\boldsymbol{r},\boldsymbol{p}) = -\frac{4}{3}\alpha_{\rm s}\gamma_1^\mu\gamma_2^\nu g_{\mu\nu}\, V_{\rm ret}(\boldsymbol{r},\boldsymbol{p})\,, \tag{26}$$

with the spatial factor given by Eq. (24).

Note that in the static case ($\boldsymbol{p}=0$), the retardation effects disappear. In the general case, the retardation effects *reduce* (in absolute value) the strength of the interaction.

The expression of $W_{\rm ret}(\boldsymbol{r},\boldsymbol{p})$ of the last equation is apparently simple but cannot be directly "interpreted" as a quantum operator due to the non-Hermitian character of $V_{\rm ret}(\boldsymbol{r},\boldsymbol{p})$. We have the task of introducing, from that expression (obtained by means of a classical procedure), a corresponding Hermitian quantum operator. This operator will have, in any case, a non-local form; it means that the retardation effects involve the wave function of the system over all the space.

The procedure to obtain a Hermitian operator is *not unique*. In any case, it consists in *choosing an order* of the operators $\boldsymbol{p}$ and $\boldsymbol{r}$ so that a Hermitian operator is finally obtained. We shall describe in detail the following procedure because it allows us to calculate some test matrix elements in a simple way, without introducing singular terms. Another procedure is also incidentally outlined but no specific calculation is performed.

First, we recall the definition of the following non-Hermitian radial momentum operator [18]:

$$p_r = \frac{1}{r}\boldsymbol{r}\cdot\boldsymbol{p}\,. \tag{27}$$

For a wave function $\psi(r,\theta,\varphi)=\langle r,\theta,\varphi|\psi\rangle$ in spherical coordinates, we have

$$\langle r,\theta,\varphi|p_r|\psi\rangle = -i\frac{\partial}{\partial r}\psi(r,\theta,\varphi)\,. \tag{28}$$

Furthermore, we note that the operator $p_r^\dagger p_r$ *is* Hermitian. For further developments, this operator can be written, with standard procedures, in the form

$$p_r^\dagger p_r = \boldsymbol{p}^2 - \frac{\boldsymbol{L}^2}{\boldsymbol{r}^2}\,. \tag{29}$$

For transforming Eq. (24), we use the following operator ordering:

$$\left(\frac{\boldsymbol{r}\cdot\boldsymbol{p}}{r}\right)^2 \to \boldsymbol{p}\cdot\boldsymbol{r}\frac{1}{r^2}\boldsymbol{r}\cdot\boldsymbol{p} = p_r^\dagger p_r\,. \tag{30}$$

In this way, we can rewrite Eq. (26) in the form

$$W_{\rm ret}(\boldsymbol{r},\boldsymbol{p}) = -\frac{4}{3}\alpha_{\rm s}\gamma_1^\mu\gamma_2^\nu g_{\mu\nu}\,\frac{1}{r\sqrt{1+\frac{p_r^\dagger p_r}{m^2}}}\,. \tag{31}$$

The previous expression is not yet Hermitian, but we can perform a series expansion in powers of $p_r^\dagger p_r/m^2$ and write each term in the series in Hermitian form. To this aim, we make use of the following operator reordering:

$$\frac{1}{r}\left(p_r^\dagger p_r\right)^k \to \left(p_r^\dagger\right)^k \frac{1}{r}(p_r)^k\,. \tag{32}$$

In consequence, the interaction can be expanded in the following way:

$$W_{\rm ret}(\boldsymbol{r},\boldsymbol{p}) = -\frac{4}{3}\alpha_{\rm s}\gamma_1^\mu\gamma_2^\nu g_{\mu\nu}\sum_{k=0}^{\infty} c_k \left(\frac{p_r^\dagger}{m}\right)^k \frac{1}{r}\left(\frac{p_r}{m}\right)^k\,, \tag{33}$$

where the coefficients c_k are

$$c_k = (-1)^k\frac{(2k-1)!!}{(2k)!!}\,. \tag{34}$$

We can now study some developments of the previous analysis. First, we recall that, instead of the pure Coulombic potential, it is possible to use a regularized function $U(r)$ introduced in a model in which the quarks are considered extended sources of the interaction field. In general, we can perform the following formal replacement in Eq. (33):

$$\frac{1}{r} \to U(r)\,, \tag{35}$$

where $U(r)$ indicates a *generic* (pure Coulombic or regularized) spatial potential function. In particular, the author successfully used the following regularized function [4]:

$$U(r) = \frac{1}{r}\mathrm{erf}\left(\frac{r}{2d}\right)\,, \tag{36}$$

which is obtained considering a Gaussian color charge distribution of the source particles. Some test calculations performed with the pure Coulombic interaction and with the regularized one will be discussed below.

Incidentally, we also show here a different operator ordering that can be used to obtain a Hermitian interaction operator $W_{\rm ret}(\boldsymbol{p},\boldsymbol{r})$. Starting directly from Eq. (31), we define

$$W_{\rm ret}(\boldsymbol{r},\boldsymbol{p}) = -\frac{4}{3}\alpha_{\rm s}\gamma_1^\mu\gamma_2^\nu g_{\mu\nu}\left[1+\frac{p_r^\dagger p_r}{m^2}\right]^{-1/4} U(r)\left[1+\frac{p_r^\dagger p_r}{m^2}\right]^{-1/4}\,, \tag{37}$$

where the operator $p_r^\dagger p_r$ has been given explicitly in Eq. (29). In Eq. (37), it is also possible to take for $U(r)$ a generic potential function. We note

that Eq. (37) appears more compact than Eq. (33). However, for practical calculations, one has to perform, also in this case, a series expansion in powers of $p_r^\dagger p_r/m^2$. The calculation of the corresponding matrix elements is not easy to handle and, for this reason, we do not use the interaction of Eq. (37) in this work.

We now discuss a test calculation of some matrix elements of the spatial operator that appears in Eq. (33), with the generalization of Eqs. (35) and (36). To this aim, we disregard the Dirac matrices and the color coupling, focusing our attention on the spatial operator S, defined as

$$S = \sum_{k=0}^{\infty} c_k \left(\frac{p_r^\dagger}{m}\right)^k U(r) \left(\frac{p_r}{m}\right)^k . \tag{38}$$

We study the matrix elements $\langle n_b, l|S|n_a, l\rangle$, where n_b, n_a represent the energy quantum number of the states and l is the orbital angular momentum quantum number. The matrix elements do not depend on the third component of the angular momentum. For this reason, the corresponding quantum number m_l has been dropped.

For our test calculation, we take the standard harmonic oscillator wave functions in coordinate space given, for example, in Ref. [3]. By using the adimensional variable $s = r\bar{p}$, with the momentum scale $\bar{p}$, the matrix elements are written in the form

$$\langle n_b, l|S|n_a, l\rangle = \sum_{k=0}^{\infty} c_k \left(\frac{\bar{p}}{m}\right)^{2k} \int_0^\infty s^2 \mathrm{d}s \hat{R}^{(k)}_{n_b,l}(s) U\left(\frac{s}{\bar{p}}\right) \hat{R}^{(k)}_{n_a,l}(s) , \tag{39}$$

where $\hat{R}^{(k)}_{n,l}(s)$ represents the k^{th} derivative with respect to s of the adimensional radial functions. The integrals of the previous expression can be calculated analytically for the pure Coulombic case. For the regularized $U(r)$ of Eq. (36), the integrals are performed numerically. The first term of the series (that is the term with $k = 0$) gives the unretarded interaction matrix elements. By considering the calculations of the work [4], the value of $\bar{p}$ has been fixed, indicatively, at the value $\bar{p} = 0.5$ GeV; the value of d in the regularized interaction of Eq. (36), at the value $d = 0.15$ fm and the quark mass at $m = 1.27$ GeV. Some results are shown in Table 1. The sum over k has been performed up to $k = 8$ with an accuracy of the order of 6%. The difference betweeen the *unretarded* and *retarded* calculation is not big but should be taken into account in more refined models for the charmonium and bottomonium spectra.

Table 1. Test calculation of some retarded and unretarded matrix elements of the spatial operator S of Eq. (39). In the first three columns, we report n_a, n_b, and l, which represent the energy quantum number of the *ket* state, the energy quantum number of the *bra* state, and the angular momentum quantum number, respectively. In the other columns, we report the values of the matrix elements in the *unretarded* case (subscript 'unr') and in the *retarded* case (subscript 'ret'). The calculations have been performed for a pure Coulombic interaction (superscript 'C') and for the regularized interaction (superscript 'R') of Eq. (36). All the results are in GeV. Further details are given in the text.

n_a	n_b	l	$S^{\mathrm{C}}_{\mathrm{unr}}$	$S^{\mathrm{C}}_{\mathrm{ret}}$	$S^{\mathrm{R}}_{\mathrm{unr}}$	$S^{\mathrm{R}}_{\mathrm{ret}}$
0	0	0	0.564	0.525	0.449	0.412
0	2	0	0.230	0.183	0.116	0.0743
1	1	1	0.376	0.354	0.354	0.335
2	2	0	0.470	0.368	0.350	0.261
2	2	2	0.301	0.280	0.295	0.277
3	1	1	0.119	0.103	0.093	0.0786
3	3	1	0.338	0.285	0.306	0.251
3	3	3	0.258	0.245	0.256	0.243

3.1. Comments on the limitations of the model

Some critical comments regarding the general applicability of the model are necessary. The model has been developed with the aim of improving the description of *heavy* $q\bar{q}$ systems. For the case of light quark mesons, the non-perturbative character of confinement and the relativistic speeds of the quarks are likely to invalidate this treatment because, for these states, the mean values of the powers of p/m become large quantities. In particular, the expansion in powers of $\bar{p}/m$ of Eq. (39) is likely to diverge. Furthermore, different operator orderings (whose effects appear in higher-order terms of p/m) can give significantly different numerical predictions, showing the arbitrariness of the ordening choice. Finally, the constant quark velocity assumption, neglecting the radiation effects (classicaly related to acceleration) may be questionable for non-perturbative, highly relativistic bound systems. For all these reasons, we conclude that the LW construction should be merely considered as a *correction* and is probably insufficient to treat the dynamics of light quark mesons.

4. A comparison with the Feynman model

In this section, we discuss a point of conceptual interest, namely the comparison of our model of interaction (developed in the coordinate space) with the results, at tree-level, of the Feynman diagram theory obtained in the momentum space. The following analysis can also represent the starting point for constructing a momentum-space model of the quark interaction. Some comments and a possible generalization of this model will be presented in Subsection 4.1.

We develop the calculations presented in this section using the retarded interaction of Eq. (26) with $V_{\rm ret}(\boldsymbol{r},\boldsymbol{p})$ given by Eq. (24); that is, we consider point-like interacting quarks. The generalization to the case of non-point-like quarks will be discussed in Subsection 4.1.

For clarity, we first recall some standard procedures used for the *unretarded* (or *static*) interaction obtained by setting $\boldsymbol{p}=0$ in Eq. (24). In this case, we have a pure Coulombic interaction operator in the form

$$W_{\rm unr}(\boldsymbol{r}) = -\frac{4}{3}\alpha_{\rm s}\gamma_1^\mu\gamma_2^\nu g_{\mu\nu}\,\frac{1}{r}\,. \tag{40}$$

In the momentum space, we have

$$\hat{W}_{\rm unr}(\boldsymbol{q}) = \langle \boldsymbol{p}_b|W_{\rm unr}(\boldsymbol{r})|\boldsymbol{p}_a\rangle\,, \tag{41}$$

where $\boldsymbol{q}=\boldsymbol{p}_a-\boldsymbol{p}_b$ is the interaction 3-momentum transfer. For the present work, we introduce the definition of the *interaction amplitude* for the momentum space matrix elements of the interaction operator. Specifically, the unretarded interaction amplitude $\hat{W}_{\rm unr}(\boldsymbol{q})$ is obtained by means of a standard Fourier transform

$$\begin{aligned}\hat{W}_{\rm unr}(\boldsymbol{q}) &= \frac{1}{(2\pi)^3}\int {\rm d}^3r\,\exp(i\boldsymbol{q}\cdot\boldsymbol{r})W_{\rm unr}(\boldsymbol{r})\\ &= -\frac{4}{3}\alpha_{\rm s}\gamma_1^\mu\gamma_2^\nu g_{\mu\nu}\,\frac{1}{2\pi^2}\,\frac{1}{\boldsymbol{q}^2}\,,\end{aligned} \tag{42}$$

where the expression shown in the second line is obtained by means of standard handlings. We recall that, in general, the dependence of $\hat{W}_{\rm unr}(\boldsymbol{q})$ on $\boldsymbol{q}$ *only* corresponds to the local, unretarded character of $W_{\rm unr}(\boldsymbol{r})$.

On the other hand, in the Feynman theory, by using the tree-level diagram [15], one has for the interaction amplitude, an expression of the following kind:

$$\hat{W}_{\rm Fey}(\boldsymbol{p}_b,\boldsymbol{p}_a) = -\frac{4}{3}\alpha_{\rm s}\gamma_1^\mu\gamma_2^\nu g_{\mu\nu}\,\frac{1}{2\pi^2}\,\frac{1}{\boldsymbol{q}^2-(\Delta\epsilon)^2}\,N(p_b,p_a)\,, \tag{43}$$

where the energy difference $\Delta\epsilon$ represents the time component q^0 of the on-shell 4-momentum transfer. It is standardly defined as

$$q^0 = \Delta\epsilon = \epsilon(\boldsymbol{p}_a) - \epsilon(\boldsymbol{p}_b)\,. \tag{44}$$

Physically, it gives rise to the non-local and retarded character of the interaction, whereas it is neglected in unretarded calculations. We also note that the invariant (positive) squared momentum transfer

$$Q^2 = -q^\mu q_\mu = \boldsymbol{q}^2 - (\Delta\epsilon)^2 \tag{45}$$

standardly appears in the denominator of Eq. (43). We have also introduced the relativistic factor $N(p_b, p_a)$ that will be discussed in the following. We anticipate that this factor reduces to one in the non-relativistic limit.

Analogously to Eq. (41), it is possible to write *formally* the interaction amplitude $\hat{W}_{\rm Fey}(\boldsymbol{p}_b, \boldsymbol{p}_a)$ as a matrix element between two momentum states

$$\hat{W}_{\rm Fey}(\boldsymbol{p}_b, \boldsymbol{p}_a) = \langle \boldsymbol{p}_b | W_{\rm Fey} | \boldsymbol{p}_a \rangle\,, \tag{46}$$

but in this case, the explicit expression of the non-local operator $W_{\rm Fey}$ is not directly given by the theory.

In order to compare the Feynman interaction amplitude with our LW retardation model, it is also useful to write the on-shell energy difference in the following way:

$$\Delta\epsilon = -\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}_a + \boldsymbol{p}_b)}{\epsilon(\boldsymbol{p}_b) + \epsilon(\boldsymbol{p}_a)}\,. \tag{47}$$

In consequence, the Feynman interaction amplitude of Eq. (43) can be rewritten in the completely equivalent form

$$\hat{W}_{\rm Fey}(\boldsymbol{p}_b, \boldsymbol{p}_a) = -\frac{4}{3}\alpha_{\rm s}\gamma_1^\mu\gamma_2^\nu g_{\mu\nu}\,\frac{1}{2\pi^2}\,\frac{1}{\boldsymbol{q}^2 - \left[\frac{\boldsymbol{q}\cdot(\boldsymbol{p}_a+\boldsymbol{p}_b)}{\epsilon(\boldsymbol{p}_b)+\epsilon(\boldsymbol{p}_a)}\right]^2}\,N(p_b, p_a)\,. \tag{48}$$

We now turn our attention to the retarded LW interaction operator of Eq. (26), with $V_{\rm ret}(\boldsymbol{r}, \boldsymbol{p})$ of Eq. (24), and introduce the corresponding interaction amplitude

$$\hat{W}_{\rm ret}(\boldsymbol{q}, \boldsymbol{p}) = \langle \boldsymbol{p}_b | W_{\rm ret}(\boldsymbol{r}, \boldsymbol{p}) | \boldsymbol{p}_a \rangle\,. \tag{49}$$

Analogously to Eq. (42), the previous expression can be written by means of the following integral:

$$\hat{W}_{\rm ret}(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3 r\,\exp(i\boldsymbol{q}\cdot\boldsymbol{r})W_{\rm ret}(\boldsymbol{r}, \boldsymbol{p})\,. \tag{50}$$

To calculate explicitly the previous quantity, we *generalize* the Fourier transform of Eq. (42). In this calculation, the momentum $\boldsymbol{p}$ will be treated temporarily as a constant, disregarding, for the moment, the operator ordering. The procedure of this generalized calculation can be schematized as follows.

Instead of integrating with respect to $\boldsymbol{r}$, we introduce $\boldsymbol{\eta}$ as new spatial integration variable

$$\boldsymbol{\eta} = \boldsymbol{r} + \frac{(\boldsymbol{r}\cdot\boldsymbol{p})\boldsymbol{p}}{m\,[m+\epsilon(\boldsymbol{p})]}\,; \tag{51}$$

$\boldsymbol{\eta}$ satisfies the following identity:

$$\boldsymbol{\eta}^2 = \boldsymbol{r}^2 + \left(\frac{\boldsymbol{r}\cdot\boldsymbol{p}}{m}\right)^2 . \tag{52}$$

By means of this expression, we can write the retarded spatial function of Eq. (24) in the form

$$V_{\rm ret}(\boldsymbol{r},\boldsymbol{p}) = \frac{1}{\eta}\,. \tag{53}$$

We also need to express $\boldsymbol{r}$ as a function of $\boldsymbol{\eta}$. One has

$$\boldsymbol{r} = \boldsymbol{\eta} - \left(\frac{\boldsymbol{\eta}\cdot\boldsymbol{p}}{m}\right)\frac{\boldsymbol{p}}{m}\,\frac{m^2}{[m+\epsilon(\boldsymbol{p})]\,\epsilon(\boldsymbol{p})}\,. \tag{54}$$

From this equation, we obtain the Jacobian determinant J for the variable change of Eq. (51)

$$J = \frac{\partial(\boldsymbol{r})}{\partial(\boldsymbol{\eta})} = \frac{m}{\epsilon(\boldsymbol{p})}\,. \tag{55}$$

Then we introduce the constant vector $\boldsymbol{\xi}$

$$\boldsymbol{\xi} = \boldsymbol{q} - \frac{(\boldsymbol{q}\cdot\boldsymbol{p})\,\boldsymbol{p}}{[m+\epsilon(\boldsymbol{p})]\,\epsilon(\boldsymbol{p})} \tag{56}$$

that satisfies the two following identities:

$$\boldsymbol{q}\cdot\boldsymbol{r} \;=\; \boldsymbol{\xi}\cdot\boldsymbol{\eta}\,, \tag{57}$$

$$\boldsymbol{\xi}^2 \;=\; q^2 - \left[\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{\epsilon(\boldsymbol{p})}\right]^2 . \tag{58}$$

In Eq. (50), by means of Eq. (57), we can replace $\boldsymbol{q}\cdot\boldsymbol{r}$ with $\boldsymbol{\xi}\cdot\boldsymbol{\eta}$. In this way, in the final result of the integration over $\boldsymbol{\eta}$, the factor $1/\boldsymbol{q}^2$ of the unretarded calculation is replaced by $1/\boldsymbol{\xi}^2$. By using Eq. (58), one obtains the final result

$$\hat{W}_{\rm ret}(\boldsymbol{q},\boldsymbol{p}) = -\frac{4}{3}\alpha_{\rm s}\gamma_1^{\mu}\gamma_2^{\nu}g_{\mu\nu}\,\frac{1}{2\pi^2}\,\frac{1}{\boldsymbol{q}^2 - \left[\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{\epsilon(\boldsymbol{p})}\right]^2}\,\frac{m}{\epsilon(\boldsymbol{p})}\,. \tag{59}$$

We can now put this result of our LW model of retardation (transformed to momentum space) in the same form as the Feynman interaction amplitude, given in Eq. (48). To this aim, in our result of Eq. (59), we make the following replacements:

$$\begin{aligned} \boldsymbol{p} &\rightarrow \frac{1}{2}\left(\boldsymbol{p}_b + \boldsymbol{p}_a\right) , \\ \epsilon(\boldsymbol{p}) &\rightarrow \frac{1}{2}\left[\epsilon(\boldsymbol{p}_b) + \epsilon(\boldsymbol{p}_a)\right] . \end{aligned} \tag{60}$$

With these replacements, one recovers the same expression given by the Feynman theory of Eq. (48). In particular, from the factor $m/\epsilon(\boldsymbol{p})$ (that represents the Jacobian determinant of Eq. (55)), one obtains for the relativistic factor $N(p_b, p_a)$ the following expression:

$$N(p_b, p_a) = \frac{2m}{\epsilon(\boldsymbol{p}_b) + \epsilon(\boldsymbol{p}_a)} . \tag{61}$$

This entire procedure shows a substantial physical agreement between the two models. Further discussions and comments will be given in Subsection 4.1.

4.1. Comments on the model and possible generalizations

We analyze here the physical content of the obtained results and discuss some possible applications and generalizations.

Starting from the LW interaction operator of our model, we have obtained, in Eq. (59), the corresponding interaction amplitude. By using, as *prescriptions*, the replacements of Eq. (60), we have reconstructed the standard interaction amplitude of the tree-level Feynman theory. In consequence, we tentatively conclude that the latter theory has the same physical content as our LW retardation model.

The Feynman theory gives a non-local interaction amplitude from which, at the moment, a standard Hermitian operator (depending on $\boldsymbol{r}$ and $\boldsymbol{p}$) cannot be explicitly derived. However, one can use *directly* the Feynman interaction amplitude to calculate, in the momentum space, the bound-state matrix elements. In this case, it is necessary to use the momentum-space wave functions of the bound states; the price to pay, with respect to standard coordinate-space calculations, is that numerical integrals of higher dimensionality must be performed.

On the other hand, our coordinate-space LW retarded interaction allows us to understand more intuitively the role of retardation in bound systems. However, to obtain a Hermitian expression, an operator ordering, not univocally defined, must be introduced. Furthermore, in any case, for calculating the bound-state matrix elements, an expansion in powers of p/m (not convergent for light quarks) must be performed.

We conclude that, for introducing retardation in hadronic models, the Feynman interaction amplitude seems preferable to the LW coordinate-space interaction which, on the other hand, can be used to check more sophisticated momentum-space models.

Some words of caution are needed about the relativistic factor $N(p_b, p_a)$ explicitly given in Eq. (61). That expression is the result of the replacements given by Eq. (59) which, as stated before, represent a prescription for reconstructing the Feynman interaction amplitude. Other forms could be hypothesized for that quantity; for example, one could "symmetrize" separately, in the following way, the Jacobian determinant of Eq. (55), obtaining

$$N'(p_b, p_a) = \frac{m}{2}\left[\frac{1}{\epsilon(\boldsymbol{p}_b)} + \frac{1}{\epsilon(\boldsymbol{p}_a)}\right] . \tag{62}$$

Different forms of $N(p_b, p_a)$ can give significantly different numerical results. This point should be carefully considered when the theoretical models are developed and compared with the experimental data of the hadronic spectra.

Another point of interest is that for the phenomenological quark interaction, it is usually necessary to introduce vertex form factors. In the unretarded models, one has a color (non-point-like) charge distribution of the quarks that gives rise to a *regularized* quark interaction [2, 4] depending only on r. In the Feynman model, one can multiply the interaction amplitude of Eq. (43) by two vertex form factors that depend on the *invariant* (squared) momentum transfer Q^2 of Eq. (45), taking directly into account the retardation contributions given by $q^0 = \Delta\epsilon$, defined in Eq. (44).

Furthermore, we recall that also the strong coupling constant $\alpha_{\rm s}$ is also a *running* coupling constant $\alpha_{\rm s}(Q^2)$ [8], whose dependence on Q^2 must be correctly taken into account.

All these additional dependences on Q^2 can be straightforwardly included in a momentum-space calculation.

In summary, a momentum-space calculation based on the Feynman interaction amplitude of Eq. (43), with the form factors and the $\alpha_{\rm s}(Q^2)$, should be able to treat retardation and relativistic effects that, otherwise, would be partially ignored in coordinate-space models.

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