THE SZEKERES METRICS WITH $M < 0$

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The evolution equation of the Szekeres metrics allows for solutions with the mass function $M < 0$. They exist in both classes of the Szekeres metrics, have no Big Bang singularity and no origin. In both classes, the conditions for no shell crossings ensure that the mass density ρ of the dust source in the Einstein equations is negative at all times. Thus, these metrics do not qualify as cosmological models. In the Friedmann limit, the implication $M < 0 \Longrightarrow \rho < 0$ is immediate. In the general Szekeres metrics, it follows by tuning conclusions from different equations.

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1. Motivation and summary

The signature of the metric will be $(+, -, -, -)$, the derivatives will be denoted as $\partial f/\partial z = f,_z$. We assume that the cosmological constant $\Lambda = 0$.

The Szekeres metrics obey the Einstein equations with a dust source and have, in general, no symmetry [1]. They were found by Szekeres in 1975 [2, 3] beginning with the following Ansatz:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{2\alpha}\mathrm{d}z^2 - \mathrm{e}^{2\beta}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) , \tag{1.1}$$

where $\alpha(t, z, x, y)$ and $\beta(t, z, x, y)$ are to be determined by solving the Einstein equations, and the velocity field of the dust is $u^\alpha = {\delta^\alpha}_0$, $(x^0, \ldots, x^3) = (t, z, x, y)$. For a derivation of the solutions of the Einstein equations, see Ref. [4]. They split into Class I, where $\beta,_z \neq 0$ and Class II, where $\beta,_z = 0$. Both classes contain the Friedmann metrics [5, 6] as limits and Class I found numerous applications in theoretical cosmology [7].

The metric of Class I (in the parametrisation of Hellaby [8]) is

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}t^2 - \frac{\left(\Phi,_z - \Phi\mathcal{E},_z/\mathcal{E}\right)^2}{\varepsilon - k}\,\mathrm{d}z^2 - \frac{\Phi^2}{\mathcal{E}^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)\,, && (1.2)\\ \mathcal{E} &\overset{\text{def}}{=} \frac{S}{2}\left[\left(\frac{x-P}{S}\right)^2 + \left(\frac{y-Q}{S}\right)^2 + \varepsilon\right], && (1.3)\end{aligned}$$

where k, P, Q, and S are arbitrary functions of z, $\varepsilon = \pm 1$ or 0, and the function $\Phi(t,z)$ must obey

$$\Phi,_t{}^2 = -k + 2M/\Phi\,. \tag{1.4}$$

The $M(z)$ is an arbitrary function, and the mass density ρ of the dust is

$$\kappa c^2 \rho = \frac{2\left(M,_z - 3M\mathcal{E},_z/\mathcal{E}\right)}{\Phi^2\left(\Phi,_z - \Phi\mathcal{E},_z/\mathcal{E}\right)}\,, \qquad \kappa \overset{\text{def}}{=} 8\pi G/c^4\,. \tag{1.5}$$

The value of ε determines the type of *quasisymmetry* of the metric: with $\varepsilon = +1$, the surfaces of constant t and z in (1.2) and (1.3) are nonconcentric spheres, with $\varepsilon = -1$, they have constant negative curvature, and with $\varepsilon = 0$, they are intrinsically flat (although it is questionable whether they can be interpreted as Euclidean planes, a more plausible interpretation is that they are flat tori [9]). The corresponding Szekeres metrics are called quasispherical, quasihyperbolic, and quasiplane, respectively[1].

The Friedmann metrics result from (1.2)–(1.4) when $k = k_0 z^2$, $M = M_0 z^3$, and $\Phi = zR(t)$ (k_0 and M_0 being arbitrary constants); then R obeys an equation identical in form to (1.4) with (k, M, Φ) replaced by (k_0, M_0, R). In the Friedmann limit, $\mathcal{E},_z = 0$ and (1.5) reduces to $\kappa c^2 \rho = 6M_0/R^3$, so $\rho > 0$ immediately implies $M_0 > 0$. In the general Szekeres case, a solution of (1.4) exists for $M < 0$ and $k < 0$, and it is not immediately clear from (1.5) what consequences $M < 0$ has.

In Class II, the metric is (1.1) with

$$\begin{aligned} \mathrm{e}^{\beta} &= \Phi(t)\mathrm{e}^{\nu}\,, \qquad \mathrm{e}^{\alpha} = \Phi(t)\,\sigma(z,x,y) + \lambda(t,z)\,, && (1.6)\\ \sigma &= \mathrm{e}^{\nu}\left[\frac{1}{2}U(z)\left(x^2+y^2\right) + V_1(z)x + V_2(z)y + 2W(z)\right], && (1.7)\\ \mathrm{e}^{-\nu} &\overset{\text{def}}{=} 1 + \frac{1}{4}k\left(x^2+y^2\right)\,, && (1.8)\end{aligned}$$

[1] The Hellaby parametrisation of (1.2) and (1.3) obscures the fact (which is evident in the original Szekeres parametrisation) that *the same* Szekeres spacetime may be quasispherical in one region and quasihyperbolic elsewhere, the boundary between these regions being quasiplane, see Ref. [10] for an illustration.

where U, V_1, V_2, and W are arbitrary functions of z. The function $\Phi(t)$ obeys (1.4), but here k and M are constants. We define

$$-k + 2M/\Phi \overset{\text{def}}{=} \mathcal{U}(\Phi)\,. \tag{1.9}$$

The function $\lambda(t, z)$ obeys

$$\lambda,_t \Phi\Phi,_t + \lambda M/\Phi = (U + kW)\Phi + X(z)\,, \tag{1.10}$$

where $X(z)$ is an arbitrary function. (Equation (1.10) is valid only when $\Phi,_t \neq 0$; this will be assumed.) We will consider only the case $k < 0$; then λ and W can be reparametrised so that the new $\widetilde{W}$ obeys $U + k\widetilde{W} = 0$ [4]. Then, omitting the tildes

$$\lambda,_t \Phi\Phi,_t + \lambda M/\Phi = X(z)\,. \tag{1.11}$$

The solution of (1.11) can be written as

$$\lambda = \sqrt{\mathcal{U}}\left[\int \frac{X(z)}{\Phi\mathcal{U}^{3/2}}\mathrm{d}\Phi + Y(z)\right]. \tag{1.12}$$

The integral is elementary (see further). The mass density ρ is here

$$\kappa c^2 \rho = \frac{2X + 6M\sigma}{\Phi^2\ \mathrm{e}^{\alpha}}\,. \tag{1.13}$$

Class II metrics can be obtained from Class I by a limiting transition invented by Hellaby [11], see Ref. [4] for a derivation.

Although the Szekeres metrics have been investigated in numerous papers with several applications in view (see Refs. [12, 13] for overviews), the solutions of (1.4) with $M < 0$ have not been looked at up to now, so their geometrical and physical interpretation is unknown. The aim of the present paper is to fill this gap.

In Section 2, the solutions corresponding to $M < 0$ and $k < 0$ are displayed. In Section 3, the conditions for avoiding shell crossings in Class I metrics are investigated. It is proved that shell crossings can be avoided, but the conditions of the arbitrary functions that guarantee the avoidance imply that the mass density of the dust must be negative at all times.

In Section 4, the no-shell-crossing conditions are investigated for Class II metrics. Here, if $\rho > 0$ is fulfilled at the instant of maximum compression, $t = t_B$, then $\rho \to \infty$ at a finite time t_0 and $\rho < 0$ at $t \to \infty$. However, the parameters of the metric can be chosen so that $\rho < 0$ and $|\rho| < \infty$ everywhere.

Section 5 is a summary of the results. The properties listed above exclude the Szekeres metrics with $M < 0$ as models of any physical object. This paper is meant to fill a formal gap in the study of the Szekeres metrics. Its result can be stated as the following theorem:

Theorem 1. *In the whole family of the Szekeres metrics, the inequality $M<0$ combined with the conditions of no shell crossings implies that the mass density of the dust source in the Einstein equations is negative everywhere.*

2. The solutions of (1.4) with $M < 0$

With $M < 0$, Eq. (1.4) has solutions only when $k < 0$. In both classes of the Szekeres metrics, we denote

$$M = -\mathcal{M}\,, \qquad k = -K\,, \tag{2.1}$$

so that $\mathcal{M} > 0$ and $K > 0$. Then (1.4) becomes

$$\varPhi,_t{}^2 = K - 2\mathcal{M}/\varPhi\,. \tag{2.2}$$

The solution of (2.2) is

$$\varPhi = (\mathcal{M}/K)\ (\cosh\eta + 1)\,, \tag{2.3}$$

$$t - t_B = \left(\mathcal{M}/K^{3/2}\right)\ (\sinh\eta + \eta)\,, \tag{2.4}$$

where η is the parameter, t_B is an arbitrary function of z in Class I and an arbitrary constant in Class II. The $\varPhi$ of (2.3) does not vanish at any $\eta \in (-\infty, +\infty)$ and at any z (because $\mathcal{M} > 0$), so there is no Big Bang/Big Crunch and no origin. It has the lower bound $2\mathcal{M}/K$ at $\eta = 0 \Longleftrightarrow t = t_B$. Equation (2.2) implies that $\varPhi,_{tt} = \mathcal{M}/\varPhi^2 > 0$, *i.e.* the dust expands with acceleration for $t > t_B$/contracts with deceleration for $t < t_B$.

Equation (1.5) now becomes

$$\kappa c^2 \rho = \frac{2\left(3\mathcal{M}\mathcal{E},_z/\mathcal{E} - \mathcal{M},_z\right)}{\varPhi^2\left(\varPhi,_z - \varPhi\mathcal{E},_z/\mathcal{E}\right)}\,. \tag{2.5}$$

From (2.3) and (2.4), we find

$$\varPhi,_t = \frac{\sqrt{K}\sinh\eta}{\cosh\eta + 1}\,. \tag{2.6}$$

From this point on, the two classes have to be considered separately.

3. Conditions for no shell crossings in Class I ($\beta,_z \neq 0$) metrics

Differentiating (2.3) and (2.4) by z and using (2.6), we find the equation identical in form to (20.71) in Ref. [4]

$$\Phi,_z = \left(\frac{\mathcal{M},_z}{\mathcal{M}} - \frac{K,_z}{K}\right)\Phi + \left[\left(\frac{3K,_z}{2K} - \frac{\mathcal{M},_z}{\mathcal{M}}\right)(t - t_B) - t_{B,z}\right]\Phi,_t\ . \qquad (3.1)$$

For ρ to be finite, the denominator of (2.5) may vanish only together with the numerator. If $\Phi,_z - \Phi\mathcal{E},_z/\mathcal{E} = 0$ anywhere else, then this is a shell crossing singularity. Thus, to ensure $|\rho| < \infty$, we must impose the condition

$$\text{either} \quad \Phi,_z - \Phi\mathcal{E},_z/\mathcal{E} > 0\,, \qquad (3.2)$$

$$\text{or} \quad \Phi,_z - \Phi\mathcal{E},_z/\mathcal{E} < 0\,. \qquad (3.3)$$

Now the three types of quasisymmetry must be considered separately.

3.1. The quasispherical case

In this case, the set $\mathcal{E},_z = 0$ must exist and is a great circle of every (x, y) sphere, while the extrema of $\mathcal{E},_z/\mathcal{E}$ are at the poles, where [14]

$$\left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\text{extreme}} = \pm\frac{1}{S}\sqrt{S,_z{}^2 + P,_z{}^2 + Q,_z{}^2}\,, \qquad S > 0\,. \qquad (3.4)$$

Let us investigate (3.2) first. It will be fulfilled everywhere when

$$\frac{\Phi,_z}{\Phi} > \left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max} = +\frac{1}{S}\sqrt{S,_z{}^2 + P,_z{}^2 + Q,_z{}^2}\,. \qquad (3.5)$$

Using (2.3), (2.4), (2.6), and (3.1), the above can be written as

$$\frac{\mathcal{M},_z}{\mathcal{M}} - \frac{K,_z}{K} + \left(\frac{3K,_z}{2K} - \frac{\mathcal{M},_z}{\mathcal{M}}\right)F(\eta) - t_{B,z}\frac{K^{3/2}}{\mathcal{M}}G(\eta) > \left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max}\,, \qquad (3.6)$$

where

$$F(\eta) \overset{\text{def}}{=} \frac{\sinh\eta(\sinh\eta + \eta)}{(\cosh\eta + 1)^2}\,, \qquad G(\eta) \overset{\text{def}}{=} \frac{\sinh\eta}{(\cosh\eta + 1)^2}\,. \qquad (3.7)$$

These functions have the following properties:

$$\begin{aligned} F(\eta) &> 0 \text{ for } \eta \neq 0\,, & F(0) &= 0\,, & \lim_{\eta\to\pm\infty} F(\eta) &= 1\,, \\ \text{sign}\ (G) &= \text{sign}\ (\eta)\,, & G(0) &= 0\,, & \lim_{\eta\to\pm\infty} G(\eta) &= 0\,. \end{aligned} \qquad (3.8)$$

We want to ensure that (3.6) holds at all η. It will hold at $\eta = 0$ when

$$\frac{K,_z}{K} < \frac{\mathcal{M},_z}{\mathcal{M}} - \left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max}, \tag{3.9}$$

and will hold at $\eta \to \pm\infty$ when

$$\frac{K,_z}{K} > 2\left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max}. \tag{3.10}$$

A necessary condition for (3.9) and (3.10) to hold simultaneously is

$$\frac{\mathcal{M},_z}{\mathcal{M}} > 3\left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max}. \tag{3.11}$$

When (3.5) holds for all η, (2.5) with (3.11) imply that $\rho < 0$ for all η. Consequently, this is not a model of the observed Universe. Nevertheless, we will now show that there exists a choice of $\mathcal{M}, K$, and t_B that ensures (3.6) to hold at all η (which means that ρ is finite at all times).

Let (3.11) hold and let us choose K, so that (3.9) holds and

$$\frac{3K,_z}{2K} - \frac{\mathcal{M},_z}{\mathcal{M}} > 0\,. \tag{3.12}$$

It can be verified that $K,_z/K$ and $\mathcal{M},_z/\mathcal{M}$ obeying (3.9), (3.11), and (3.12) do exist, for example by visualising $(\mathcal{E},_z/\mathcal{E})_{\max}$ as a unit of length. Then, the expression in the first line of (3.6) alone is greater than $(\mathcal{E},_z/\mathcal{E})_{\max}$ in consequence of (3.9) and (3.12).

The $G(\eta)$ of (3.7) has the further property

$$-\sqrt{3}/9 \leq G(\eta) \leq \sqrt{3}/9\,. \tag{3.13}$$

Thus, the $G(\eta)$ term in (3.6) will enhance the inequality at $\eta > 0$ if

$$t_{B,z} < 0\,. \tag{3.14}$$

Equation (3.6) will hold at $\eta < 0$ if it holds at the minimum of $G(\eta)$, which occurs at such $\eta_0 < 0$ for which $\cosh\eta_0 = 2$ ($\Longrightarrow \sinh\eta_0 = -\sqrt{3}$), thus we require

$$\begin{aligned}&\frac{\mathcal{M},_z}{\mathcal{M}} - \frac{K,_z}{K} + \left(\frac{3K,_z}{2K} - \frac{\mathcal{M},_z}{\mathcal{M}}\right)\frac{\sqrt{3}\left(\sqrt{3}+\eta_0\right)}{9}\\ &- (-t_{B,z})\frac{K^{3/2}}{\mathcal{M}}\frac{\sqrt{3}}{9} > \left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max},\end{aligned} \tag{3.15}$$

which is equivalent to

$$0 < -t_{B,z} < \frac{9\mathcal{M}}{\sqrt{3}K^{3/2}} \left[\frac{\mathcal{M},_z}{\mathcal{M}} - \frac{K,_z}{K} - \left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max}\right] + \frac{\mathcal{M}}{K^{3/2}} \left(\frac{3K,_z}{2K} - \frac{\mathcal{M},_z}{\mathcal{M}}\right) \left(\sqrt{3} + \eta_0\right) . \qquad (3.16)$$

The expression in the second line is positive in consequence of (3.9), (3.12), and of $\sqrt{3} \approx 1.732, \eta_0 \approx -1.317$.

The final verdict in the (3.2) case is thus: if K and $\mathcal{M}$ obey (3.9) and (3.12) and $t_{B,z}$ obeys (3.16), then the metric (1.2) with $\varepsilon = +1$ is nowhere singular, but defines a dust distribution with $\rho < 0$ for all η.

Now let us verify whether (3.3) can co-exist with $\rho > 0$. By the same reasoning as before, we obtain, instead of (3.6),

$$\frac{\mathcal{M},_z}{\mathcal{M}} - \frac{K,_z}{K} + \left(\frac{3K,_z}{2K} - \frac{\mathcal{M},_z}{\mathcal{M}}\right) F(\eta) - t_{B,z}\frac{K^{3/2}}{\mathcal{M}}\ G(\eta) < \left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\min} = -\left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max} . \qquad (3.17)$$

Requiring that (3.17) holds at $\eta = 0$, we obtain

$$\frac{K,_z}{K} > \frac{\mathcal{M},_z}{\mathcal{M}} + \left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max} . \qquad (3.18)$$

Requiring that (3.17) holds at $\eta \to \pm\infty$, we obtain

$$-\frac{K,_z}{K} > 2\left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max} . \qquad (3.19)$$

From (3.18) and (3.19), it follows that

$$-\frac{\mathcal{M},_z}{\mathcal{M}} > 3\left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max} . \qquad (3.20)$$

But for any value of $\mathcal{E},_z/\mathcal{E}$, the following is true:

$$\frac{\mathcal{E},_z}{\mathcal{E}} \geq \left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\min} = -\left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max} . \qquad (3.21)$$

From (3.20) and (3.21), we obtain

$$3\frac{\mathcal{E},_z}{\mathcal{E}} - \frac{\mathcal{M},_z}{\mathcal{M}} > 0\,, \qquad (3.22)$$

and this, together with (3.3), implies that $\rho < 0$. Thus, even if $\mathcal{M}, K$, and t_B can be chosen so that $\Phi,_z/\Phi - \mathcal{E},_z/\mathcal{E} < 0$ holds, $\rho < 0$ follows anyway.

3.2. The quasihyperbolic case

This case was worked out in Refs. [10] and [15] for $M > 0$. When $\varepsilon = -1$, Eq. (3.4) is replaced by

$$\left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\text{extreme}} = -\varepsilon_2 \nu \frac{1}{S} \sqrt{{S,_z}^2 - {P,_z}^2 - {Q,_z}^2}\,, \tag{3.23}$$

where $\varepsilon_2 = \pm 1$ is the sign of $S,_z$ and $\nu = \pm 1$ is the sign of $\mathcal{E}$. These extrema exist only if

$${S,_z}^2 > {P,_z}^2 + {Q,_z}^2\,. \tag{3.24}$$

In the opposite case, shell crossings are inevitable with any sign of M [10], so we assume (3.24) to hold.

The space of constant t consists of two parts, $\mathcal{E},_z/\mathcal{E}$ having a fixed sign in each part, and nowhere being zero [10]. The $(\mathcal{E},_z/\mathcal{E})_{\text{extreme}}$ is a maximum where $\mathcal{E},_z/\mathcal{E} < 0$ and a minimum where $\mathcal{E},_z/\mathcal{E} > 0$. (These two parts are different coordinate coverings of the same space [15].)

Let us first consider (3.2), separately for $\mathcal{E},_z/\mathcal{E} < 0$ and $\mathcal{E},_z/\mathcal{E} > 0$.

3.2.1. Case I: $\mathcal{E},_z/\mathcal{E} < 0$ with (3.2)

In this part of the spacetime, the condition for (3.2) is

$$\frac{\Phi,_z}{\Phi} > \left(\frac{\mathcal{E},_z}{\mathcal{E}}\right)_{\max} = -\frac{1}{S} \sqrt{{S,_z}^2 - {P,_z}^2 - {Q,_z}^2} < 0\,, \tag{3.25}$$

and then the condition for $\rho > 0$ in (2.5) is

$$\mathcal{M},_z < 3\mathcal{M}\frac{\mathcal{E},_z}{\mathcal{E}} < 0\,. \tag{3.26}$$

We can still use (3.1) and (3.6), $(\mathcal{E},_z/\mathcal{E})_{\max}$ now being (3.25) in place of (3.5). The reasoning here proceeds in analogy to the one that follows (3.8) and again leads to (3.11), which contradicts (3.26). Thus, $\rho < 0$ also here.

3.2.2. Case II: $\mathcal{E},_z/\mathcal{E} > 0$ with (3.2)

In this case, $\mathcal{E},_z/\mathcal{E}$ has the positive minimum given by (3.23) with $-\varepsilon_2\nu = +1$, and is unbounded from above. Thus, there exists no choice for the functions $K, \mathcal{M}$, and t_B in (3.1) that would guarantee (3.2) in the whole region where $\mathcal{E},_z/\mathcal{E} > 0 \Longrightarrow$ this case is empty.

We now consider (3.3), and again two cases separately:

3.2.3. Case III: $\mathcal{E}_{,z}/\mathcal{E} < 0$ with (3.3)

Here, $\mathcal{E}_{,z}/\mathcal{E}$ has a negative maximum, but is not bounded from below. Consequently, it is impossible to fulfil (3.3) in this part, and this case is empty, just like Case II.

3.2.4. Case IV: $\mathcal{E}_{,z}/\mathcal{E} > 0$ with (3.3)

Here, $\mathcal{E}_{,z}/\mathcal{E}$ has a positive minimum, and the condition for (3.3) is

$$\frac{\Phi_{,z}}{\Phi} < \left(\frac{\mathcal{E}_{,z}}{\mathcal{E}}\right)_{\min} = \frac{1}{S}\sqrt{{S_{,z}}^2 - {P_{,z}}^2 - {Q_{,z}}^2} > 0\,. \tag{3.27}$$

Using (3.6) and (3.7) for $\Phi_{,z}/\Phi$, the reasoning presented below (3.8) leads to

$$\mathcal{M}_{,z}/\mathcal{M} < 3\left(\mathcal{E}_{,z}/\mathcal{E}\right)_{\min}\,. \tag{3.28}$$

This guarantees that $\rho < 0$. So, the conclusion here is the same as in Case I. The equivalence of the results obtained for $\mathcal{E}_{,z}/\mathcal{E} < 0$ and for $\mathcal{E}_{,z}/\mathcal{E} > 0$ confirms that the two parts of each $t =$ constant space are different coordinate coverings of the same object [15].

3.3. The quasiplane case

This case was worked out in Refs. [10] and [9] for $M > 0$. When $\varepsilon = 0$, the metric (1.2) and (1.3) and Eqs. (1.4) and (1.5) do not change in form when the functions in them are redefined as follows:

$$(\Phi, \mathcal{E}) = \frac{1}{S}\left(\widetilde{\Phi}, \widetilde{\mathcal{E}}\right), \qquad K = \frac{\widetilde{K}}{S^2}, \qquad \mathcal{M} = \frac{\widetilde{\mathcal{M}}}{S^3}\,. \tag{3.29}$$

The result is the same as if $S = 1$, and we shall assume this.

As shown in Ref. [9], a surface of constant t and z in the quasiplane Szekeres model can be interpreted in two ways:

1. As an infinite plane.

2. As a torus-like closed surface that is created by identifying opposite edges of a flat basic square[2].

With the first interpretation, shell crossings are inevitable [9, 10], so we shall not consider it here.

[2] Actually, the basic square consists of four smaller squares whose edges are identified with a twist [9]. The twist is necessary to fulfil the matching conditions.

With the second interpretation, to avoid shell crossings, the inequality[3]

$$\frac{\Phi,_z}{\Phi} \geq \frac{\pi}{\sqrt{2}} \sqrt{{P,_z}^2 + {Q,_z}^2} \tag{3.30}$$

must be fulfilled at all t and z [9]; then the set where $\Phi,_z/\Phi - \mathcal{E},_z/\mathcal{E} = 0$ lies outside the basic square, *i.e.* is not part of the spacetime. The calculation here is analogous to the one that we applied to (3.5) and (3.6), Eq. (3.11) follows again, and $\rho < 0$ is also implied here.

4. Conditions for no shell crossings in Class II ($\beta,_z = 0$) metrics

Unlike in Class I, the sign of k determines the type of quasisymmetry, and with $k < 0$ the metric is quasihyperbolic. Equations (2.2)–(2.4) still hold, but now $K > 0$ and $\mathcal{M} > 0$ are constants. The integral in (1.12) reads (recall: $\mathcal{U} = K - 2\mathcal{M}/\Phi$)

$$\int \frac{\mathrm{d}\Phi}{\Phi \mathcal{U}^{3/2}} = -\frac{2}{K\sqrt{\mathcal{U}}} + \frac{1}{K^{3/2}} \ln\left|\frac{\sqrt{K}+\sqrt{\mathcal{U}}}{\sqrt{K}-\sqrt{\mathcal{U}}}\right| . \tag{4.1}$$

Note that $0 \leq \mathcal{U} \leq K$ from (1.9), so $(\sqrt{K}+\sqrt{\mathcal{U}})/(\sqrt{K}-\sqrt{\mathcal{U}}) \geq 1$, and consequently the ln term in (4.1) is non-negative at all t. In fact, the whole right-hand side of (4.1) is non-negative[4].

The explicit formula for λ becomes now

$$\lambda = X(z)\left[-\frac{2}{K} + \frac{\sqrt{\mathcal{U}}}{K^{3/2}} \ln\left|\frac{\sqrt{K}+\sqrt{\mathcal{U}}}{\sqrt{K}-\sqrt{\mathcal{U}}}\right|\right] + Y(z)\sqrt{\mathcal{U}} . \tag{4.2}$$

The formula for mass density (1.13) is here

$$\kappa c^2 \rho = \frac{2X - 6\mathcal{M}\sigma}{\Phi^2\ \mathrm{e}^{\alpha}} , \tag{4.3}$$

and (1.8) is

$$\mathrm{e}^{-\nu} = 1 - \frac{1}{4}\, K\left(x^2 + y^2\right) . \tag{4.4}$$

Seemingly, this spacetime consists of two disjoint regions, $x^2 + y^2 > 4/K$ and $x^2 + y^2 < 4/K$. However, the argument used in Ref. [15] for the quasi-hyperbolic Class I metrics applies also here: the two regions are different coordinate coverings of the same spacetime, so it suffices to consider one of them.

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[3] Tildes are dropped for better readability. The Φ below is the $\widetilde{\Phi}$ of (3.29).

[4] This is an easy exercise for the reader. Begin by defining $x \overset{\text{def}}{=} \sqrt{K/\mathcal{U}} \geq 1$.

We verify whether it is possible to choose the arbitrary functions so that $|\rho| < \infty$ and $\rho > 0$ everywhere. We begin by assuming

$$X > 3\mathcal{M}\sigma \tag{4.5}$$

and $\mathrm{e}^\alpha > 0$ in (4.3). From (1.6) and (4.2), we have

$$\mathrm{e}^\alpha = \varPhi\sigma - \frac{2X}{K} + \frac{X\sqrt{\mathcal{U}}}{K^{3/2}} \ln\left|\frac{\sqrt{K} + \sqrt{\mathcal{U}}}{\sqrt{K} - \sqrt{\mathcal{U}}}\right| + Y\sqrt{\mathcal{U}}\,. \tag{4.6}$$

Taking this at $t = t_B$, where $\varPhi = 2\mathcal{M}/K$ and $\mathcal{U} = 0$, we obtain from $\mathrm{e}^\alpha > 0$

$$X < \mathcal{M}\sigma\,. \tag{4.7}$$

This can hold simultaneously with (4.5) only when $\sigma < 0$, so $X < 0$.

To see what happens as $t \to \infty$, where $\varPhi \to \infty$, let us note that

$$\lim_{t\to\infty}\left[\frac{1}{\varPhi} \ln\left|\frac{\sqrt{K} + \sqrt{\mathcal{U}}}{\sqrt{K} - \sqrt{\mathcal{U}}}\right|\right] \equiv \lim_{t\to\infty}\left[\frac{2}{\varPhi} \ln\left|\frac{\varPhi}{2\mathcal{M}} \left(\sqrt{\mathcal{U}} + \sqrt{K}\right)\right|\right] = 0\,, \tag{4.8}$$

so, at large t, $|\varPhi\sigma|$ dominates over the other terms in (4.6). With $\sigma < 0$, this means $\lim_{t\to\infty} \mathrm{e}^\alpha = -\infty$, *i.e.* ρ becomes negative in the infinite future. However, if $\mathrm{e}^\alpha > 0$ at $t = t_B$ and $\mathrm{e}^\alpha < 0$ at $t \to \infty$, then $\mathrm{e}^\alpha = 0$ at some $t_0 > t_B$ [5]. Then (4.3) shows that $\rho \underset{t\to t_0}{\longrightarrow} \infty$, so a singularity in ρ is inevitable.

Let us now assume that, instead of (4.5),

$$X < 3\mathcal{M}\sigma\,. \tag{4.9}$$

Is this compatible with $\mathrm{e}^\alpha < 0$? Taking $\mathrm{e}^\alpha < 0$ and (4.6) at $t = t_B$, we obtain

$$X > \mathcal{M}\sigma\,. \tag{4.10}$$

Equations (4.9) and (4.10) are consistent only when $\sigma > 0$. But then, from (4.8), $\varPhi\sigma$ dominates over the other terms in (4.6) as $t \to \infty$, so $\mathrm{e}^\alpha \underset{t\to\infty}{\longrightarrow} +\infty$, and again e^α must go through zero somewhere between $t = t_B$ and $t \to \infty$.

Thus, if $\rho > 0$ at the minimum of $\varPhi$, then ρ becomes infinite at some $t = t_0 > t_B$ and negative at $t \to \infty$.

Let us now see whether ρ can be nonsingular when $\rho < 0$. Assuming $\mathrm{e}^\alpha < 0$ and taking e^α at $t = t_B$, we obtain (4.10) again. At $t \to \infty$, the sign of e^α is the same as the sign of σ. Thus, to ensure $\rho < 0$ at large t, we must have $X > 3\mathcal{M}\sigma$, when $\sigma > 0$ and $X > \mathcal{M}\sigma$, when $\sigma < 0$.

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[5] From (4.6), $\mathrm{d}\mathrm{e}^\alpha/\mathrm{d}t$ is well-defined for all $t_B < t < \infty$, so e^α is continuous.

Let us take $\sigma > 0$ for the beginning. The inequality $X > 3\mathcal{M}\sigma$ will hold throughout the whole (x, y) surface, when σ has an upper bound $\sigma_{\max}$ and $X > 3\mathcal{M}\sigma_{\max}$. Let us consider the region in which $\mathrm{e}^{-\nu} > 0$ in (1.8)

$$x^2 + y^2 < 4/K \tag{4.11}$$

and let us denote

$$\mathcal{P} \overset{\text{def}}{=} \frac{1}{2}\, U \left(x^2 + y^2\right) + V_1 x + V_2 y + 2U/K \tag{4.12}$$

(recall: we use the parametrisation in which $W = -U/k = U/K$), so

$$\sigma = \frac{\mathcal{P}}{1 - \frac{1}{4}\, K\left(x^2 + y^2\right)}\,. \tag{4.13}$$

Let the overbar denote the limit at $x^2 + y^2 \nearrow 4/K$. Then $\bar{\sigma} = \pm\infty$ and the sign of σ is the sign of $\mathcal{P}$. Note that $\bar{\mathcal{P}}$ is finite. If $\bar{\mathcal{P}} > 0$, then σ has no upper bound. If $\bar{\mathcal{P}} < 0$, then $\bar{\sigma} = -\infty$, contrary to the assumed $\sigma > 0$. Thus, $\sigma > 0$ throughout the disk (4.11) is impossible.

Now let $\sigma < 0$. Then $\mathcal{P} < 0$ within the disk (4.11), so $\bar{\mathcal{P}} \leq 0$ ($\bar{\sigma} = -\infty$ if $\bar{\mathcal{P}} < 0$) and $X > \mathcal{M}\sigma$ (this still permits $X < 0$ in (4.6)). The $\Phi(t)$ is an increasing function at $t > t_B$, so $\Phi(t)\sigma < \Phi(t_B)\sigma < 0$. Moreover, the co-factor of X in (4.6), which is proportional to the right-hand side of (4.1), is positive for all t. Then a sufficient condition for $\mathrm{e}^{\alpha} < 0$ in the disk $x^2 + y^2 < 4/K$ is $X < 0$ and $Y < 0$.

To ensure $\sigma < 0$ in the disk (4.11), we must have $\mathcal{P} < 0$, even at the maximum of $\mathcal{P}$. The maximum is at $(x, y)_{\max} = -(1/U)(V_1, V_2)$, $U < 0$, and the point $(x, y)_{\max}$ will be within the disk (4.11) when

$${V_1}^2 + {V_2}^2 < 4U^2/K\,, \tag{4.14}$$

and then $\bar{\mathcal{P}} < 0$, as it should be.

Thus, the final conclusion is that in Class II, $\rho > 0$ and $|\rho| < \infty$ is impossible, but $\rho < 0$ and $|\rho| < \infty$ is achieved by choosing the constants and functions so that (4.14) holds, and

$$U < 0\,, \qquad \mathcal{M}\sigma_{\max} < X < 0\,, \qquad \text{and} \qquad Y < 0\,. \tag{4.15}$$

5. Summary and conclusions

The evolution equation of the Szekeres metrics, (1.4) (which is identical to the Friedmann equation), has a well-defined solution when the mass function M is negative ([4, Sec. 20.2.1]; then also $k < 0$). In the Friedmann limit, $M < 0$ immediately implies $\rho < 0$ for the mass density, which is impossible in any real object. However, in the Szekeres metrics, where the relation between M and ρ is (1.5) in Class I and (1.13) in Class II, the consequences of $M < 0$ are not immediately visible. The aim of the present paper was to investigate them.

In brief, when $M < 0$, for both classes of the Szekeres metrics the conditions for no shell crossing lead to $\rho < 0$ in the whole spacetime. Consequently, the Szekeres metrics with $M < 0$ cannot describe any physical entity.

Section 1 presented the motivation for the present paper and the basic properties of both classes of the Szekeres metrics. In Section 2, the explicit solution of the Szekeres evolution equation with $M < 0$ was given.

In Section 3, the conditions for no shell crossings in Class I metrics were investigated for the quasispherical, quasihyperbolic, and quasiplane cases with $M < 0$ and $k < 0$. In each case, the shell crossing is avoided at the cost of having $\rho < 0$ throughout the spacetime.

In Section 4, the reasoning of Section 3 was applied to Class II metrics. Here, with $k < 0$, only the quasihyperbolic metric exists. The conclusion was that if $\rho > 0$ at the instant of maximum compression, then necessarily $\rho \to \infty$ at a finite $t = t_0$, so a singularity is unavoidable. However, the parameters of the metric can be chosen so that $|\rho| < \infty$ and $\rho < 0$ everywhere.

The property $\rho < 0$ excludes the Szekeres metrics with $M < 0$ as models of any physical object, and $\rho \to \infty$ at a finite t is an even worse offence. This paper was meant to fill a formal gap in the investigation of the Szekeres family of metrics: the case $M < 0$ has not been looked at so far.

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