ASYMPTOTIC BEHAVIOUR OF THE PLANAR ONE-LOOP CORRECTION TO THE REGGE TRAJECTORY IN THE DUAL MODEL

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The planar one-loop amplitude of the dual resonance model gives a first-order correction to the Regge trajectory. The analytic structure of this correction is investigated and its asymptotic behaviour calculated. The problem of analytic continuation into the right half-plane is discussed.

1. Introduction

In the present paper we continue our investigation [1] of the correction to the linear input trajectory caused by the one-loop contribution to unitarity in the dual resonance model. Neveu and Scherk [2] have found the following expression for the asymptotic behaviour of the planar one-loop diagram

$$F(\alpha_s, \alpha_t) \sim g^4(-\alpha_s)^{\alpha_t}[\ln(-\alpha_s)\Gamma(-\alpha_t)\Sigma(\alpha_t)+$$
$$+\Gamma(-\alpha_t)\beta(t)-\Gamma'(-\alpha_t)\Sigma(\alpha_t)],$$
$$\alpha_s \to -\infty. \tag{1}$$

This can be interpreted as the g^4 term of the expression

$$g^2\beta_{\text{new}}(t)\Gamma(-\alpha_{\text{new}}(t))\,(-\alpha_s)^{\alpha_{\text{new}}(t)} \tag{2}$$

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expanded in powers of the coupling constant g. We are using

$$\alpha_{\text{new}}(t) = \alpha_t + g^2 \Sigma(\alpha_t) + O(g^4),$$

$$\beta_{\text{new}}(t) = 1 + g^2 \beta(t) + O(g^4),$$

$$\alpha_t = \alpha_0 + t/2. \tag{3}$$

If the dual loop approach has indeed some resemblance to a perturbation theory one must expect *i*) that the corrections at the one-loop level are small compared with the linear input trajectory and *ii*) that higher loop corrections are in turn small compared with the one-loop terms.

We will investigate the analytic structure and the asymptotic behaviour of the correction term $\Sigma(\alpha_t)$, in particular whether the condition $\alpha_{\text{new}}(t)/\alpha_t \to 1$ is fulfilled for asymptotic values of t. $\Sigma(\alpha_t)$ contains all the corrections produced by planar loops because, for $s \to \infty$, t fixed, the u–t box diagram has the same behaviour as the s–t box but α_s is replaced by $-\alpha_s$ in (1). The s–u contribution vanishes exponentially. The case of nonplanar loops was studied by Alessandrini, Amati, and Morel [3]. The analytic continuation of the planar amplitude $F(\alpha_s, \alpha_t)$ into the right half-plane of s is much more difficult than in the twisted case [3], so we will consider only the continuation of $\Sigma(\alpha_t)$.

2. *Analytic structure of* $\Sigma(\alpha_t)$

We start from the renormalized expression for $\Sigma(\alpha_t)$ given in [2]. We will use the variables

$$\omega = xy, \qquad v = \frac{\ln x}{\ln \omega}, \qquad q = e^{2\pi^2/\ln \omega} \tag{4}$$

and the ψ functions of [4]

$$\psi(x, y) \equiv \psi(v|\omega) = \omega^{v(v-1)/2} \exp\left[\sum_1^\infty \frac{2\omega^n - \omega^{nv} - \omega^{n(1-v)}}{n(1-\omega^n)}\right] =$$

$$= -\ln \omega \frac{\theta_1\left(v \middle| -\dfrac{2\pi i}{\ln \omega}\right)}{\theta_1'\left(0 \middle| -\dfrac{2\pi i}{\ln \omega}\right)}. \tag{5}$$

This allows us to write

$$\Sigma(\alpha_t) = 4\pi^2 \int_0^1 d\omega \int_0^1 dv \frac{\omega^{-\alpha_0 - 1}}{-\ln \omega} [(1-\omega^v)(1-\omega^{1-v})]^{\alpha_0 - 1} \times$$

$$\times (1-\omega) [f(\omega)]^{-4} e^{\alpha_0 h_1} [e^{\alpha_t H} - 1], \tag{6}$$

where

$$h_1 = -2\ln\psi, \quad \tau = -\frac{2\pi i}{\ln\omega},$$

$$f(\omega) = \prod_1^\infty (1-\omega^n),$$

$$H = \ln\left[\frac{\psi'^2-\psi\psi''}{\ln^2\omega}\right] = \ln\left[\frac{\theta_1'(v|\tau)^2-\theta_1(v|\tau)\theta_1''(v|\tau)}{\theta_1'(0|\tau)^2}\right] =$$

$$= 2\ln\psi+\ln\left[\sum_1^\infty n\,\frac{\omega^{nv}+\omega^{n(1-v)}}{1-\omega^n} - \frac{1}{\ln\omega}\right]. \tag{7}$$

The notations for the theta functions are taken from Bateman, and the prime denotes differentiation with respect to the first argument.

As we show in the Appendix, h_1 and H have no singularities in the interior of the integration region. Therefore, divergencies of the integral (6) can arise only from the boundaries. Thus we have to examine the behaviour of the integrand near the boundaries.

a) Near $\omega = 1$ no divergencies do appear. This is due to the choice of renormalization [2].

b) Near $\omega = 0$, and for $v \neq (0$ or $1)$ we can approximate

$$e^{\alpha_0 h_1} = \psi^{-2\alpha_0} \approx \omega^{\alpha_0 v(1-v)},$$

$$e^{\alpha_t H} = \psi^{2\alpha_t}(-\ln\omega)^{-\alpha_t}\left[1-\ln\omega\sum_1^\infty n\,\frac{\omega^{nv}+\omega^{n(1-v)}}{1-\omega^n}\right]^{\alpha_t} \approx \omega^{-\alpha_t v(1-v)}(-\ln\omega)^{-\alpha_t},$$

$$f(\omega) \approx 1. \tag{8}$$

This gives for the integrand the behaviour

$$\omega^{-\alpha_0-1-tv(1-v)/2}(-\ln\omega)^{-1-\alpha_t} - \omega^{-\alpha_0-1+\alpha_0 v(1-v)}(-\ln\omega)^{-1}. \tag{9}$$

From (9) follows convergence near $\omega = 0$ if

$$\operatorname{Re} t < \frac{-2\alpha_0}{v(1-v)} \quad \text{and} \quad \alpha_0 < 0.$$

Minimisation of the r. h. s. with respect to v yields the convergence condition $\operatorname{Re} t < -8\alpha_0 \equiv 4\mu^2$.

c) Near $v = 0$ we use $\theta_1(0|\tau) = \theta_1''(0|\tau) = 0$ and derive

$$e^{\alpha_0 h_1} = \left[-\ln\omega\,\frac{\theta_1(v)}{\theta_1'(0)}\right]^{-2\alpha_0} \approx (-\ln\omega)^{-2\alpha_0} v^{-2\alpha_0},$$

$$e^{\alpha_t H} \approx 1+\alpha_t h(\omega) v^4,$$

$$[(1-\omega^v)(1-\omega^{1-v})]^{\alpha_0-1} \approx [-v\ln\omega]^{\alpha_0-1}(1-\omega)^{\alpha_0-1}. \tag{10}$$

The integrand, therefore, behaves as $v^{-\alpha_0+3}$ and consequently the integral is convergent near $v = 0$ if $\alpha_0 < 0$.
d) Near $v = 1$ the behaviour of the integrand is the same as near $v = 0$ because of the symmetry under $v \to 1-v$.

Thus we have shown that (6) determines for $\alpha_0 < 0$ a function of t holomorphic in the left half-plane Re $t < 4\mu^2$. Now we change the integration path in (6) to obtain an analytic continuation into the right half-plane. This change is necessary only in the neighbourhood of $\omega = 0$. In terms of the variable $\xi = \ln \omega$ the integrand behaves near $\xi = -\infty$ (i. e. $\omega = 0$) as

$$e^{\xi(-\alpha_0 - tv(1-v)/2)}(-\xi)^{-1-\alpha_t} - e^{\xi(-\alpha_0+\alpha_0 v(1-v))}(-\xi)^{-1}. \tag{11}$$

If we change the integration contour from

$$\xi = |\xi| e^{i\pi}, \qquad 0 \leqslant |\xi| < \infty$$

to a contour which becomes parallel to $|\xi|\, e^{i(\pi+\varphi)}$ at infinity, the integral along the new contour is convergent if

$$\text{Re}\,(e^{i\varphi} t) < 4\mu^2 \cos \varphi. \tag{12}$$

We can change the contour without crossing singularities of the integrand[1] provided φ is in the range $-\pi/2 < \varphi < \pi/2$. The case $|\varphi| \geqslant \pi/2$ must be excluded because this would imply $|\omega| = |e^{\xi}| \geqslant 1$ and lead out of the region of existence for the theta functions.

Thus we have proved that $\Sigma(\alpha_t)$ is holomorphic in the whole t plane except on the real positive axis $t \geqslant 4\mu^2$. The singularities there are branchpoints (normal thresholds). The discontinuities across the cut have been investigated in the previous paper [1].

3. *Asymptotic behaviour of* $\Sigma(\alpha_t)$

It is shown in the Appendix that $H(v|\omega) \geqslant 0$ in the interval $0 \leqslant (v, \omega) \leqslant 1$ and that $H = 0$ if and only if $v = (0 \text{ or } 1)$, or $\omega = 1$. For Re $t \to -\infty$ the asymptotic behaviour of $\Sigma(\alpha_t)$ is determined, therefore, by the behaviour of the integrand in the vicinity of $v = (0, 1)$, $\omega = 1$. If $t \to \infty$ parallel to the imaginary axis in the left half-plane, we use the method of stationary phase [3, 5].

To get a Fourier-type integral we differentiate $\Sigma(\alpha_t)$:

$$\Sigma'(\alpha_t) = 4\pi^2 \int_0^1 d\omega \int_0^1 dv \, \frac{\omega^{-\alpha_0-1}}{-\ln \omega} \left[(1-\omega^v)(1-\omega^{1-v})\right]^{\alpha_0-1}(1-\omega) \times$$
$$\times \left[f(\omega)\right]^{-4} H e^{\alpha_0 h_1} e^{\alpha_t H}. \tag{13}$$

[1] No singularities are crossed in changing the contour because the theta functions are holomorphic in

$$|q| \equiv \exp(2\pi^2/\xi) < 1,$$

hence singularities of the integrand in the left half-plane could arise only from zeros of exp H. We can restrict the change of the contour to small ω, where exp $H \sim \omega^{v(1-v)}(-\ln \omega)$. Obviously this expression is without zeros except at $\omega = 0$.

The asymptotic behaviour is determined by the critical points of $H(v|\omega)$. We show in the Appendix that there are no critical points of the first kind ($\partial H/\partial v = \partial H/\partial \omega = 0$) in the interior of the integration domain. Furthermore, by investigating the limits of H on the boundaries we find that $v = (0, 1)$, $\omega = 1$ are critical surfaces and that there are no critical points on the surface $\omega = 0$. Therefore, the asymptotic behaviour parallel to the imaginary axis is controlled by the same contributions as for Re $t \to -\infty$.

Let us now consider the asymptotic behaviour in the right half-plane. To get an integral representation of $\Sigma'(\alpha_t)$ valid for

$$t = |t| e^{i\left(\frac{\pi}{2} - \alpha\right)}, \qquad 0 \leqslant \alpha < \pi/2,$$

we change the integration path in the ω plane in the way indicated in Sect. 2. As has been already discussed, this change is necessary only near $\omega = 0$. This allows us to demand arg $H = \alpha$ as a second condition on the changed path. Then $\alpha_t H$ is purely imaginary and we have a pure Fourier-type integral in the real asymptotic variable $|t|$. The asymptotic

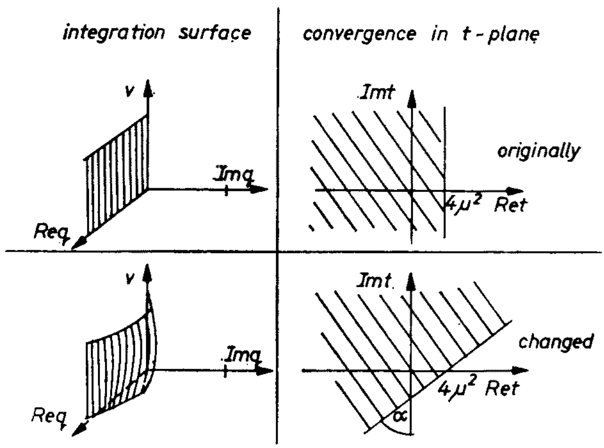

Fig. 1. Integration surface in the $v \otimes \omega$ plane and convergence domain in the t plane

behaviour is then governed by the critical points on the integration surface in the complex $\omega\otimes$ real v space (Fig. 1).

This procedure poses the following questions:

(*i*) Is the condition arg $H(v, q) = \alpha$ compatible with the convergence condition near $\omega = 0$ i. e. $q = 1$?

(*ii*) Are there really curves in the q plane for each $0 \leqslant v \leqslant 1$ satisfying arg $H = \alpha$ and passing through $q = 0$ and $q = 1$ so that an analytical integration surface is defined in the $q\otimes v$ space?

(*iii*) Is the planned change of the integration surface possible without crossing singularities of the integrand? (In Sect. 2 this was proved only for the case when the change is restricted to the vicinity of $\omega = 0$.)

(*iv*) Will there be stationary points of H on the changed surface?

Ad (i). From Eq. (7) we derive

$$H = v(v-1)\ln\omega - \ln(-\ln\omega) +$$

$$+\ln\left[1-\ln\omega\sum_{1}^{\infty} n\,\frac{\omega^{nv}-\omega^{n(1-v)}}{1-\omega^{n}}\right] + 2\sum_{1}^{\infty}\frac{2\omega^{n}-\omega^{nv}-\omega^{n(1-v)}}{n(1-\omega^{n})}. \tag{14}$$

For $0 < v < 1$ we get $H \sim v(v-1)\ln\omega$ near $\omega = 0$. That means $\arg H \sim \pi + \arg\ln\omega$. The condition of constant phase $\arg H = \alpha$ reads near $\omega = 0$ therefore $\xi = \ln\omega = |\xi|\, e^{i(\pi+\alpha)}$. From (12) convergence is guaranteed if

$$\mathrm{Re}\,[e^{i\alpha}|t|e^{i\left(\frac{\pi}{2}-\alpha\right)}] < 4\mu^{2}\cos\alpha.$$

This is true if $|\alpha| < \pi/2$. For $v = (0 \text{ or } 1)$ we have $H = 0$ (see Appendix), hence no change is necessary and on the other hand any change of the path in the q plane is compatible with convergence.

Ad (ii). In the following we understand always $\arg H = \mathrm{Im}\ln H$ as the analytic continuation along the curves under consideration. Using (14) near $q = 1$ and the power expansion of the θ_1 function near $q = 0$ we get

$$H = -\frac{2\pi^{2}v(1-v)}{\ln q} + \ln(-\ln q) - \ln 2\pi^{2} + O\left(\frac{1}{\ln q}\exp\left[\frac{2\pi^{2}}{\ln q}\min(v, 1-v)\right]\right),$$

$$q \to 1 \tag{15}$$

and

$$H = 16q^{2}\sin^{4}\pi v[1-2q^{2}(4\sin^{4}\pi v-3)+\tfrac{4}{3}q^{4}(64\sin^{8}\pi v -$$

$$-72\sin^{4}\pi v + 12\sin^{2}\pi v + 9) + O(q^{6})], \quad q \to 0 \tag{16}$$

respectively. This yields the following well behaved solution to our problem $\arg H(v, q) = \alpha$ in the vicinity of $q = 1$ and $q = 0$, respectively.

$$\varphi = \alpha + \frac{\sin\alpha}{A}R\ln R + \left[\frac{\alpha}{A}\cos\alpha - \left(\frac{1}{2}+\frac{\ln 2\pi^{2}}{A}\right)\sin\alpha\right]R + \frac{\sin 2\alpha}{2A^{2}}R^{2}\ln R +$$

$$+\left[\frac{\alpha}{A^{2}}\cos^{2}\alpha - \left(\frac{1}{12}-\frac{1}{4A}+\frac{\ln 2\pi^{2}}{2A^{2}}\right)\sin 2\alpha\right]R^{2} + O(R^{3}\ln^{3}R) + O\left(\frac{e^{-2\pi^{2}v/R}}{R}\right),$$

$$q = 1 - Re^{-i\varphi}, \quad A \equiv 2\pi^{2}v(1-v), \tag{17}$$

(for $q \to 1$).

$$\chi = \frac{\alpha}{2} + \sin\alpha(4\sin^{4}\pi v - 3)r^{2} - 4\sin 2\alpha\left(\frac{8}{3}\sin^{8}\pi v + 2\sin^{2}\pi v - 3\right)r^{4} + O(r^{6}),$$

$$q = re^{i\chi}, \quad (\text{for } q \to 0). \tag{18}$$

For general q we have investigated the function arg $H(v, q)$ by computer calculations. Here one must take care in choosing the sheet of the logarithm in order to get the proper analytic continuation of arg H if in the $[\theta_1'^2 - \theta_1\theta_1'']/\theta_1'(0)^2$ plane the origin is encircled.

A limiting case is $v \to 0$. Here we can expand

$$\arg H = \arg\left[\left.\frac{\theta_1'''^2 - \theta_1'\theta_1^{(5)}}{\theta_1'^2}\right|_{v=0} + \frac{v^2}{15}\left.\frac{\theta_1'''\theta_1^{(5)} - \theta_1'\theta_1^{(7)}}{\theta_1'^2}\right|_{v=0} + O(v^4)\right]. \tag{19}$$

H is holomorphic in the neighbourhood of $v = 0$ and in $0 \leqslant |q| \leqslant p, p < 1$. Therefore (19) is valid uniformly in q for $0 \leqslant |q| \leqslant$ p. The curve obtained from

$$\alpha = \arg\left[\frac{\theta_1'''^2 - \theta_1'\theta_1^{(5)}}{\theta_1'^2}\right]_{v=0}$$

is for $v \to 0$ approached uniformly in $0 \leqslant |q| \leqslant p$ too. Due to the fact that H has an essential singularity at $|q| = 1$ this uniformity is lost at $q = 1$. The behaviour of the limiting curve is given by

$$\varphi = \frac{\alpha}{3} + \left(\frac{1}{\pi^2} - \frac{1}{2}\right) R \sin\frac{\alpha}{3} + O(R^2),$$

$$q = 1 - Re^{-i\varphi}, \quad v = 0, \tag{20}$$

near $q = 1$ and by

$$\chi = \frac{\alpha}{2} - 3r^2 \sin\alpha + O(r^4),$$

$$q = re^{i\chi}, \quad v = 0, \tag{21}$$

near $q = 0$.

Fig. 2 shows the curves arg $H = \alpha$ for several values of v and α. Note the symmetry $H(1-v) = H(v)$. The figure very strongly suggests a positive answer to (*ii*).

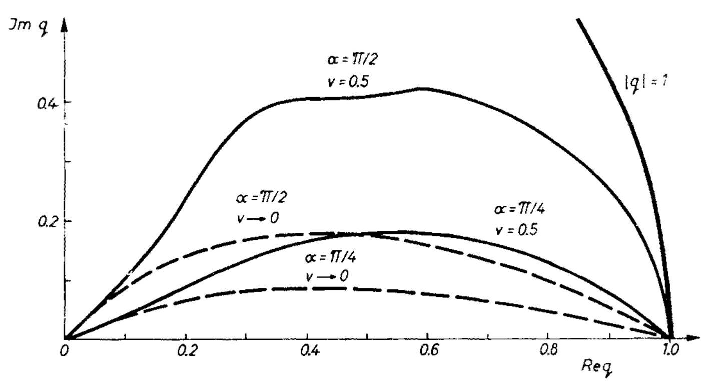

Fig. 2. Curves of constant phase α of the function H in the q plane (v as a parameter)

Ad (*iii*). H is holomorphic in the circle $|q| < 1$ except at the zeros of $\theta_1'^2 - \theta_1\theta_1''$. As it is proved in the Appendix, no such zeros exist for real v and real q. However, we must further exclude the possibility that zeros for complex values of q, crossed in changing the integration path, will reach the real v axis.

Assume for a moment that there are zeros of $\theta_1'^2 - \theta_1\theta_1''$ at $q = q_1, q_2, \ldots, q_n$ in the region crossed by changing the integration surface. Then by expanding $[\theta_1'^2 - \theta_1\theta_1'']/\theta_1'(0)^2$ we get $H = \ln(q-q_1) + \ldots + \ln(q-q_n) + \hat{H}$, where $\hat{H}$ is holomorphic in the region under investigation. Encircling all the zeros $q_1, \ldots, q_n$ would increase H by $2n\pi i$ i. e. change arg H. Using the behaviour of the trajectories arg $H = \alpha$ (Fig. 2) we can choose for fixed v a closed path ABCDA in the q plane (see Fig. 3) and return to the same arg H.

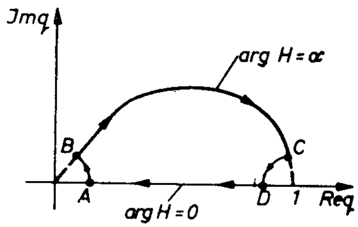

Fig. 3. Illustration to the proof that the function H has no singularities inside the region bounded by the original and the changed integration surface

Consequently, we have no zero q_i inside ABCDA. Note that we could move the segments AB and CD arbitrarily to $q = 0$ respectively. $q = 1$ because, from (15) and (16), there are no zeros q_i near $q = 0$ or $q = 1$ (except $q = 1$ itself).

Ad (*iv*). The phase arg $H = \operatorname{Im} \ln H$ as the imaginary part of an analytic function is a potential function of Re q, Im q. The stationary points of $\ln H$ as a function of q are the points where $\partial H/\partial q = 0$, $H \neq 0$. Therefore the profile of arg H near stationary points of H at which $H \neq 0$ would look like Fig. 4. Stationary points of H at which $H = 0$ can

Fig. 4. Profile of a potential function near a saddle point

be excluded by noticing that a zero of H at some point in the interior of ABCDA (Fig. 3) would cause an increase of arg H by a multiple of 2π if one goes from A via BCD back to A. Our results (Fig. 2) exclude the saddle behaviour (Fig. 4) between arg $H = 0$ and arg $H = \pi/2$.

Thus we have shown that in the right half-plane the asymptotic behaviour is, owing to the absence of interior critical points, governed, as before, by the end-point contributions. The asymptotic expression derived below is therefore valid in the whole t plane except the positive real axis.

a) Asymptotic contributions from $\omega = 1$

In terms of the variable

$$q = \exp(2\pi^2/\ln \omega)$$

and using

$$[f(\omega)]^{-4} = \frac{\ln^2 \omega}{4\pi^2}\, \omega^{1/6} q^{-1/3}[f(q^2)]^{-4}, \tag{22}$$

we get from (6) for the contribution under investigation

$$\Sigma_1(t) \sim 4\pi^4 \int_0^1 dv \int_0^\varepsilon dq\omega^{1/6-\alpha_0}(1-\omega)\,[(1-\omega^v)\,(1-\omega^{1-v})]^{\alpha_0-1} q^{-4/3}(-\ln q)^{-3} \times$$
$$\times [f(q^2)]^{-4} e^{\alpha_0 h_1}[e^{\alpha_t H}-1]. \tag{23}$$

Expanding for small q

$$1-\omega \approx \frac{2\pi^2}{-\ln q},$$

$$[(1-\omega^v)\,(1-\omega^{1-v})]^{\alpha_0-1} \approx [v(1-v)]^{\alpha_0-1}\left(\frac{-\ln q}{2\pi^2}\right)^{-2\alpha_0+2},$$

$$e^{\alpha_0 h_1} = \left[\frac{2\pi^2\theta_1(v|\tau)}{-\ln q\theta_1'(0|\tau)}\right]^{-2\alpha_0} \approx \left(\frac{2\pi^2}{-\ln q}\right)^{-2\alpha_0}\left(\frac{\sin \pi v}{\pi}\right)^{-2\alpha_0},$$

$$H \approx 16q^2 \sin^4 \pi v, \tag{24}$$

we arrive at

$$\Sigma_1(t) \sim 2\pi^2 \int_0^1 dv[v(1-v)]^{\alpha_0-1}\left(\frac{\sin \pi v}{\pi}\right)^{-2\alpha_0} \times$$

$$\times \int_0^\varepsilon dqq^{-4/3}(-\ln q)^{-2}\,[\exp(16q^2\alpha_t \sin^4 \pi v)-1]. \tag{25}$$

To perform the q-integration we calculate ($A = 16 \sin^4 \pi v$)

$$\int_0^\varepsilon dqq^{-4/3}(-\ln q)^{-2}[e^{A\alpha_t q^2}-1] =$$

$$= 2(-\alpha_t A)^{1/6} \int_0^{-\varepsilon^2 A\alpha_t} dx(e^{-x}-1)x^{-7/6}[-\ln x+\ln(-\alpha_t A)]^{-2} =$$

$$= 2(-\alpha_t A)^{1/6}\Bigg\{\int_0^{(-\alpha_t A)^{-1}} dx(e^{-x}-1)x^{-7/6}[-\ln x+\ln(-\alpha_t A)]^{-2} +$$

$$+ \sum_{n=0}^{\infty}\binom{-2}{n}(-1)^n[\ln(-\alpha_t A)]^{-2-n}\frac{\partial^n}{\partial\beta^n}\int_{(-\alpha_t A)^{-1}}^{-\varepsilon^2\alpha_t A} dx(e^{-x}-1)x^{\beta-1}\Big|_{\beta=-1/6}\Bigg\} \sim$$

$$\underset{(\mathrm{Re}\,\varepsilon^2 t\to-\infty)}{\sim} 2(-\alpha_t A)^{1/6}\sum_{n=0}^{\infty}\binom{-2}{n}[\ln(-\alpha_t A)]^{-2-n}(-1)^n\Gamma^{(n)}\left(-\frac{1}{6}\right)\sim$$

$$\sim 2(-\alpha_t)^{1/6}[\ln(-\alpha_t)]^{-2}\Gamma(-\tfrac{1}{6})A^{1/6}. \tag{26}$$

In this manner we find from (25) finally

$$\Sigma_1(t)\sim 4\pi^2(-\alpha_t)^{1/6}[\ln(-\alpha_t)]^{-2}\Gamma(-\tfrac{1}{6})\times$$

$$\times\int_0^1 dv[v(1-v)]^{\alpha_0-1}\left(\frac{\sin\pi v}{\pi}\right)^{-2\alpha_0}(2\sin\pi v)^{2/3}. \tag{27}$$

This expression is well-defined for $\alpha_0 < 2/3$. The behaviour $(-t)^{1/6}$ was, for $t\to-\infty$, already anticipated in [2].

One could try to derive the higher terms of the series in powers of $1/\ln(-\alpha_t)$. To do this one has to expand the integrand of (23) near $q = 0$ in powers of q and $1/\ln q$. Each term of the form $q^{-4/3}(\ln q)^{-m}$ yields a series

$$(-\alpha_t)^{1/6}\sum_{n=m}^{\infty}c_{nm}[\ln(-\alpha_t)]^{-n}, \tag{28}$$

all of them have to be combined in order to get the $1/\ln(-\alpha_t)$ expansion of (23). It does not seem to be entirely hopeless to calculate the coefficients of $(\ln(-\alpha_t))^{-n}$ in the limit $n\to\infty$, which should allow a closer inspection of the analytic properties of $\Sigma_1(t)$ to be made.

b) Asymptotic contributions from $v = 0$ and $v = 1$

Because of the symmetry $\theta_1(1-v) = \theta_1(v)$ the whole integrand is symmetric under $v\to 1-v$. The asymptotic contributions from $v = 0$ and $v = 1$ are therefore equal and we can restrict ourselves to the investigation of

$$\Sigma_2(t)\sim 8\pi^4\int_0^1 dq\int_0^{\varepsilon} dv\omega^{\frac{1}{3}-\alpha_0}(1-\omega)\,[(1-\omega^v)(1-\omega^{1-v})]^{\alpha_0-1}q^{-4/3}(-\ln q)^{-3}\times$$

$$\times[f(q^2)]^{-4}e^{\alpha_0 h_1}[e^{\alpha_t H}-1]. \tag{29}$$

Using (A19) we can approximate

$$[(1-\omega^v)(1-\omega^{1-v})]^{\alpha_0-1}\approx[v(-\ln\omega)(1-\omega)]^{\alpha_0-1},$$

$$e^{\alpha_0 h_1}\approx(-\ln\omega)^{-2\alpha_0}v^{-2\alpha_0},$$

$$e^{\alpha_t H} = e^{\alpha_t[v^4 h+O(v^6)]},$$

$$h = \frac{1}{12}\left.\frac{\theta_1'''^2-\theta_1'\theta_1^{(5)}}{\theta_1'^2}\right|_{v=0}, \tag{30}$$

which gives for (29)

$$\Sigma_2(t) \sim 4\pi^2(2\pi^2)^{-\alpha_0} \int_0^1 dq\omega^{\frac{1}{6}-\alpha_0} q^{-4/3}(-\ln q)^{\alpha_0-2}[f(q^2)]^{-4} \times$$

$$\times (1-\omega)^{\alpha_0} \int_0^{\varepsilon} dv v^{-\alpha_0-1}[e^{\alpha_t h v^4}-1], \tag{31}$$

and finally

$$\Sigma_2(t) \sim \pi^2(2\pi^2)^{-\alpha_0}(-\alpha_t)^{\alpha_0/4}\Gamma\left(-\frac{\alpha_0}{4}\right) \times$$

$$\times \int_0^1 dq\omega^{\frac{1}{6}-\alpha_0}(1-\omega)^{\alpha_0} q^{-4/3}(-\ln q)^{\alpha_0-2}[f(q^2)]^{-4} h^{\alpha_0/4}. \tag{32}$$

For $\alpha_0 < 2/3$ this contribution is dominated by $\Sigma_1(t)$. The result (32) needs some discussion as it appears to be divergent for $\alpha_0 < 2/3$ since $h = 16\pi^4 q^2 + O(q^4)$. The origin of this divergence lies in a double counting of the vicinity of $(v, q) = (0, 0)$. A more careful examination of this corner yields the result that the correct asymptotic contributions from the boundaries $v = (0, 1)$ and $q = 0$ are given by the series (28) for $\Sigma_1(t)$ and by (32) replaced by the real part of the analytic continuation of (32) from above the branch point $\alpha_0 = 2/3$ where the integral is convergent.

4. *Conclusion*

The main results of our investigation of the planar one-loop correction $\Sigma(\alpha_t)$ to the Regge trajectory are

a) the calculation of the explicit form of the leading term in the asymptotic behaviour of $\Sigma(\alpha_t)$ and

b) the proof that the result holds in the whole t plane except the positive real axis.

It would be interesting to extend the calculations beyond the leading term and to determine the general asymptotic term of the series in $1/\ln t$. This would allow one to achieve a better insight in the analytic properties of $\Sigma(\alpha_t)$. Maybe one could then find a correspondence with the previously calculated [1] discontinuity disc $\Sigma(\alpha_t)$. Of course, the discontinuity of the leading term of an asymptotic expansion may in general differ from the asymptotic behaviour of the discontinuity.

The leading term of $\Sigma(\alpha_t)$ was calculated along the rays

$$t = |t| \exp\left(i\left(\frac{\pi}{2} - \alpha\right)\right), \quad 0 \leqslant \alpha < \frac{\pi}{2}, \quad |t| \to \infty.$$

The investigation should be supplemented by a calculation of the asymptotic behaviour along parallels to the positive real axis.

A further remark regards the proof of the possibility to continue the asymptotic expression of $\Sigma(\alpha_t)$ into the right half-plane. In our argument we had to refer to numerical calculations of arg H, which should be avoided in a straightforward proof.

Finally, we should stress again that our investigation pertains only to $\Sigma(\alpha_t)$, i. e. to the asymptotic loop contribution for $\alpha_s \to -\infty$. No claim is made as regards the asymptotic behaviour of the full planar one-loop amplitude in the right s half-plane.

APPENDIX

Properties of the function H

In this Appendix we compile some properties of the function

$$H = \ln \frac{\theta_1'(v|\tau)^2 - \theta_1(v|\tau)\theta_1''(v|\tau)}{\theta_1'(0|\tau)^2}, \tag{A1}$$

in particular for real ω, $0 \leqslant \omega \leqslant 1$ and complex v. The notations for the theta functions are taken from Bateman, furthermore

$$\tau = -\frac{2\pi i}{\ln \omega}, \qquad q = e^{i\pi\tau} = e^{2\pi^2/\ln\omega}. \tag{A2}$$

The derivative with respect to the first argument is

$$\frac{\partial}{\partial v} H \equiv H'(v|\tau) = \frac{\theta_1'\theta_1'' - \theta_1\theta_1'''}{\theta_1'^2 - \theta_1\theta_1''}, \tag{A3}$$

for the other derivative we use $\dot{\theta}_1 = \frac{1}{4\pi i}\theta_1''$, and find

$$\frac{\partial}{\partial \tau} H \equiv \dot{H}(v|\tau) = \frac{1}{4\pi i}\frac{2\theta_1'\theta_1''' - \theta_1''^2 - \theta_1\theta_1''''}{\theta_1'^2 - \theta_1\theta_1''} - \frac{1}{2\pi i}\frac{\theta_1'''(0)}{\theta_1'(0)}. \tag{A4}$$

From the periodicity properties of the theta functions

$$\begin{aligned} \theta_1(v+1|\tau) &= \theta_1(-v|\tau) = -\theta_1(v|\tau), \\ \theta_1(v+\tau|\tau) &= e^{-i\pi(2v+\tau)}\theta_1(v|\tau), \end{aligned} \tag{A5}$$

we derive

$$\begin{aligned} H'(v+1|\tau) &= H'(v|\tau) = -H'(-v|\tau), \\ H'(v+k\tau|\tau) &= H'(v|\tau) - 4\pi i k, \qquad k = 0, \pm 1, \pm 2, \ldots, \end{aligned} \tag{A6}$$

and

$$\begin{aligned} \dot{H}(v+1|\tau) &= \dot{H}(v|\tau) = \dot{H}(-v|\tau), \\ \dot{H}(v+k\tau|\tau) &= \dot{H}(v|\tau) - kH'(v|\tau) + 2\pi i k^2, \qquad k = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{A7}$$

Both H' and $\dot{H}$ are meromorphic functions of v with poles at the zeros of $\theta_1'^2 - \theta_1\theta_1''$. To find these zeros we look at the function

$$g(v) = -\frac{\partial^2}{\partial v^2}\ln\theta_1(v) = \frac{\theta_1'(v)^2 - \theta_1(v)\theta_1''(v)}{\theta_1^2(v)}, \tag{A8}$$

which is a double periodic function with periods 1 and τ. It has a double pole in each periodicity cell namely the zeros of θ_1 at $v = m+n\tau$, consequently there must be two zeros of g in the periodicity cell. We will prove now that the zeros of g are — for real q — located on the parallels to the real axis going through $(m+1/2)\tau$. We notice

$$\theta_1'^2 - \theta_1\theta_1''|_{v+\tau/2} = e^{-2\pi i(v+\tau/4)}(\theta_4'^2 - \theta_4\theta_4'')|_v \tag{A9}$$

and calculate

$$\theta_4'^2 - \theta_4\theta_4''|_{v=0,1} = -8\pi^2 q_0^2 \sum_{n=1}^{\infty} (1-q^{2n-1})^4 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1-q^{2n-1})^2} < 0,$$

$$\theta_4'^2 - \theta_4\theta_4''|_{v=1/2} = 8\pi^2 q_0^2 \prod_{n=1}^{\infty} (1+q^{2n-1})^4 \sum_{n=1}^{\infty} \frac{q^{2n-1}}{(1+q^{2n-1})^2} > 0,$$

$$0 < q < 1. \tag{A10}$$

It follows that the zeros of g are situated at

$$v = (m+\tfrac{1}{2})\tau + \begin{cases} \delta \\ 1-\delta \end{cases}, \quad 0 \leqslant \delta \leqslant \tfrac{1}{2}. \tag{A11}$$

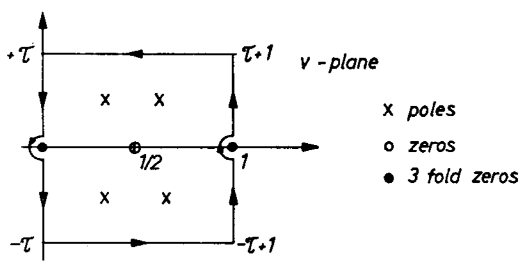

Fig. 5. Periodicity cells of the functions $\dot{H}$ and H' in the v plane

Now we are ready to determine the zeros of H'. Their number N in the double cell (Fig. 5) is given by (see (A6))

$$N-4 = \frac{1}{2\pi i} \oint dv \frac{H''}{H'} =$$

$$\frac{1}{2\pi i} \int_0^1 dv \left[\frac{H''(v)}{H'(v)+4\pi i} - \frac{H''(v)}{H'(v)-4\pi i} \right]. \tag{A12}$$

The related indefinite integrals are $\ln(H'(v) \pm 4\pi i)$. Since H' is real in the interval $0 \leqslant v \leqslant 1$, the image of (0,1) in the $(H' \pm 4\pi i)$ plane does not encircle the origin. Thus from $H'(0) = H'(1)$ we conclude that the integrals vanish, and $N = 4$, consequently.

It follows easily from

$$\theta_1(0) = \theta_1''(0) = \theta_1''''(0) = \theta_1'(\tfrac{1}{2}) = \theta_1'''(\tfrac{1}{2}) = 0 \tag{A13}$$

that there is a threefold zero at $v = 0$ and a zero at $v = 1/2$:

$$H' = \frac{\theta_1'''(0)^2 - \theta_1'(0)\theta_1^{(5)}(0)}{3\theta_1'(0)^2}\, v^3 + O(v^5), \qquad v \approx 0,$$

$$H' = \frac{\theta_1''(\tfrac{1}{2})^2 - \theta_1(\tfrac{1}{2})\theta_1''''(\tfrac{1}{2})}{\theta_1(\tfrac{1}{2})\theta_1''(\tfrac{1}{2})}\left(v - \frac{1}{2}\right) + O\left(\left(v - \frac{1}{2}\right)^3\right), \qquad v \approx \frac{1}{2}. \tag{A14}$$

By determining the sign of $H''(1/2)$

$$H''\left(\frac{1}{2}\right) = \frac{\theta_1''(\tfrac{1}{2})^2 - \theta_1(\tfrac{1}{2})\theta_1''''(\tfrac{1}{2})}{-\theta_1(\tfrac{1}{2})\theta_1''(\tfrac{1}{2})} =$$

$$= \frac{[2\pi^2 \sum_0^\infty q^{n(n+1)}(2n+1)^2]^2 - 4\pi^4 \sum_0^\infty q^{n(n+1)} \sum_0^\infty q^{n(n+1)}(2n+1)^4}{4\pi^2 \sum_0^\infty q^{n(n+1)} \sum_0^\infty q^{n(n+1)}(2n+1)^2} < 0 \tag{A15}$$

$$0 < q < 1$$

we find that $H'(v)$ behaves qualitatively as shown in Fig. 6.

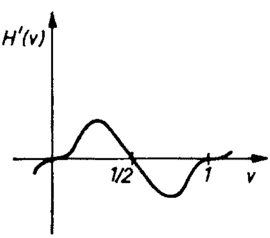

Fig. 6. Qualitative behaviour of the function $H'(v)$

The zeros of $\dot{H}(v)$ can be located in a similar manner. Their number N in the double cell (Fig. 5) is

$$N - 4 = \frac{1}{2\pi i}\oint dv\,\frac{\dot{H}'}{\dot{H}} =$$

$$= \frac{1}{2\pi i}\int_0^1 dv \left[\frac{\frac{\partial}{\partial v}[\dot{H}(v) + H'(v) + 2\pi i]}{\dot{H}(v) + H'(v) + 2\pi i} - \frac{\frac{\partial}{\partial v}[\dot{H}(v) - H'(v) + 2\pi i]}{\dot{H}(v) - H'(v) + 2\pi i}\right]. \tag{A16}$$

The related indefinite integrals are

$$\ln(\dot{H}(v) \pm H'(v) + 2\pi i).$$

Since $\dot{H}(v)$ is purely imaginary we see using Fig. 6 that the image of [0, 1] in the $(\dot{H} \pm H' + 2\pi i)$ plane does encircle the origin if and only if $i\dot{H}(1/2) > 2\pi$. In particular we find

$$N-4 = \begin{cases} 0 \\ \\ -2 \end{cases} \quad \text{for} \quad i\dot{H}(\tfrac{1}{2}) \begin{cases} < 2\pi \\ \\ > 2\pi \end{cases} . \tag{A17}$$

By explicit calculation we find a fourfold zero at $v = 0$. Then, from (A17), no other zeros are situated in the double cell.

Summarizing we can state that for real ω, $0 < \omega < 1$, there are no saddle points $(\dot{H} = H' = 0)$ in the interior of the double cell (Fig. 5), and in particular no saddle points in the interior of the integration domain of (6).

Finally, we study the properties of H on the boundaries. From (A13) we see that

$$H(0|\tau) = 0. \tag{A18}$$

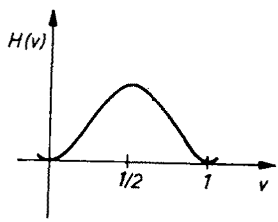

Fig. 7. Qualitative behaviour of the function $H(v)$

Together with Fig. 6 this yields the qualitative behaviour sketched in Fig. 7. Furthermore, from (A14) we get

$$H = \frac{\theta_1'''(0)^2 - \theta_1'(0)\theta_1^{(5)}(0)}{12\theta_1'(0)^2} v^4 + O(v^6), \tag{A19}$$

and (cf. (16)) in the limit $\omega \to 1$, i.e. $q \to 0$

$$\mathrm{H}(v|\tau) \to 16\, q^2 \sin^4 \pi v + \mathrm{O}(q^4). \tag{A20}$$

The critical surfaces are, therefore, $q = 0$, $v = 0$, and $v = 1$. In the case $\omega \to 0$, $0 < v < 1$ we find $H \to \infty$. Extending the arguments of [5] to unbound functions multiplying the asymptotic variable we see that $\omega = 0$ is not a critical surface.

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