THE MELOSH TRANSFORMATION FOR INTERACTING QUARKS*

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Recent developments in the problem of the Melosh transformation for interacting quarks are discussed.

CONTENT

1. Melosh transformation for free quarks

a) Introduction

Two schemes based on algebraic or group theoretic properties have been very successful in particle physics. Both are known under the heading of $SU(6)_W$ [1–3]. One, called current or charge algebra, starts from the anticommutation relations of free quark fields, and then abstracts properties which are supposed to be more general. The other one, the classification group, considers quarks as carrying spin and unitary spin quantum numbers.

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It has been pointed out that the two algebras $SU(6)_w$ cannot be identified [2]. Melosh [4] has given an explicit unitary transformation which connects the two groups. However, his arguments are based only on the properties of free quarks. Therefore, we shall consider in Section 2 quarks interacting with an external field [5].

This first section gives a short review of current algebra, light-like charges, the classification group and the need for the Melosh transformation.

b) Current or charge algebra

We start from a SU(3) triplet of spin $\frac{1}{2}$ *free quark fields*. Suppressing spinor and SU(3) indices, their canonical anticommutation relations are

$$\{q(x), q^+(y)\}_{x^0=y^0} = \delta(\vec{x}-\vec{y}). \tag{1}$$

With these quark fields, one defines a set of 144 *currents*:

$$j^\mu(x) = \tfrac{1}{2} : \bar{q}(x)\gamma^\mu O\lambda q(x):, \tag{2}$$

O is one of the 16 Dirac matrices, λ one of the 9 Gell-Mann matrices.

These currents are observables and one gets for example for:

$O = I$; $\lambda = \lambda_3 + \frac{1}{\sqrt{3}}\lambda_8$: electromagnetic current,

$O = I, \gamma_5$; $\lambda = \lambda_1 \pm i\lambda_2$: $\Delta S = 0$ weak current

$O = I, \gamma_5$; $\lambda = \lambda_4 \pm i\lambda_5$: $|\Delta S| = 1$ weak current.

Finally, one defines the 144 space-like charges

$$Q = \int_{x^0=t} d^3x j^0(x). \tag{3}$$

Because of the anticommutation relation (1), these charges obey the commutation relation of a *U(6,6) algebra*

$$[Q_A, Q_B] = iC_{ABC}Q_C, \qquad A = 1,\ldots, 144. \tag{4}$$

Gell-Mann postulated that these relations, or a subset of them, are true even for interacting fields. This hypothesis can be tested through the corresponding sum rules

$$\sum_n \langle 1 \text{ particle } |Q_A|n\rangle \langle n|Q_B|1 \text{ particle}\rangle - A \leftrightarrow B$$
$$= iC_{ABC}\langle 1 \text{ particle } |Q_C|1 \text{ particle}\rangle. \tag{5}$$

The most celebrated is the Adler-Weisberger [6] sum rule, which uses the chiral SU(2) × × SU(2) subalgebra. The space-like charges Q have defects which were pointed out by Coleman [7], who proved that if Q leaves the vacuum invariant, Q must commute with the Hamiltonian. Now, most of the charges (3), do *not* commute with H. Therefore, they will not leave the vacuum invariant. This means that in the interacting case they will not be well defined operators.

It has been pointed out by Fubini and Furlan [8] that the sum rules (5) converge better and enhance one particle contributions in the sum over intermediate states if they are evaluated in the infinite momentum frame ($p_z \to \infty$).

c) Light-like charges [9]

The observations at the end of the last paragraph can be taken care of if, instead of space-like charges Q, one introduces light-like charges $\hat{Q}$. The latter can be considered as being obtained from the former by a Lorentz boost in the z direction in the limit where the velocity approaches the velocity of light c. Hence the matrix elements of $\hat{Q}$ are somehow equivalent to the matrix elements of space-like charges in the infinite momentum limit.

The virtue of the light-like charges is that they annihilate the vacuum even if they do not commute with the Hamiltonian.

They are defined by:

$$\hat{Q} = \int_{x^0+x^3=\tau} d^3x_L j_+(x),$$

$$2j_+ = j^0+j^3, \quad \vec{x}_L = (x^1, x^2, x^0-x^3). \tag{6}$$

For free fields, this can be written as

$$\hat{Q} = \tfrac{1}{2}\int_\tau d^3x_L : \bar{q}\gamma_+ O\lambda q:$$

$$= \tfrac{1}{2}\int_\tau d^3x_L : q^+ \frac{1+\alpha_3}{2} O\lambda q: . \tag{7}$$

Furthermore, one considers only the subset *$SU(6)_w$ of good charges* defined by

$$[O, \alpha_3] = 0. \tag{8}$$

Recall that α_3 generates a boost in the z-direction and therefore the charges restricted by (8) commute with Lorentz transformations in the z direction. Only for these charges does the integration on the light-plane $x^0+x^3 = \tau$ make sense [9]. $SU(6)_w$ is the collinear group introduced by Lipkin and Meshkov [1]. For a simple proof that the charges $\hat{Q}$ annihilate the vacuum, see for example Leutwyler [9].

Therefore, we are now left with the $SU(6)_w^{\hat{Q}}$ algebra

$$[\hat{Q}_A, \hat{Q}_B] = iC_{ABC}\hat{Q}_C \quad (A, B, C = 1, \ldots, 35). \tag{9}$$

d) Classification group $SU(6)_w^c \otimes SO(3)^L$

As mentioned in the introduction, quarks are considered here only as carriers of spin $\frac{1}{2}$ and SU(3) triplet quantum numbers. One has the correspondance:

$SU(2)_w^c$: spin of quarks,

$SO(3)^L$: orbital angular momentum.

The first one comes from the subset

$$O = \beta\sigma_1, \beta\sigma_2, \sigma_3$$

One gets the quantum numbers of mesons by considering quark-antiquark systems, those of baryons from three quark states. For the least massive particles:

Particles	Representation of $SU(6)_W^c$	Representation of $SO(3)^L$
Mesons 0^-, 1^-	35+1	$L = 0$
0^+, 1^+, 2^+	35+1	$L = 1$
Baryons $^1/_2{}^+$, $^3/_2{}^+$	56	$L = 0$
$^1/_2{}^-$, $^3/_2{}^-$, $^5/_2{}^-$	70	$L = 1$
etc.		

e) Need for the Melosh transformation

One may be tempted to identify the charge group (9) with the classification group, i. e. assign physical particle states to irreducible representations of the charge group. But this is untenable from the empirical point of view. For example, the axial vector coupling constant is predicted to be $^5/_3$ instead of 1.24. Various mixing schemes have been proposed to circumvent this kind of difficulty [10]. Gell-Mann has pointed out [2] that the anomalous magnetic moment of the proton becomes zero. Theoretical arguments have also been given by Bell and Hey [11].

The most aesthetic mixing scheme has been proposed by Melosh [4]. He argued that already in the free field case hadrons made of wave-packets of current quarks (and antiquarks) do not have the right total spin. This is because light-like charges transform in a complicated way under rotations and contain "orbital excitations". Melosh gave an explicit unitary operator which transforms one group into the other. It has been shown that the matrix elements of this operator are just Clebsch Gordan coefficients for the Poincaré group [12].

In the next section we shall reanalyze some of the above arguments in the case of quarks interacting with an external field. We shall find the same "diseases" if one tries to identify the charge group with the classification group and discover that they are cured by the same Melosh transformation, at least within a certain approximation. In order to avoid duplication, we shall therefore not develop the arguments of this paragraph any more [13]. It suffices to say that the phenomenological analysis based on the Melosh transformation has been very successful [14].

2. *Melosh transformation for interacting quarks* [5]

a) Introduction

The aim of this section is to study the relations between the charge group and the the classification group when quarks move in an external field.

We first observe that for studying algebraic properties and symmetry properties of the Hamiltonian, it will be enough to study the first quantized Dirac equation with a given potential. Firstly, it can be shown [9] that for the good components q_+ of the quark fields

$$q_+ = \frac{1+\alpha_3}{2} q \tag{10}$$

one has again canonical anticommutation relations for equal $\hat{x}^0$ values,

$$\{q_+(x), q_+^+(y)\}_{\hat{x}^0=\hat{y}^0} = \delta(\vec{x}_L - \vec{y}_L), \tag{11}$$

where

$$\hat{x}^0 = x^0 + x^3, \qquad \vec{x}_L = (x^1, x^2, x^0 - x^3).$$

Secondly, for light-like charges

$$\hat{Q} = \int_{\hat{x}^0=\tau} d^3x_L q_+^+ O q_+ \tag{12}$$

one gets for the commutator of two charges

$$\begin{aligned}[\hat{Q}_1, \hat{Q}_2] &= [\int_{\hat{x}^0=\tau} d^3x_L q_+^+ O_1 q_+, \int_{\hat{y}^0=\tau} d^3y_L q_+^+ O_2 q_+] \\ &= \int_{\hat{x}^0=\tau} d^3x_L q_+^+ [O_1, O_2] q_+,\end{aligned} \tag{13}$$

where O_1 and O_2 are now Dirac matrices (from now on we leave out SU(3) matrices and indices which are inessential in this context).

So the problem is reduced to the consideration of the Dirac equation

$$E\psi = H\psi = (\vec{\alpha}\vec{p}c + \beta mc^2 + \phi + \beta V)\psi \tag{14}$$

with two arbitrary potentials ϕ and V (for the sake of generality we consider both potentials ϕ and V; we shall see that for our purposes there is no essential difference between the two).

To get the analogue of (12), we shall translate (14) into light-like language. Now, the general features of the solutions of (14) are of course well known. In particular, there is a spin-orbit coupling which, compared to the rest energy $E_0 = mc^2$, is of order c^{-4}. Hence, only terms of H which are of order c^{-3} or less have a two-fold degeneracy due to spin.

We shall calculate in the next paragraph the Hamiltonian $\hat{H}_+$ for good components ψ_+ on the light-plane. It will be seen that terms of $\hat{H}_+$ up to order c^{-2} are invariant under a group $SU(2)_w$ which is a realization, because of (13), of the light-like charge group $SU(2)_w^{\hat{Q}}$ of equation (9). So it seems that $\hat{H}_+$ has less symmetry than H. Furthermore, if one assigns particles to irreducible representations of the charge group, we shall see explicitly that their magnetic moment will be zero.

These two facts: apparent lower symmetry of $\hat{H}_+$ and vanishing of the magnetic moment, will motivate the introduction of a unitary transformation U acting on the states ψ_+.

To a $SU(2)_w$ transformation of ψ_+ will correspond a $SU(2)_M$ transformation for $U\psi_+$. U will be chosen in such a way that H_+ will be invariant under $SU(2)_M$, up to order c^{-3}. Furthermore the matrix elements of the magnetic moment between the new states will be shown to be equal to the Dirac magnetic moment.

It will turn out, and this is the main result of this section, that U is equal to the Melosh [4] transformation, to the order c^{-1}.

We shall also show that while the good components of ψ on the light-plane transform in a complicated way under rotations, and therefore should not be identified with the wave function of a particle with spin $\frac{1}{2}$, the Melosh transformed states behave correctly under rotations.

This point will be confirmed by exhibiting a close relationship between the Melosh and the Foldy-Wouthuysen transformation.

b) Dirac equation on the light-plane for good components

We start with

$$E\psi = H\psi = (c\vec{\alpha}\vec{p}+\beta(mc^2+V)+\phi)\,\psi, \tag{14}$$

where

$$p^i = \frac{1}{i}\frac{\partial}{\partial x^i}$$

and change to *light-plane variables*

$$\hat{x}^3 = x^3, \quad p^3 = \hat{p}^3-\hat{p}^0,$$

$$\hat{x}^0 = x^3+x^0, \quad p^0 = \hat{p}^0 = \frac{1}{c}H. \tag{15}$$

Notice that we do not change p^0.

We also define the new wave function

$$\hat{\psi}(\hat{x}) = \psi(x). \tag{16}$$

Furthermore we need the *projection*

$$\psi_\pm = \frac{1\pm\alpha_3}{2}\,\psi. \tag{17}$$

Using the Dirac equation (14) we can eliminate the "bad" components $\hat{\psi}_-$

$$\hat{\psi}_- = (c\hat{p}^3-\phi)^{-1}\,[(\alpha_\perp p_\perp+\beta(mc^2+V)]\,\hat{\psi}_+, \tag{18}$$

$$p_\perp = (p^1, p^2)$$

and get

$$\hat{H}_+\hat{\psi}_+ = \{[c\alpha_\perp p_\perp+\beta(mc^2+V)]\,(c\hat{p}^3-\phi)^{-1}$$
$$\times[c\alpha_\perp p_\perp+\beta(mc^2+V)]+c\hat{p}^3+\phi\}\,\hat{\psi}_+. \tag{19}$$

This, according to whether it is the first or second-quantized theories under consideration, is the light-plane Schrödinger equation for the wave-function $\hat{\psi}_+$ or the light-plane equation of motion for the Heisenberg operator $\hat{\psi}_+$.

To exhibit the symmetry properties of (19), as outlined in the preceding paragraph, we expand $\hat{H}_+$ in powers of c^{-1}.

Let us define the operator η of order c^0

$$\eta = \frac{\hat{p}^3}{mc}. \tag{20}$$

Then

$$\frac{2\hat{H}_+}{mc^2} = \eta^{-1}+\eta+\left\{\frac{p_\perp^2}{m}\eta^{-1}+V\eta^{-1}+\eta^{-1}V+\phi+\eta^{-1}\phi\eta^{-1}\right\}\frac{1}{mc^2}$$

$$+[\eta^{-1}\phi\eta^{-1}, \beta\alpha_\perp p_\perp]\frac{1}{m^2c^3}+[V, \beta\alpha_\perp p_\perp\eta^{-1}]\frac{1}{m^2c^3}+O(c^{-4})$$

c) Symmetry of the light-plane Hamiltonian and magnetic moment

It is clearly seen that terms of $\hat{H}_+$ including the order c^{-2} are invariant under the group SU(2) generated by

$$\delta\hat{\psi}_+ = \frac{i\vec{\sigma}\vec{\varepsilon}}{2}\hat{\psi}_+. \tag{22}$$

Because (22) acts on the good components (17) and using equation (13), we identify this group with a realization of the *charge group* $SU(2)_W^{\hat{Q}}$ of equation (9).

In order to compute the *magnetic moment*, we make in (14) the minimal replacement

$$\vec{p} \to \vec{p}-\frac{e}{c}\vec{A}, \qquad p_0 \to p_0-\frac{e}{c}A_0 \tag{23}$$

and choose the potential

$$A^1 = A^2 = 0,\ A^0 = -A^3 \tag{24}$$

which does not change equation (18).

Furthermore, if we put

$$A^3 = H_1 x^2$$

corresponding to a uniform magnetic field in the x^1 direction, the component of the magnetic moment $\mathscr{M}_1$ is

$$\mathscr{M}_1 = ex^2 \tag{25}$$

Thus static dipole magnetic moments are given by the appropriate matrix elements of ex^2 evaluated between good components of light-plane wave functions.

The fact that (24) also implies an applied electric field does not matter when parity conservation is assumed, so that static electric moments vanish.

Now, since rotations about the 3-axis leave the light plane $\hat{x}^0 = x^0+x^3 = \tau$ invariant, and furthermore commute with α_3, one still has for such a rotation

$$\delta\hat{\psi}_+ = -i\omega(x^1p^2-x^2p^1+\tfrac{1}{2}\sigma_3)\hat{\psi}_+ \tag{26}$$

(it will be shown later on that rotations about the two other axis are much more complicated).

Hence, for the ground state, the 3-component of W-spin can be identified with the 3-component of total angular momentum.

The *ground state magnetic moment* is then given by the W_3-spin-flip matrix element of $\mathcal{M}_1 = ex^2$, which is clearly zero.

$$\langle L_3 = 0\uparrow |x^2| L_3 = 0\downarrow\rangle = 0. \tag{27}$$

This is a concrete example for the argument given in paragraph 1e).

Thus one should *not* identify $\hat{\psi}_+$ with the wave-function of a particle of total spin $\frac{1}{2}$.

d) Melosh transformation, symmetry of the Hamiltonian and magnetic moment [19]

The Hamiltonian $\hat{H}_+$ of (19) or (21) is invariant under the group $SU(2)^{\hat{Q}}_{w}$ of (22) only up to order c^{-2}. However, we know that $\hat{H}_+$ must be invariant under a $SU(2)_M$ group up to order c^{-3}, since spin-orbit coupling is of order c^{-4}. We indeed find such a group and discover that it is related to the old one by a unitary transformation U_M, which, as it turns out, is equal to the Melosh transformation up to order c^{-1}.

Thus we call this the *classification group* and assign physical particles to irreducible representations of this group. Particles will therefore belong to reducible representations of the charge group (22). This is what is meant by representation mixing [10].

We also show that the matrix element of the magnetic moment operator (25) between the new states is now equal to the conventional Dirac moment. Later on we shall see that classification states have total angular momentum $j = \frac{1}{2}$.

Instead of transforming the group and the states, it is equivalent to transform the Hamiltonian and the magnetic moment and work with the old group and old states. So we define a new Hamiltonian

$$H_M = U_M\hat{H}_+U_M^{-1}, \tag{28}$$

where U_M is defined by the requirement that H_M be invariant under the old group (22). Recall, (21), that

$$\begin{aligned}\frac{2}{mc^2}\hat{H}_+ = \frac{2}{mc^2}H_0 &+ \frac{1}{m^2c^3}[\eta^{-1}\phi\eta^{-1}, \beta\alpha_\perp p_\perp] \\ &+ \frac{1}{m^2c^3}[V, \beta\alpha_\perp p_\perp\eta^{-1}] \\ &+ O(c^{-4}),\end{aligned}$$

where

$$\frac{2}{mc^2} H_0 = \eta^{-1}+\eta+ \frac{1}{mc^2}\left[\frac{p_\perp^2}{m}\eta^{-1}+V\eta^{-1}+\eta^{-1}V+\phi+\eta^{-1}\phi\eta^{-1}\right].$$

In order to eliminate spin dependent terms, we find

$$U_{\mathrm{M}} = 1+\tfrac{1}{2}\frac{\beta\alpha_\perp p_\perp}{mc}+O(c^{-2}). \tag{29}$$

The Melosh transformation [4] is:

$$U_{\mathrm{M}} = \frac{1+\sqrt{\eta^2}+\dfrac{\beta\alpha_\perp p_\perp}{mc}}{\sqrt{(1+\sqrt{\eta^2})^2+p_\perp^2/m^2c^2}} \tag{30}$$

which agrees with (29) to the given order

With (28) and (29) one gets

$$\frac{1}{mc^2} H_{\mathrm{M}} = \frac{1}{mc^2} H_0+O(c^{-4}). \tag{31}$$

So, indeed, H_{M} is invariant under (22) up to order c^{-3}. Alternatively $\hat{H}_+$ is invariant under the classification group $\mathrm{SU(2)_M}$ acting on the new states ψ_{M}

$$\delta\psi_{\mathrm{M}} = U_{\mathrm{M}}^{-1}\frac{i\vec{\sigma}\vec{\varepsilon}}{2}U_{\mathrm{M}}\psi_{\mathrm{M}},$$

$$\psi_{\mathrm{M}} = U_{\mathrm{M}}^{-1}\psi_+. \tag{32}$$

To calculate the matrix elements of the magnetic moment, we transform the operator $\mathcal{M}_1 = ex^2$ with U_{M}:

$$U_{\mathrm{M}}\mathcal{M}_1U_{\mathrm{M}}^{-1} = ex^2+\tfrac{1}{2}e(-i)\frac{\beta\alpha_2}{mc}+O(c^{-2}).$$

It is convenient to use a representation of the Dirac matrices such that

$$\alpha_3' = \frac{1}{\sqrt{2}}(\alpha_3+\beta)\alpha_3\frac{1}{\sqrt{2}}(\alpha_3+\beta) = \beta = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix},$$

$$\beta'\alpha_2' = \frac{1}{\sqrt{2}}(\alpha_3+\beta)\beta\alpha_2\frac{1}{\sqrt{2}}(\alpha_3+\beta) = \alpha_2\alpha_3 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1\end{pmatrix},$$

such that

$$U_{\mathrm{M}}\mathcal{M}_1U_{\mathrm{M}}^{-1} = ex^2+\tfrac{1}{2}\frac{e\sigma_1}{mc}+O(c^{-2}) \tag{33}$$

In this representation, the ground state magnetic moment is now given by the W_3 spin flip matrix element of (33)

$$\langle L_3 = 0\uparrow | U_{\mathrm{M}} \mathscr{M}_1 U_{\mathrm{M}}^{-1} | L_3 = 0\downarrow\rangle$$

$$= \langle L_3 = 0\uparrow | ex^2 + \tfrac{1}{2}\frac{e\sigma_1}{mc} + \ldots | L_3 = 0\downarrow\rangle = \tfrac{1}{2}\frac{e}{mc} + O(c^{-2}) \tag{34}$$

which is the conventional Dirac moment.

e) Behaviour of $\hat{\psi}_+(\hat{x})$ under rotations

As shown in (26), the good components $\hat{\psi}_+$ on the light-plane behave in the usual way under rotations about the 3-axis. This is not the case for the rotation about the 1 and 2-axis. For example, consider J_1 defined on usual wave functions

$$J_1 \psi(x) = (x^2 p^3 - x^3 p^2 + \tfrac{1}{2}\sigma_1)\, \psi(x). \tag{35}$$

Changing to light-plane variables

$$\hat{x}^i = x^i,\ \hat{x}^0 = x^0 + x^3,\ p^3 = \hat{p}^3 - \hat{p}^0,\ p^0 = \hat{p}^0$$

and projecting

$$\psi_\pm = \frac{1 \pm \alpha_3}{2}\psi$$

one gets now

$$\hat{J}_1 \hat{\psi}_+(\hat{x}) = [x^2(\hat{p}^3 - \hat{p}_0) - x^3 p^2]\, \hat{\psi}_+(\hat{x}) + \tfrac{1}{2}\sigma_1 \hat{\psi}_-(\hat{x}). \tag{36}$$

We use (18) and (21) in the form

$$\frac{\hat{p}_0}{mc} = \tfrac{1}{2}(\eta + \eta^{-1}) + \tfrac{1}{2}\frac{p_\perp^2 \eta^{-1}}{mc} + \ldots,$$

$$\hat{\varphi}_- = \eta^{-1}\left(\beta + \alpha_1 \frac{p^1}{mc} + \alpha_2\alpha_3 \frac{p^2}{mc} + \ldots\right)\hat{\varphi}_+ + \ldots . \tag{37}$$

The non-hermitian terms

$$-\tfrac{1}{2}x^2 \frac{(p^2)^2}{mc}\eta^{-1} \quad \text{and} \quad \sigma_1\alpha_2\alpha_3 \frac{p^2}{mc}\eta^{-1}$$

compensate.

Furthermore, to order c^{-1}, we write

$$(\eta - \eta^{-1})\hat{\varphi}_+(\hat{x}) = \frac{1}{i}\frac{\partial \hat{\varphi}_+}{\partial z}\frac{1}{mc},$$

$$\eta^{-1}\hat{\varphi}_+(\hat{x}) = \hat{\varphi}_+ - \frac{1}{i}\frac{\partial \hat{\varphi}_+}{\partial z}\frac{1}{mc}, \tag{38}$$

so that

$$\hat{J}_1\hat{\psi}_+(\hat{x}) = (x^2p_z - x^3p^2 + \tfrac{1}{2}\sigma_1\beta - \tfrac{1}{2}\sigma_1\beta p_z + \tfrac{1}{2}\sigma_1\alpha_1 p^1)\hat{\psi}_+,$$

$$p_z\hat{\psi}_+(x^1, x^2, z) = \frac{1}{i}\frac{\partial\hat{\psi}_+}{\partial z}. \tag{39}$$

The presence of the two last terms shows that $\hat{\psi}_+$ does not transform irreducibly under rotations.

So let us try again a Melosh transformation such that

$$J_1^{\mathrm{M}} = U_{\mathrm{M}}J_1\hat{U}_{\mathrm{M}}^{-1},$$

$$J_1^{\mathrm{M}} = \hat{J}_1 - \tfrac{1}{2}[\hat{J}_1, \beta\alpha_\perp p_\perp] + \dots . \tag{40}$$

The new terms are

$$\{-\tfrac{1}{2}p_z\beta\alpha_2[x^2, p^2] - \tfrac{1}{4}[\sigma_1\beta, \beta\alpha_1]p^1 + \dots\}\hat{\psi}_+ = \{\tfrac{1}{2}\sigma_1\beta p_z - \tfrac{1}{2}\sigma_1\alpha_1 p^1 + \dots\}\hat{\psi}_+$$

and compensate exactly the extra terms in (39) so that

$$J_1^{\mathrm{M}}\hat{\psi}_+(\hat{x}) = (x^2p_z - x^3p^2 + \tfrac{1}{2}\sigma_1\beta)\hat{\psi}_+(\hat{x}) \tag{41}$$

and the states

$$\psi_{\mathrm{M}} = U_{\mathrm{M}}^{-1}\psi_+$$

transform correctly under rotations.

f) Foldy-Wouthuysen transformation [15]

The Foldy-Wouthuysen transformation takes

$$H_{\mathrm{D}} = \vec{\alpha}\vec{p} + \beta m + \phi$$

into

$$H_{\mathrm{FW}} = e^{iS}H_0e^{-iS},$$

such that

$$[H_{\mathrm{FW}}, \beta] = 0. \tag{42}$$

In the free case:

$$e^{iS} = e^{iS_0} = \frac{m + \sqrt{m^2 + \vec{p}^2} + \beta\vec{\alpha}\vec{p}}{\sqrt{(m + \sqrt{m^2 + \vec{p}^2})^2 + \vec{p}^2}}$$

$$= m + \tfrac{1}{2}\beta\vec{\alpha}\vec{p} - \tfrac{1}{8}\vec{p}^2 + O(c^{-3}), \tag{43}$$

$$e^{iS_0}(\beta m + \vec{\alpha}\vec{p})e^{-iS_0} = \beta\sqrt{m^2 + \vec{p}^2}. \tag{44}$$

In the interacting case, one has to add a piece

$$iS_1 = \tfrac{1}{2}\beta[\tfrac{1}{2}\beta\vec{\alpha}\vec{p}, \phi]$$

such that

$$e^{iS_1}e^{iS_0}H_D e^{-iS_0}e^{-iS_1} = \beta\sqrt{m^2+\vec{p}^2}+\phi-\tfrac{1}{8}\{\vec{p}^2,\phi\}-\tfrac{1}{4}\beta\vec{\alpha}\vec{p}\phi\vec{\alpha}\vec{p}+O(c^{-5}). \tag{45}$$

The spin dependent term is of order c^{-4} which of course contains the well known spin--orbit coupling. Up to order c^{-3}, H_{FW} commutes with all matrices which commute with β. So we have a $U(2)\times U(2)$ approximate symmetry with generators

$$1,\ \beta,\ \vec{\sigma},\ \vec{\sigma}\beta. \tag{46}$$

The Melosh group $SU(2)_M$ just corresponds to the subset

$$\beta\sigma_1,\ \beta\sigma_2,\ \sigma_3 \tag{47}$$

which also commutes with α_3.

To see this one goes through the steps

$$\phi = e^{iS}\psi,$$

$$\phi_+ = \tfrac{1}{2}(1+\alpha_3)\phi = \tfrac{1}{2}(1+\alpha_3)e^{iS}\psi = A\psi_+ + B\psi_-,$$

where A and B are definite, but complicated expressions. Going over to light-plane variables

$$\hat{\phi}_+(\hat{x}) = \hat{A}\hat{\psi}_+(\hat{x})+\hat{B}\hat{\psi}_-(\hat{x})$$

and using the Dirac equation, the calculation shows that

$$\hat{\phi}_+ = C\hat{\psi}_+,$$

where C is up to a normalization factor equal to the Melosh transformation, so that

$$\hat{\phi}_+ \sim \psi_M. \tag{48}$$

3. *Dirac equations with exact higher symmetry* [16]

Working out higher order terms in (45), one finds that a Coulomb-like potential ϕ appears with opposite sign compared to a Lorentz scalar potential V. This makes one suspicious that a higher symmetry is at work. Since potentials like that are used in quark models [17], it may be worth while investigating the matter further.

So let us consider the special case

$$V = \phi = \tfrac{1}{2}W. \tag{49}$$

The Dirac equation now reads

$$E\psi = H\psi = \left(\vec{\alpha}\vec{p}+\beta m+\frac{1\pm\beta}{2}W\right)\psi. \tag{50}$$

One gets for the projections

$$\psi_\pm = P_\pm\psi,\quad P_\pm = \frac{1\pm\beta}{2},$$

$$(E-m-W)\psi_+ = \vec{\alpha}\vec{p}\psi_- \tag{51}$$

$$(E+m)\psi_- = \vec{\alpha}\vec{p}\psi_+.$$

The potential has disappeared from the last equation. Eliminating ψ_- yields

$$(E-m-W)\,\psi_+ = (E+m)^{-1}\vec{p}^2\,\psi_+. \tag{52}$$

The Dirac matrices no longer appear, so that we have the infinitesimal symmetry [18]

$$\delta\psi_+ = \tfrac{1}{2}\, i\vec{\sigma}\vec{\varepsilon}\psi_+, \tag{53}$$

$$2i\vec{\sigma} = \hat{\alpha}\times\hat{\alpha}. \tag{54}$$

From (51) then

$$\delta\psi_- = (E+m)^{-1}\vec{\alpha}\vec{p}\delta\psi_+ = \vec{\alpha}\vec{p}\,\frac{\vec{\sigma}\vec{\varepsilon}}{2i}\,\vec{\alpha}\vec{p}(\vec{p}^2)^{-1}\psi_-. \tag{55}$$

Combining (53) and (55)

$$\delta\psi = \frac{\vec{\varepsilon}\vec{S}}{2i}\,\psi,$$

$$2\vec{S} = \frac{\vec{\alpha}\times\vec{\alpha}}{i}\,\frac{1+\beta}{2} + \vec{\alpha}\vec{p}\,\frac{\vec{\alpha}\times\vec{\alpha}}{i\vec{p}^2}\,\vec{\sigma}\vec{p}\,\frac{1-\beta}{2}. \tag{56}$$

One checks that

$$[\hat{S}, H] = 0,\ [S_x, S_y] = 2iS_z, \text{ etc.}$$

So we indeed have an *exact* SU(2) symmetry group of the Hamiltonian.

This can be generalized to Dirac hamiltonians of the form

$$H = \vec{\alpha}\vec{p}+\beta m+\tfrac{1}{2}W(1+\vec{\alpha}\vec{e}+\beta e_0), \tag{57}$$

where $\vec{e}$ and e_0 are constants such that

$$e_0^2+\vec{e}^2 = 1. \tag{58}$$

Let us introduce 4-vectors

$$\boldsymbol{\alpha} = (\beta, \vec{\alpha}), \qquad \boldsymbol{p} = (m, \vec{p}), \qquad \boldsymbol{e} = (e_0, \vec{p}) \tag{59}$$

in an Euclidean vector space with inner products

$$\boldsymbol{\alpha p} = \beta m+\vec{\alpha}\vec{p}$$

etc. Then

$$H = \boldsymbol{\alpha p}+\tfrac{1}{2}W(1+\boldsymbol{\alpha e}) \tag{60}$$

$$= \vec{\alpha}'p'+\beta' m'+\tfrac{1}{2}W(1+\beta'), \tag{61}$$

where the primed quantities are obtained by a rotation which brings the vector $e = (e_0 = 1,\ \vec{e} = 0)$ into general position. Hence, we have symmetry operators

$$2\vec{S}' = \frac{\vec{\alpha}'\times\vec{\alpha}'}{i}\,\frac{1+\beta'}{2} + \vec{\alpha}'\vec{p}'\,\frac{\vec{\alpha}'\times\vec{\alpha}'}{i\vec{p}'^2}\vec{\alpha}'\vec{p}'\,\frac{1-\beta'}{2}. \tag{62}$$

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[19] The arguments presented here for external potentials have been extended to quasi-potentials. See: J. S. Bell, H. Ruegg, *Positronium on the Null-Plane and Melosh Transformation*, CERN Preprint TH 2095 (1975).