ON THE APPLICATION OF THE METHOD OF THE FORMAL SERIES TO THE VARIABLE PHASE APPROACH

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The method of formal series can be applied to solving the variable phase equation for partial amplitude, to relativistic potential, theory and quasipotential approaches. The partial amplitude is represented by the ratio of two power series of the coupling constant.

1. Introduction

It is well-known that for the determination of the phase-shift δ_l, S_l-matrix and partial amplitude of the scattering f_l in the nonrelativistic quantum theory one can use regular and irregular solutions of the radial Schrödinger equation. During the last years for the finding of these magnitudes the method of variable phase approach was formulated as presented in the papers by Calogero [1] and Babikov [2]. The variable phase approach in the case of local spherically symmetric potentials is based on an ordinary nonlinear first-order differential equation for the phase shift or for the partial amplitude. The method of variable phase approach was generalized to describe relativistic problems in the quasi-potential approach [3], [4].

In the present paper we apply the method of formal series to solve a nonlinear variable phase equation for partial amplitude. This method, due to Dubois-Violette [5], [6], seems to be well suited for programming by computers and also for theoretical considerations.

An exact expression for the partial amplitude may be represented by the ratio of two power series of the coupling constant. The obtained approximation is valid for strong and weak coupling. When the series in the numerator and denominator are approximated by two polynomials, instead of the corresponding series the Padè approximant is obtained.

In Section 2 the partial amplitude for the nonrelativistic local potential theory is considered. The variable phase approach for the nonlocal potential problem is discussed in Section 3. In Section 4 relativistic partial amplitude in quasipotential approach is considered.

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2. *Partial amplitude for nonrelativistic local potential theory*

In the present context we shall consider spherically symmetric potentials

$$V = \lambda V_0(r), \tag{1}$$

with the coupling constant λ, satisfying the following conditions:

a) $V_0(r)$ is almost everywhere continuous function in the interval $0 \leqslant r < \infty$,

b) $$J(r_0) = \int_0^{r_0} x|V_0(x)|dx < \infty,$$

c) $$J(r_1) = \int_{r_1}^{\infty} x|V_0(x)|dx < \infty,$$

for any positive numbers r_0 and r_1.

The nonrelativistic partial amplitude $f_l(r)$ defined by the relation

$$f_l(r) = e^{i\delta_l(r)} \sin \delta_l(r), \tag{2}$$

where $\delta_l(r)$ is the phase shift produced by the potential up to the distance r, satisfies the non-linear differential equation

$$\frac{df_l(r)}{dr} = -\frac{V(r)}{k}[j_l(kr) + if_l(r)h_l^{(1)}(kr)]^2, \tag{3}$$

$$f_l(0) = 0, \quad (2\mu = \hbar = 1). \tag{4}$$

The functions $j_l(kr)$ and $h_l^{(1)}(kr)$ are two linearly independent solutions of the corresponding radial Schrödinger equation when $V(r) = 0$:

$$j_l(kr) = \sqrt{\frac{\pi kr}{2}} J_{l+1/2}(kr), \quad h_l^{(1)}(kr) = \sqrt{\frac{\pi kr}{2}} H_{l+1/2}^{(1)}(kr). \tag{5}$$

Introducing the new unknown function $F_l(r)$ by the relation

$$F_l(r) = f_l(r) - \frac{ij_l(kr)}{h_l^{(1)}(kr)}, \tag{6}$$

we obtain instead of the equation (3) and the condition (4)

$$\frac{dF_l(r)}{dr} = -\frac{k}{[h_l^{(1)}(kr)]^2} + \frac{V(r)}{k}[h_l^{(1)}(kr)]^2 F_l^2(r), \tag{7}$$

$$F_l(0) = 0. \tag{8}$$

The problem of solving the equation (7) with the initial condition (8) is reduced to the discussion of the integral equation

$$F_l(r) + \lambda \mathscr{G}[F_l(r), r] = \varphi_l(r), \tag{9}$$

where

$$\varphi_l(r) = -\frac{ij_l(kr)}{h_l^{(1)}(kr)}, \tag{10}$$

$$\mathscr{G}[F_l(r), r] = -\int_0^r \frac{V_0(x)}{k}[h_l^{(1)}(kx)]^2 F_l^2(x)dx. \tag{11}$$

Let us assume that the formal series in the variable U is given

$$S(U) = U + \lambda\mathscr{G}(U), \tag{12}$$

where λ is a real or complex number and $\mathscr{G}$ is a given formal series of order 2 at least. Suppose that U is also the formal series in the other variable U_0:

$$U = U(U_0), \tag{13}$$

with

$$U(0) = 0, \tag{14}$$

and such that the U formal series is the inverse of the $S(U)$ series with respect to substitution of formal series:

$$S(U(U_0)) = S \circ U(U_0) =$$

$$= S(U(U_0) + \lambda\mathscr{G}(U(U_0))) = U_0. \tag{15}$$

According to a classical theorem of the substitution of formal series [5], [6] there exists a unique inverse U of the S such that

$$U \circ S(U_0) = U(U_0 + \lambda\mathscr{G}(U_0)) = U_0, \tag{16}$$

or

$$\hat{\tau}_{\lambda\mathscr{G}(U_0)} U(U_0) = U_0, \tag{17}$$

where $\hat{\tau}_{\lambda\mathscr{G}(U_0)}(U_0)$ is the translation operator for U corresponding to a translation $\lambda\mathscr{G}(U_0)$ of its argument.

For the non-linear integral equation (9) the composition of formal series in a well--chosen functional space is

$$S \circ U(U_0) = F_l(r) + \lambda\mathscr{G}[F_l(r), r] = \varphi_l(r), \tag{18}$$

and its solution takes the form

$$F_l(r) = \frac{P_l(\lambda, k; r)}{Q_l(\lambda, k)} = \frac{\Gamma \exp\{-\lambda \int dy \delta\mathscr{G}[\varphi_l, y]/\delta\varphi_l(y)\}\varphi_l(r)}{\Gamma \exp\{-\lambda \int dy \delta\mathscr{G}[\varphi_l, y]/\delta\varphi_l(y)\} \cdot 1}. \tag{19}$$

In the last expression the symbol Γ indicates that all functional derivatives must be on the left, acting thus on all functionals which are put on their right. Keeping in mind this rule for the numerator and denominator we get

$$P_l(\lambda, k; r) = \Gamma \exp\{-\lambda \int dy \delta\mathscr{G}[\varphi_l, y]/\delta\varphi_l(y)\}\varphi_l(r) =$$

$$= \sum_{n=0}^{\infty} P_l^{(n)}(\lambda, k; r) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^{\infty} \dots \int_0^{\infty} dx_1 \dots dx_n \times$$

$$\times \frac{\delta^n}{\delta\varphi_l(x_1) \dots \delta\varphi_l(x_n)} \int_0^{x_1} \dots \int_0^{x_n} dx'_1 \dots dx'_n \mathscr{L}_l(k, x'_1; \varphi_l(x'_1)) \dots \mathscr{L}_l(k, x'_n; \varphi_l(x'_n))\varphi_l(r), \tag{20}$$

$$Q_l(\lambda, k) = \Gamma \exp\{-\lambda \int dy \delta\mathscr{G}[\varphi_l, y]/\delta\varphi_l(r)\} \cdot 1 =$$

$$= \sum_{n=0}^{\infty} Q_l^{(n)}(\lambda, k) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^{\infty} \dots \int_0^{\infty} dx_1 \dots dx_n \times$$

$$\times \frac{\delta^n}{\delta\varphi_l(x_1) \dots \delta\varphi_l(x_n)} \int_0^{x_1} \dots \int_0^{x_n} dx'_1 \dots dx'_n \mathscr{L}_l(k, x'_1; \varphi_l(x'_1)) \dots \mathscr{L}_l(k, x'_n; \varphi_l(x'_n)), \tag{21}$$

where

$$\mathscr{L}_l(k, x; \varphi_l(x)) = \frac{V_0(x)}{k} [h_l^{(1)}(kx)]^2 \varphi_l^2(x). \tag{22}$$

The solution of partial amplitude may be represented by the ratio

$$f_l(r) = \frac{P_l(\lambda, k; r) - \varphi_l(r) Q_l(\lambda, k)}{Q_l(\lambda, k)}. \tag{23}$$

It is an easy exercise to verify that

$$P_l^{(0)}(\lambda, k; r) = \varphi_l(r) = -\frac{i j_l(kr)}{h_l^{(1)}(kr)},$$

$$P_l^{(1)}(\lambda, k; r) = -\lambda \int_0^r \frac{V_0(x)}{k} j_l^2(kx) dx - \frac{\lambda j_l(kr)}{h_l^{(1)}(kr)} \int_0^{\infty} \frac{V_0(x)}{k} \times$$

$$\times j_l(kx) h_l^{(1)}(kx) dx, \qquad Q_l^{(0)}(\lambda, k) = 1,$$

$$Q_l^{(1)}(\lambda, k) = -i\lambda \int_0^{\infty} dx V_0(x) j_l(kx) h_l^{(1)}(kx)/k,$$

$$Q_l^{(2)}(\lambda, k) = -\frac{\lambda^2}{2k^2}\int_0^\infty dx\{V_0(x)h_l^{(1)}(kx)j_l(kx)\times$$

$$\times \int_0^\infty V_0(y)j_l(ky)h_l^{(1)}(ky)dy + 2V_0(x)\,[h_l^{(1)}(kx)]^2 \int_0^x V_0(y)j_l^2(ky)dy\}. \tag{24}$$

In the approximation obtained by stopping to the first degree of the numerator and denominator we get

$$f_l(r) = \frac{-\lambda \int_0^r V_0(x)j_l^2(kx)dx}{k-\lambda \int_0^\infty V_0(x)\,\{(-1)^l j_{-l-1}(kx)j_l(kx)+ij_l^2(kx)\}dx}\,. \tag{25}$$

A relation analogical to (25) was obtained by Drukarev [7] by another method. The approximation (25) may be found by Drukarev's expression after multiplying the numerator and denominator by the appropriate expression.

Since the denominator of (23) is a regular function of λ the partial amplitude can be developed formally as a power series

$$f_l(r) = \sum_{n=0}^\infty f_{l,n}\lambda^n. \tag{26}$$

The $[N, M]$ Padè approximant to the function f_l is the quotient $P_M(\lambda)/Q_N(\lambda)$ of two polynomials of degree M and N, respectively. These polynomials are defined in a unique way by the relation

$$f_l(r)Q_N(\lambda) = P_M(\lambda)+0(\lambda^{M+N+1}). \tag{27}$$

For the [1,1] Padè approximant obtained by (23) we have

$$f_l^{[1,1]}(r) = \frac{-\lambda \int_0^r V_0(x)j_l^2(kx)dx}{1+Q_l^{(1)}(\lambda, k)-\dfrac{P_l^{(2)}(\lambda, k; r)-\varphi_l(r)Q_l^{(2)}(\lambda, k)}{P_l^{(1)}(\lambda, k; r)-\varphi_l(r)Q_l^{(1)}(\lambda, k)}}\,. \tag{28}$$

The proposed method for determining the partial amplitude can be applied to the variable phase equation in the quasipotential approach, describing the relativistic two-body system obtained and discussed by Todorov [8]

$$\frac{df_l(r)}{dr} = -\frac{2m_1m_2}{W}V(r)\,[j_l(br)+ih_l^{(1)}(br)f_l(r)]^2. \tag{29}$$

The magnitudes m_1 and m_2 in the equation (29) are the masses of the two particles. W is the total energy of the system and

$$b^2 = (2W)^{-2}[W^4-2(m_1^2-m_2^2)W^2+(m_1^2-m_2^2)^2], \tag{30}$$

is the on-shell value of the center of mass momentum squared of each of the two particles.

3. *Partial amplitude for non-local potential*

The present method for solving the variable phase equation for partial amplitude is also applicable to the nonrelativistic problem with non-local potential. In this case the radial equation of Schrödinger is an integro-differential equation

$$\frac{d^2u_l(r)}{dr^2}+\left[k^2-\frac{l(l+1)}{r^2}\right]u_l(r)-\int_0^\infty V_l(r,r')u_l(r')dr' = 0, \tag{31}$$

where

$$V_l(r,r') = 2\pi\lambda rr' \int_{-1}^{+1} P_l(\cos\theta)V_0(\vec{z},\vec{z}')d\cos\theta = \lambda V_{0l}(r,r'). \tag{32}$$

The equation for partial amplitude in this case is of the type:

$$\frac{df_l(r)}{dr} = -\frac{1}{k}[j_l(kr)+if_l(r)h_l^{(1)}(kr)]\int_0^\infty dr' V_l(r,r')\,[j_l(kr')+if_l(r')h_l^{(1)}(kr')]. \tag{33}$$

After introducing the new unknown function by (6) we have

$$\frac{dF_l(r)}{dr} = \frac{d\varphi_l(r)}{dr}+\frac{\lambda}{k}F_l(r)h_l^{(1)}(kr)\int_0^\infty dr' V_{0l}(r,r')h_l^{(1)}(kr')F_l(r'), \tag{34}$$

$$f_l(0) = F_l(0) = 0. \tag{35}$$

So, the problem is reduced to the discussion of the integral equation

$$F_l(r)+\lambda g[F_l(r),r] = \varphi_l(r), \tag{36}$$

where

$$g[F_l(r),r] = -\frac{1}{k}\int_0^r dxF_l(x)h_l^{(1)}(kx)\int_0^\infty dr' V_{0l}(x,r')h_l^{(1)}(kr')F_l(r'). \tag{37}$$

The solution of the equation (36) takes the form

$$F_l(r) = \frac{p_l(\lambda,k;r)}{q_l(\lambda,k)} = \frac{\Gamma\exp\{-\lambda\int dx\delta g[\varphi_l,x]/\delta\varphi_l(x)\}\varphi_l(r)}{\Gamma\exp\{-\lambda\int dx\delta g[\varphi_l,x]/\delta\varphi_l(x)\}\cdot 1} =$$

$$= \sum_{n=0}^{\infty} p_l^{(n)}(\lambda,k;r)\{\sum_{n=0}^{\infty} q_l^{(n)}(\lambda,k)\}^{-1}, \tag{38}$$

where

$$p_l^{(n)}(\lambda, k; r) = \frac{\lambda^n}{n!} \int_0^\infty \dots \int_0^\infty dx_1 \dots dx_n \frac{\delta^n}{\delta\varphi_l(x_1) \dots \delta\varphi_l(x_n)} \int_0^{x_1} \dots \int_0^{x_n} dx'_1 \dots dx'_n \times$$

$$\times \tilde{\mathscr{L}}_l(k, x'_1; \varphi_l(x'_1)) \dots \tilde{\mathscr{L}}_l(k, x'_n; \varphi_l(x'_n))\varphi_l(r), \tag{39}$$

$$q_l^{(n)}(\lambda, k) = \frac{\lambda^n}{n!} \int_0^\infty \dots \int_0^\infty dx_1 \dots dx_n \frac{\delta^n}{\delta\varphi_l(x_1) \dots \delta\varphi_l(x_n)} \int_0^{x_1} \dots \int_0^{x_n} dx'_1 \dots dx'_n \times$$

$$\times \tilde{\mathscr{L}}_l(k, x'_1; \varphi_l(x'_1)) \dots \tilde{\mathscr{L}}_l(k, x'_n; \varphi_l(x'_n)), \tag{40}$$

$$\tilde{\mathscr{L}}_l(k, x; \varphi_l(x)) = \frac{1}{k} \int_0^\infty \varphi_l(x) h_l^{(1)}(kx) h_l^{(1)}(kr') V_{0l}(x, r') \varphi_l(r') dr'. \tag{41}$$

After the evaluation of the functional derivatives in (40) and (41) we have

$$p_l^{(0)}(\lambda, k; r) = \varphi_l(r) = -\frac{ij_l(kr)}{h_l^{(1)}(kr)}, \tag{42}$$

$$p_l^{(1)}(\lambda, k; r) = \frac{\lambda}{k} \left\{ \int_0^r dx \varphi_l(x) h_l^{(1)}(kx) \int_0^\infty dx' h_l^{(1)}(kx') V_{0l}(x, x') \times \right.$$

$$\times \varphi_l(x') + \varphi_l(r) \int_0^\infty dx h_l^{(1)}(kx) \int_0^x dx' h_l^{(1)}(kx') V_{0l}(x', x) \varphi_l(x') +$$

$$\left. + \tfrac{1}{2} \varphi_l(r) \int_0^\infty dx h_l^{(1)}(kx) \int_0^\infty dx' h_l^{(1)}(kx') V_{0l}(x, x') \varphi_l(x') \right\}, \tag{43}$$

$$q_l^{(0)}(\lambda, k) = 1, \tag{44}$$

$$q_l^{(1)}(\lambda, k) = \frac{\lambda}{k} \left\{ \frac{1}{2} \int_0^\infty dx h_l^{(1)}(kx) \int_0^\infty dx' h_l^{(1)}(kx') V_{0l}(x, x') \varphi_l(x') + \right.$$

$$\left. + \int_0^\infty dx h_l^{(1)}(kx) \int_0^x dx' h_l^{(1)}(kx') V_{0l}(x', x) \varphi_l(x') \right\}. \tag{45}$$

By using the expressions (42)–(45) we obtain the partial amplitude

$$f_l(r) = \frac{P_l(\lambda, k; r) - \varphi_l(r) q_l(\lambda, k)}{q_l(\lambda, k)}. \tag{46}$$

When we stop in the first degree of λ in (46) we get

$$f_l(r) = -\lambda \int_0^r j_l(kx)dx \int_0^\infty dx' V_{0l}(x, x') j_l(kx') \{k - i\lambda[\tfrac{1}{2} \int_0^\infty dx \times$$

$$\times h_l^{(1)}(kx) \int_0^\infty dx' V_{0l}(x, x') j_l(kx') + \int_0^\infty dx h_l^{(1)}(kx) \int_0^x dx' V_{0l}(x', x) j_l(kx')]\}^{-1}. \tag{47}$$

The considered method for solving the phase variable equation for the non-local potential may be very useful for the discussion of the quasipotential equation of Logunov-Tavkhelidze [9]–[11] for partial wave function

$$\frac{d^2u_l}{dr^2} + \left[k^2 - \frac{l(l+1)}{r^2}\right] u_l = V(r, k^2) \int_0^\infty dr' K_l(r, r') u_l(r'), \tag{48}$$

where

$$K_l(r, r') = \pi \sqrt{\frac{r}{r'}} \int_0^\infty \frac{qdq}{\sqrt{q^2+m^2}} J_{l+1/2}(qr) J_{l+1/2}(qr'). \tag{49}$$

4. Partial amplitude for a relativistic problem

Several years ago a quasipotential equation for partial wave function with local potential was proposed [8]

$$\left[2 \operatorname{ch} i\frac{d}{dr} + \frac{l(l+1)}{r(r+i)} e^{i\frac{d}{dr}} - 2E_K + V(r)\right] \psi_l(X_K, r) = 0, \tag{50}$$

$$E_K = \sqrt{1+k^2} = \operatorname{ch} X_K,$$

which described relativistic two body system of identical masses. On the basis of equation (50) the corresponding variable phase equation for partial amplitude was obtained and discussed [9]

$$\frac{d}{dr} f_l(X_K, r) = -\frac{V(r)}{W_l(r, X_K)} [s_l(r, X_K) + f_l(X_K, r) e_l^{(1)}(r, X_K)]^2. \tag{51}$$

The functions $s_l(r, X_K)$ and $e_l^{(1)}(r, X_K)$ are two linearly independent solutions of equation (50) when $V(r) = 0$

$$s_l(r, X_K) = \sqrt{\frac{\pi}{2} \operatorname{sh} X_K} (-i)^{l+1} \frac{\Gamma(ir+l+1)}{\Gamma(ir)} P_{ir-1/2}^{-l-1/2}(\operatorname{ch} X_K), \tag{52}$$

$$e_l^{(1)}(r, X_K) = \sqrt{\frac{\pi}{2} \operatorname{sh} X_K} (-i)^{l+1} \frac{\Gamma(ir+l+1)}{\Gamma(ir)} Q_{-ir-1/2}^{-l-1/2}(\operatorname{ch} X_K), \tag{53}$$

and $W_l(r, X_K)$ are their Wronskians. It is not difficult to show that in the non-relativistic limit

$$s_l(r, X_K) \to \sqrt{\frac{\pi kr}{2}}\, J_{l+1/2}(kr), \qquad e_l^{(1)}(r, X_K) \to -i\sqrt{\frac{\pi kr}{2}}\, H_{l+1/2}^{(1)}(kr). \tag{54}$$

Analogically to the nonrelativistic case, the solution of (51) with the initial condition $f_l(X_K, 0) = 0$ may be represented by

$$f_l(X_K, r) = -\varphi_l(r, X_K) + \frac{\Gamma \exp\{-\lambda \int_0^\infty dx \delta H_l[\varphi_l, x]/\delta\varphi_l(x, X_K)\}\varphi_l(r, X_K)}{\Gamma \exp\{-\lambda \int_0^\infty dx \delta H_l[\varphi_l, x]/\delta\varphi_l(x, X_K)\} \cdot 1}, \tag{55}$$

$$\varphi_l(r, X_K) = s_l(r, X_K)/e_l^{(1)}(r, X_K), \tag{56}$$

$$H_l[\varphi_l(r, X_K), r] = \int_0^r \frac{V_0(x)}{W_l(x, X_K)} [e_l^{(1)}(x, X_K)]^2 \varphi_l^2(x, X_K) dx, \tag{57}$$

if we assume that the integral

$$\int_0^\infty \dots \int_0^\infty dx_1 \dots dx_n \frac{\delta^n}{\delta\varphi_l(x_1, X_K) \dots \delta\varphi_l(x_n, X_K)} \times$$
$$\times \int_0^{x_1} \dots \int_0^{x_n} dx_1' \dots dx_n' \left\{ \frac{V_0(x_1')}{W_l(x_1', X_K)} [e_l^{(1)}(x_1', X_K)]^2 \varphi_l^2(x_1', X_K) \dots \right.$$
$$\left. \times \left\{ \frac{V_0(x_n')}{W_l(x_n', X_K)} [e_l^{(1)}(x_n', X_K)]^2 \varphi_l^2(x_n', X_K) \right\}, \qquad n = 0, 1, \dots, \right. \tag{58}$$

exists. It is an easy exercise to verify that

$$f_l(X_K, r) = \frac{-\lambda \int_0^r \frac{V_0(x)}{W_l(x, X_K)} s_l^2(x, X_K) dx + \dots}{1 - \lambda \int_0^\infty dx \frac{V_0(x)}{W_l(x, X_K)} s_l(x, X_K) e_l^{(1)}(x, X_K) + \dots}. \tag{59}$$

The expression for partial amplitude (59) may be very useful for the problems of bound states and resonances of the two-particle relativistic system. For the appropriate model of potentials we can also get some information about high-energy scattering of the particles.

5. *Conclusion*

The method of formal series allows us to solve the approximate variable phase equations for the nonrelativistic potential theory and the relativistic two-body problem in the quasipotential approach of Logunov-Tavkhelidze, Kadyshevsky and Todorov. The approximation is valid for strong and weak coupling constants, and can be applied to several physical problems.

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