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NUCLEAR FORCES AND THE QUANTISATION OF M.I.T. BAGS

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We discuss the origin of inter-hadronic forces in the M.I.T. bag model and are led to consider a method of quantisation in which the boundary of the bag is a quantum variable.

1. Introduction

In recent years the quark model has had an impressive record of successes in explaining the interactions of hadrons, so the hypothesis that hadrons are in some way "made out of quarks" has gained much support. Unfortunately some of the required quark properties are contradictory; for example, deep inelastic scaling suggests that quarks are light and essentially non-interacting, whereas the non-appearance of free quarks can most readily be explained if they are very massive and have very strong interactions.

The M.I.T. bag model [1] provides a convenient relativistic way of confining approximately free and massless quarks inside hadrons in such a way that only the observed hadronic states can exist. The confining mechanism is defined by a "pressure" parameter B which is fitted to the mass of the baryonic state (the nucleon) and which then yields approximately the correct size. In this first approximation the quarks do not interact so it is pertinent to enquire about the origin of nuclear forces, i.e. the forces between hadrons, in the model. Of course these forces could arise from the explicit quark-quark interactions which are ignored in first approximation (but which certainly have to be present) and indeed it is under this condition that we obtain the "additive quark model" rules for high energy hadronic interactions [2]. We shall show however that, even without the inter-quark forces, there is a large hadronic interaction arising from the binding effect in the original Lagrangian. Indeed this term in the classical model appears to be too large and we are therefore led to revise the quantisation procedure of Ref. [1], so that the "length" becomes a quantum variable and has a kinetic energy associated with it.

In so far as this model gives a significant contribution to the hadronic interaction it appears at first sight that the additive quark model rules are purely accidental. However it should be noted that these rules in practice usually involve t-channel Pomeron exchange

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which, being a diffractive effect, has no place in classical scattering. It is unlikely, therefore, that our calculations have anything to do with the Pomeron, which could well arise primarily from direct quark-quark collisions.

In this paper we are interested mainly in general ideas rather than in making specific predictions so we shall, except for a few remarks, work in one space dimension and consider bags containing charged scalar fields. In the next section we describe the classical one-dimensional M.I.T. bag ground state (the "nucleon") and show that the model gives an unrealistic "deuteron". We suggest that this is because we have done a classical rather than a quantum calculation.

In Section 3 we analyse the scattering of two one-dimensional bags [3] and show how we can use this to extract information about the interaction potential between two bags [4]. We use this information in a "Schrödinger equation for the deuteron" which we discuss in Section 4. Some corrections to this, which might yield a reasonable value for the deuteron bound state are discussed in Section 5.

The Schrödinger equation used in Section 4 was obtained by a plausibility argument (classical scattering $\Rightarrow$ potential $\Rightarrow$ Schrödinger equation) rather than by direct quantisation of the original Lagrangian. In Section 6 we show how it is possible to quantise the theory in such a way that the length appears as a quantum variable. We obtain a Schrödinger equation in which the "field" and "length" are coupled. We discuss approximate solutions to this, in simplified situations, in Section 7.

Finally we draw attention to a problem which arises through the infinite zero-point-energy which inevitably arises when all modes of the field are included in the calculation. We suggest that it is not adequate to simply subtract an overall (infinite) constant from the final answer, but that it is necessary to formulate the problem in such a way that it does not appear — the crucial point being that one normally solves problems in nuclear physics for example with *finite mass* nucleons.

2. *The classical one-dimensional bag*

The one-dimensional bag, with complex scalar fields ϕ_α, is described by the Lagrangian [1]

$$L = \int_{\text{Bag}\,(s)} dx \left[\sum_\alpha \left| \frac{\partial \phi_\alpha}{\partial t} \right|^2 - \sum_\alpha \left| \frac{\partial \phi_\alpha}{\partial x} \right|^2 - B \right]. \tag{2.1}$$

Here α labels, for example, the type of quark. We use units in which $\hbar = c = 1$. This leads to the free field equation

$$\frac{\partial^2 \phi_\alpha}{\partial t^2} - \frac{\partial^2 \phi_\alpha}{\partial x^2} = 0 \tag{2.2}$$

inside the bag, and to the boundary conditions

$$\phi_\alpha = 0 \tag{2.3}$$

and

$$\sum_{\alpha} \left| \frac{\partial \phi_\alpha}{\partial t} \right|^2 - \sum_{\alpha} \left| \frac{\partial \phi_\alpha}{\partial x} \right|^2 = -B \tag{2.4}$$

at the ends. Note that we are using the Dirichlet boundary conditions rather than those that follow directly from (2.1). This point is discussed in References [1] and [3].

The charge normalisation conditions for a single particle to be associated with the field ϕ_α is

$$i \int_{\text{Bag}} \left\{ \phi_\alpha^* \frac{\partial \phi_\alpha}{\partial t} - \frac{\partial \phi_\alpha^*}{\partial t} \phi_\alpha \right\} dx = 1. \tag{2.5}$$

We consider a static bag with ends at $x = \pm a/2$. The ground state solution of (2.2), satisfying the boundary condition (2.3) and normalised to unity according to (2.5) is

$$\phi_\alpha = (1/\sqrt{\pi}) \exp(-i\pi t/a) \cos(\pi x/a). \tag{2.6}$$

The non-linear boundary condition (2.4) then gives

$$a^2 = \frac{N\pi}{B}. \tag{2.7}$$

where N is the number of fields in the bag, i.e. α runs from 1 to N. Thus, when we have chosen B, the length and the fields are determined.

The energy, which is the mass since the bag is at rest, is given by

$$M = \int_{\text{Bag}} dx \left[\sum_{\alpha} \left| \frac{\partial \phi_\alpha}{\partial t} \right|^2 + \sum_{\alpha} \left| \frac{\partial \phi_\alpha}{\partial x} \right|^2 + B \right] \tag{2.8}$$

$$= 2BaN \tag{2.9}$$

$$= 2(\pi B N)^{1/2} \tag{2.10}$$

on using (2.7).

To form a "proton" we would put three different quarks in a bag, i.e. $N = 3$. Then

$$M_{\text{p}} = 2(3\pi B)^{\frac{1}{2}}, \tag{2.11}$$

and we can choose B to obtain the correct result. However a bag can also contain 6 quark fields (still in a colour singlet as required — see Ref. [1]) and, by choosing these suitably we would have the quantum numbers of the deuteron, with mass

$$M_{\text{d}} = 2(6\pi B)^{\frac{1}{2}}. \tag{2.12}$$

The resulting "binding energy" is

$$\mathscr{E} = (2 - 2^{1/2}) M_{\text{p}} \simeq 0.59\, M_{\text{p}}. \tag{2.13}$$

This binding energy is much too large and the situation is not improved much if we do the corresponding classical calculation in 3 dimensions using spinor quark fields [5].

The resulting binding energy is then

$$\mathscr{E} = (2-2^{3/4})M_p \simeq 0.32\, M_p. \tag{2.14}$$

It is clear that the state we have obtained is very unlike the deuteron. Its binding energy is too large and its radius ($\sqrt[4]{2}$ times the proton radius) is too small. This, however, is not surprising since we would not expect to obtain the deuteron correctly in a classical calculation. The classical deuteron would have the neutron and proton stationary at the deepest part of the interaction potential. This would be at the centre, i.e. zero separation, unless there is a hard core. Thus, in our opinion, the reason why this calculation gives the wrong result is that we have ignored the kinetic energy of relative motion which is essential in a proper quantum mechanical treatment. Note that in a standard nuclear physics calculation of the deuteron the hard core plays very little role — it is *quantum mechanics* not *repulsive forces* that give the deuteron a large radius.

Thus we wish to turn from a classical treatment to a quantum mechanical one. Clearly to make this simple we would like to separate the problem of the internal dynamics of the two bags (the neutron and the proton) from that of their relative motion. The standard way of doing this is to find the interaction, say from scattering, and then insert this into a Schrödinger equation. To this end we study scattering of two bags [3].

3. The scattering of two one-dimensional bags

The general problem of the scattering of bags is the solution of the wave equation (2.2) subject to (2.3) and (2.4), with given initial conditions. The initial conditions can be obtained by taking the static bag solutions, and performing Lorentz transformations on them so that we have two bags which, at $t = 0$, join. From then on we have one bag until some later time when it again separates into two. It turns out that the solution to this classical problem is trivial in one space dimension but is extremely difficult (or impossible?) in more.

To see why the problem is easy in one space dimension we define, following Ref. [1], light-cone variables

$$z^{\pm} = \frac{1}{\sqrt{2}}(t \pm x) \tag{3.1}$$

in terms of which the wave-equation becomes

$$\frac{\partial^2 \phi_\alpha}{\partial z^+ \partial z^-} = 0 \tag{3.2}$$

with general solution

$$\phi_\alpha = \chi_\alpha^+(z^+) + \chi_\alpha^-(z^-) \tag{3.3}$$

for any functions χ^+, χ^-. The linear boundary condition becomes

$$\chi_\alpha^+(z_i^+(z^-)) + \chi_\alpha^-(z^-) = 0, \tag{3.4}$$

where $z_i(z^-)$, $i = 1$ and 2, are the end points. If we differentiate this w.r.t. z^- we get

$$\frac{d\chi_\alpha^+}{dz^+}\frac{dz_i^+}{dz^-} + \frac{d\chi_\alpha^-}{dz^-} = 0. \tag{3.5}$$

The non-linear b.c. is

$$2\,\mathrm{Re}\sum_\alpha \frac{d\chi_\alpha^+}{dz^+}\frac{d\chi_\alpha^-}{dz^-} = -2B. \tag{3.6}$$

Eliminating $\frac{dx_\alpha^+}{dz^+}$ from (3.5) and (3.6) we obtain

$$\frac{dz_i^+}{dz^-} = \frac{2\sum_\alpha \left|\frac{d\chi_\alpha^-}{dz^-}\right|^2}{B}. \tag{3.8}$$

Since the r.h.s. of this equation is independent of i, it shows that $\frac{dz_1^+}{dz^-} = \frac{dz_2^+}{dz^-}$, so that the "length" measured at fixed z^- — and hence by the same argument also at fixed z^+ — is constant. Note that this argument is unaffected by the presence of several fields. We shall see that these conditions determine the scattering.

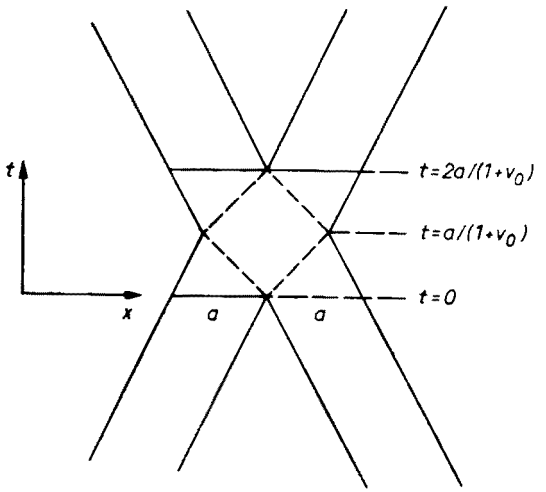

Fig. 1. Showing the collision of two one-dimensional bags

For this purpose we refer to Fig. 1 in which we see two (identical) bags moving with velocities v_0 and $^-v_0$ respectively. At $t = 0$ the two bags collide. The outer boundaries cannot know about this collision till a light signal from the point of collision can reach them, i.e. at $t = \frac{a}{1+v_0}$. So, up till this time they continue to move with velocities v_0

and $-v_0$ as before. What happens after this time is then determined by the constant length condition, for example, the right-hand boundary of the right-hand bag has to move with the same velocity as that of the left-hand boundary of the left-hand bag prior to the collision, i.e. v_0. Thus the situation until $t = \dfrac{2a}{1+v_0}$ is as shown in Fig. 1. At this time the bags are free to move apart again and we assume that this is what happens. (They do not *have* to move apart [3] but could remain together and "breathe". The Lagrangian does not uniquely determine the motion at points such as $t = \dfrac{2a}{1+v_0}$.)

Although this is not clear from Fig. 1 it is easy to see, by considering bags with different fields or bags of different length, that the two bags pass through each other (rather than bounce back). The diagram then clearly shows that they travel on average faster when they are overlapping than when they are separated — thus we have an attractive force. At least the model will give the correct sign to the nuclear force!

4. Quantised model of interacting bags

We now take the "potential" obtained from our solution of the scattering and use it in a Schrödinger equation [4]. Let x be the separation of the centres of the two bags, then this equation is

$$-\frac{h^2}{2(M_p/2)}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x). \tag{4.1}$$

We relate $V(x)$ to the relative velocity $2v$ by using conservation of energy:

$$\tfrac{1}{2}\left(\frac{M_p}{2}\right)(2v)^2 = \tfrac{1}{2}\left(\frac{M_p}{2}\right)(2v_0)^2 - V(x). \tag{4.2}$$

Now the time taken by the two bags to pass through each other $\left(\dfrac{2a}{1+v_0}\right)$ is related to $2v$ by

$$\frac{2a}{1+v_0} = \int_{-a}^{+a}\frac{dx}{2v} \tag{4.3}$$

$$= \int_{-a}^{+a}\frac{dx}{[4v_0^2+(4V/M_p)]^{1/2}} \tag{4.4}$$

where, to obtain the last equation, we have substituted for v from (4.2). Thus our scattering solution gives us an "average" of the potential. It is clear from (4.4) that the potential is

velocity dependent so, since we are interested in low velocity we put $v_0 = 0$, when (4.4) becomes

$$\int_0^a \frac{dx}{\sqrt{V}} = \frac{2a}{\sqrt{M_p}}. \tag{4.5}$$

To proceed we must know the shape of the potential. In principle, if we could attach an unambiguous meaning to the position of the individual bags during the time when they are in contact, this could be determined from the scattering solution. However this is not possible so we must guess the shape. If we take a square well, with range a, then we have a depth of $(0.25\, M_p)$, which is about 6 times too large to give a zero binding energy deuteron.

To make a more realistic guess we recall that the classical solution appeared to correspond to putting the particles at the centre of the potential, in which case $V(x = 0)$ would be equal to the binding energy, i.e.

$$V(0) = (2-2^{\frac{1}{2}})M_p. \tag{4.6}$$

Then, parametrising the shape by

$$V(x) = V(0)\left[1-\frac{x}{a}\right]^{2p}, \quad 0 < x < a \tag{4.7}$$

we find from (4.5)

$$p \simeq 1/9. \tag{4.8}$$

The resulting potential is now too large by an overall factor of about 5; it gives a binding energy $\sim$ 100 MeV.

It is important to note that the effect we are considering is not small. The classical binding energy (2.13) was $0.59\, M_p$, whereas the binding energy is now $\sim 0.1\, M_p$. Thus we claim that it is essential to do "quantum mechanics" in order to understand nuclear forces in bag models.

5. Towards a more realistic deuteron

(*i*) The essential step to make the previous calculation realistic is to solve the classical scattering problem in 3-space dimension. This is a problem in partial differential equations. We know the ground state solutions for static spherical bags. By a Lorentz transformation we can arrange for these to move with a given velocity and can therefore calculate the fields inside two bags at the moment when they collide for any given impact parameter. This gives us the initial condition and we then solve the wave-equation (actually the Dirac equation if we use spinor fields) with fields that are zero on a given boundary. This boundary moves with time and the problem is to find the boundary as a function of time such that the non-linear boundary condition remains true.

Unfortunately we have not managed to do this calculation. If we had we would know the differential cross-section for the scattering of the two bags and could therefore compute

the effective interaction potential as a function of separation. (There would, incidentally be other aspects of calculation which would be interesting. Do the bags always come out in their ground state or do we have "inelastic" scattering? Do we even perhaps have a $2 \to 3$ process?).

Without such a calculation we make one crude (and perhaps meaningless) estimate of the effect of going from 1 to 3 dimensions. By comparing (2.13) and (2.14) we see that the effect is to reduce the value of $V(0)$ by a factor $\frac{2-2^{3/4}}{2-2^{1/2}} \approx 0.54$. We assume that a similar factor corrects the potential at all separations. Solution of the Schrödinger equation (in 3 dimensions) then leads to a binding energy of about 70 MeV or to a potential which is too deep by about a factor 2.

(*ii*) It is clearly foolish to expect a realistic evaluation of the mass of a deuteron in a model in which the N and the Δ for example are degenerate. There is no way in a bag model to break this degeneracy without involving quark-quark interactions. Such interactions will also contribute to the deuteron binding energy. To see the possible effects of this we consider an interaction between two quarks (1 and 2) given by

$$V = A\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + B\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + C\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2, \tag{5.1}$$

where A, B, C are functions of the separation of the quarks. The contribution of this to the Δ, N mass difference is

$$\Delta - \mathrm{N} = 6\langle A \rangle + 6\langle B \rangle - 12\langle C \rangle,$$

where $\langle A \rangle$, etc., are averages over the appropriate quark separations in nucleons (we assume the Δ and N have the same size). Correspondingly the contribution to the deuteron binding energy is

$$\delta\mathscr{E} = 3\langle A \rangle_{\mathrm{d}} - \langle B \rangle_{\mathrm{d}} + \tfrac{2}{3}\tfrac{5}{} \langle C \rangle_{\mathrm{d}},$$

where $\langle A \rangle_{\mathrm{d}}$, etc., are suitable averages over quark separation in the deuteron. There is nothing more we can say without further assumptions about the relative magnitudes of A, B and C and about the force range. For example, if we assume that the three terms contribute equally to the $\Delta - \mathrm{N}$ difference and if we ignore the difference in size of the Nucleon and the Deuteron then we obtain a contribution of about -35 MeV to the deuteron binding energy. Other assumptions give very different values but unless the A term dominates we are likely to get a contribution which decreases the deuteron binding energy. Although the effect of the deuteron size will probably make $\langle A \rangle_{\mathrm{d}} < \langle A \rangle$, etc. it is, as we noted above, unrealistic to expect to calculate the deuteron energy until we have a good model for the $q-q$ interaction.

6. *Quantisation of one-dimensional bags*

Quite apart from the "technical" problems of taking into account the 3 dimensions of space and the inter-quark interactions in order to obtain a realistic description of nuclear forces, our procedure so far is inadequate at a more fundamental level. We have solved

a classical scattering problem and used the result in a Schrödinger equation to find the bound states. This is wrong because the scattering problem and the original free states should also be treated quantum mechanically, and there will be coupling between the separation and the fields during the time of overlap. What we should do is to obtain the Schrödinger equation for the energy eigenstates directly from the Lagrangian. Hopefully, in some approximation, we can reproduce results like those of Section 4 for the one-dimensional "deuteron". We shall see that our method leads to a very complicated equation and at this stage we are some way off an understanding of the deuteron problem.

In the quantisation procedure given in Ref. [1] the deuteron binding energy would not differ greatly from the classical value found in Section 2. Certainly there is no way whereby a term corresponding to a kinetic energy associated with the relative position could arise. This is basically because the length of a bag is not, in this treatment, a quantum variable.

We propose to modify the Lagrangian to remedy this "defect". To do this we impose the condition that the field $\phi(x, t)$ is zero at the end points $x = \pm a/2$. We do this before we quantise the theory. Thus, we put (for a bag containing a single scalar field ϕ):

$$\phi = \tfrac{1}{2} \sum_{n=1}^{\infty} [X_n(t) + iY_n(t)] [\exp(in\pi x/a) - (-1)^n \exp(-in\pi x/a)] \tag{6.1}$$

which is certainly zero at $x = \pm a/2$. We allow the length (a) to be a function of t. On inserting this into the Lagrangian (2.1) and doing the x-integration we find

$$L = \sum_{n=1}^{\infty} \left\{ \frac{a}{2} [\dot{X}_n^2 + \dot{Y}_n^2] + \frac{\dot{a}^2}{4a} \left(1 + \frac{n^2\pi^2}{6}\right)(X_n^2 + Y_n^2) \right.$$

$$\left. + \frac{\dot{a}}{2} [\dot{X}_n X_n + \dot{Y}_n Y_n] - \frac{\pi^2 n^2}{2a} (X_n^2 + Y_n^2) \right\} - Ba. \tag{6.2}$$

Note that an $\dot{a}^2$ term now appears in the L so that the length is associated with a "kinetic energy". Thus we can regard a, together with X_n and Y_n as quantum variables.

Using (6.1) we can express the normalisation condition (2.5) as

$$1 = a \sum_n (\dot{X}_n Y_n - \dot{Y}_n X_n). \tag{6.3}$$

The operator on the right is a "first class" constraint (it commutes with the Hamiltonian), so we impose (6.3) not as a relation between operators but as an expectation value, i.e. we require acceptable eigenstates to satisfy:

$$\langle \Psi | \Psi \rangle = \langle \Psi | a \sum_n (\dot{X}_n Y_n - X_n \dot{Y}_n) | \Psi \rangle. \tag{6.4}$$

To do this we introduce the Lagrange mutiplier λ by adding the term $\lambda[1 - a\sum_n (\dot{X}_n Y_n - X_n \dot{Y}_n)]$ to the Lagrangian, and then determining λ by (6.4). Later, we shall see that in our solutions we can put $\lambda = 0$; we do not understand why.

To find the stationary states we use the Feynman Path Integral method [6]. The details are rather messy and lead to a Schrödinger equation which we can write in the form

$$E\Psi = -\frac{1}{2\sum_n\left(1+\frac{n^2\pi^2}{3}\right)(X_n^2+Y_n^2)}\frac{\partial^2\Psi}{\partial y^2} - \frac{\sum_n\left(1+\frac{n^2\pi^2}{6}\right)(X_n^2+Y_n^2)}{y^2\sum_n\left(1+\frac{n^2\pi^2}{3}\right)(X_n^2+Y_n^2)}$$

$$\times\sum_n\left\{\frac{\partial^2\Psi}{\partial X_n^2}+\frac{\partial^2\Psi}{\partial Y_n^2}\right\} - \frac{\sum_n\left(3+\frac{n^2\pi^2}{3}\right)\left(X_n\frac{\partial\Psi}{\partial X_n}+Y_n\frac{\partial\Psi}{\partial Y_n}\right)}{2\sum_n\left(1+\frac{n^2\pi^2}{3}\right)(X_n^2+Y_n^2)y^2}$$

$$+\frac{\sum_n\left(X_n\frac{\partial^2\Psi}{\partial y\partial X_n}+Y_n\frac{\partial^2\Psi}{\partial y\partial Y_n}\right)}{\sum_n\left(1+\frac{n^2\pi^2}{3}\right)(X_n^2+Y_n^2)y} + \frac{\sum_n\left(5+\frac{n^2\pi^2}{3}\right)(X_n^2+Y_n^2)}{2\left|\sum_n\left(1+\frac{n^2\pi^2}{3}\right)(X_n^2+Y_n^2)\right|^2y^2}\Psi$$

$$+\frac{\sum_n\left(5+\frac{n^2\pi^2}{3}\right)}{3\sum_n\left(1+\frac{n^2\pi^2}{3}\right)(X_n^2+Y_n^2)y^2}\Psi + \frac{\pi^2}{2y^2}\sum_n n^2(X_n^2+Y_n^2)\Psi$$

$$+\frac{\lambda^2y^2}{2}\sum_n\{X_n^2+Y_n^2\}\Psi + By^2\Psi - \lambda\Psi, \tag{6.5}$$

where we have put

$$y = a^2. \tag{6.6}$$

This equation shows the expected coupling between the length of the bag and the amplitudes of the various nodes. Clearly it will not be a trivial problem to find solutions! We therefore turn in the next section to some particular "approximations" which illustrate what will be involved in finding a complete solution.

7. *Approximate solutions to the quantum problem*

(*i*) We first consider a solution in which a is kept fixed. Although this is contrary to our aim it helps to understand the nature of the solutions to (6.5) (and some of the problems we will encounter in finding them). We can fix a by ignoring the $\dot{a}$ terms in the

Lagrangian. Since we are interested in the ground state we also restrict ourselves to solutions with no "nodes", i.e. keep only $n = 1$. The Schrödinger equation becomes

$$(E+\lambda-Ba)\Psi(r,\theta) = -\frac{1}{2a}\left(\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}\right)+\frac{a}{2}\left(\frac{\pi^2}{a^2}+\lambda^2\right)r^2\Psi$$
$$+i\lambda\frac{\partial\Psi}{\partial\theta}-\frac{1}{2ar^2}\frac{\partial^2\Psi}{\partial\theta^2}, \tag{7.1}$$

where we have used polar coordinates (r, θ) defined by

$$X_1 = r\cos\theta, \qquad Y_1 = r\sin\theta. \tag{7.2}$$

To evaluate λ from (6.4) we need operators representing $\dot{X}_1$ and $\dot{Y}_1$. Defining, for example, the momentum conjugate to X,

$$\pi_X = -i\frac{\partial}{\partial X} \tag{7.3}$$

we have

$$\pi_X = \frac{\partial L}{\partial\dot{X}_1} = a\dot{X}_1 - a\lambda Y \tag{7.4}$$

so that

$$\dot{X}_1 = \lambda Y_1 - \frac{i}{a}\frac{\partial}{\partial X_1}$$

and

$$\dot{Y}_1 = -\lambda X_1 - \frac{i}{a}\frac{\partial}{\partial Y_1}. \tag{7.5}$$

Then (6.4) becomes

$$\langle\Psi|\Psi\rangle = \lambda\langle\Psi|ar^2|\Psi\rangle + i\left\langle\Psi\left|\frac{\partial}{\partial\theta}\right|\Psi\right\rangle. \tag{7.6}$$

It is immediately obvious that we can satisfy (7.6) with $\lambda = 0$ by taking

$$\Psi(r,\theta) = e^{-i\theta}\Psi(r). \tag{7.7}$$

Putting this into (7.1) gives

$$(E-Ba)\psi(r) = -\frac{1}{2a}\left(\frac{\partial^2\Psi}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi}{\partial r}\right)+\frac{\pi^2}{2a}r^2\Psi+\frac{1}{2ar^2}\Psi. \tag{7.8}$$

This is a harmonic oscillator in two dimensions and the lowest solution is

$$\Psi(r) = r\exp(-\pi r^2/2) \tag{7.9}$$

with energy

$$E_1 = Ba + \frac{2\pi}{a} = 3\sqrt{\pi B} \tag{7.10}$$

(where we have put $a = \sqrt{\frac{\pi}{B}}$, its classical value).

The lowest solution of (7.1) with $\lambda = 0$ is not given by (7.7) and (7.9) but is the "empty bag", i.e.

$$\Psi(r, \theta) = \exp(-\pi r^2/2) \tag{7.11}$$

with energy

$$E_0 = Ba + \pi/a = 2\sqrt{\pi B}. \tag{7.12}$$

This is the zero-point energy of our system. If we subtract this energy from the single particle state and call this the mass then it is one-half the classical single particle energy $2\sqrt{\pi B}$.

It is possible to find other solutions of (7.1) and (7.6) with $\lambda \neq 0$. In particular one can find a solution containing a single particle with a Ψ independent of θ. Its energy is $(1+2\sqrt{2})(\pi B)^{\frac{1}{2}}$ and is higher than that with $\lambda = 0$. We have no understanding of the meaning of this ambiguity so from here on we put $\lambda = 0$.

The presence of the zero point energy will prove an embarrassment particularly when we remember that we have so far included only the lowest excitation of the field ϕ, i.e. with no nodes. In fact there should be a zero point energy for each excitation state of ϕ, so the total zero-point energy will be infinite, i.e. the empty bag has an infinite mass. Just how we "subtract" this may well be important in the following discussion.

(*ii*) Here we consider the fully quantised problem for a single field without nodes. The Schrödinger equation (6.5) simplifies to

$$\begin{aligned} E\Psi = &- \frac{1}{1+\pi^2/3}\left[\frac{1}{2r^2}\frac{\partial^2\Psi}{\partial y^2} + \frac{1}{2ry^2}\frac{\partial\Psi}{\partial r} - \frac{1}{ry}\frac{\partial^2\Psi}{\partial r\partial y}\right] \\ &- \frac{1+\pi^2/6}{1+\pi^2/3}\left[\frac{1}{y^2}\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{ry^2}\frac{\partial\Psi}{\partial r} + \frac{1}{r^2y^2}\frac{\partial^2\Psi}{\partial\theta^2}\right] \\ &+ \frac{\pi^2 r^2}{2y^2}\Psi + \frac{(5+\pi^2/3)(5+2\pi^2/3)}{6(1+\pi^2/3)^2 r^2 y^2}\Psi + By^2\Psi. \end{aligned} \tag{7.13}$$

We have not been able to find exact solutions to this but have found approximate eigenstates by a variational method. For example, if we use

$$\Psi(r, \theta, y) = ry^2 \exp(-\alpha r^2 - \gamma y^2)\exp(-ip\theta), \tag{7.14}$$

where $p = 0$ for the empty bag, and $p = 1$ for the bag containing a single particle, and α and γ are variation parameters, we obtain

$$E_0 = 4.14\sqrt{\pi B} \tag{7.15}$$

with $\alpha = 1.21$; $\gamma = 0.34\sqrt{B}$ for the empty bag and

$$E_1 = 4.48\sqrt{\pi B} \tag{7.16}$$

with $\alpha = 1.04$; $\gamma = 0.32\sqrt{B}$ for the single particle bag. Note that these energies are higher than the corresponding energies when we do not include the kinetic energy of the boundary. Naturally they could be lowered by improving the trial wavefunction, but some efforts in this direction suggest that the effect of this is unlikely to be large. The difference $(E_1 - E_0)$ is actually smaller in this case than in the case of the classical length — although this result, being a small difference between two numbers, could be altered by improved trial wavefunctions.

8. *Discussion*

It is clear that we cannot make any further progress until certain problems are understood. Chief among these is the question of the zero-point energy of the various field modes. In (6.5) we see that the "mass" associated with the motion of the boundary is $\sum(r_\alpha^n)^2$ which will be infinite if all the modes are allowed. One interpretation of this is to accept the result, in which case the quantisation of the boundary would be irrelevant. However we would then be back to the classical "deuteron", which is not satisfactory. One can indeeed see what has happened here. In the model with all modes included the particles have infinite mass (the sum of all the zero-point energies). If we solved the quantum mechanical deuteron with nucleons of infinite mass then the classical solution would clearly be the correct one. (If a variable x is associated with infinite mass then the large conjugate momenta which arise through the uncertainty principle when its value is fixed at some classical number do not give any contribution to the energy.) Thus it would be wrong to calculate the deuteron with infinite mass nucleons and *then* to subtract the (infinite) sum of the zero-point energies. What is needed is some way of rewriting the Lagrangian so that the zero-point energies are automatically removed. We have not yet found a satisfactory way of doing this.

Another source of ambiguity appears to come from the method of quantisation. If instead of using the Path Integral method we use canonical quantisatión rules, starting from the Lagrangian (6.2), we seem to obtain results different from those expressed in (6.5) and (7.13). We do not understand the origin of this difference.

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