COLLECTIVE PHENOMENA IN QUANTUM FIELD THEORY*

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(Presented at the XV Cracow School of Theoretical Physics, Zakopane, June 6–19, 1975 and at the Conference on Extended Systems in Field Theory, Paris, June 16–21, 1975)

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1. Introduction

One of the first tasks for the theoretical physicist who confronts a model field theory, which is offered with the hope that it will provide explanations for natural phenomena, is the elucidation of physically interesting consequences of the model: particle spectrum, scattering amplitudes and so forth. Since it has been impossible to find exact solutions for realistic theories, approximation methods must be used, and the last quarter century has seen the development of a perturbative series whose starting point is the non-interacting

* This work was supported in part through funds provided by ERDA under contract AT (11-1-3069.

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free-field theory with readily indetifiable particle spectrum. While marvelously successful results can be achieved with some problems, in other contexts the technique yields no information, for example when the coupling constant is large. However, even for weak coupling we must expect that there exist phenomena which are not easily seen in straightforward coupling constant expansions, for example spontaneous symmetry violation, bound states, entrapment of various excitations. These cooperative, coherent effects can only be exposed by approximation procedures which do not posit the physics of the non--interacting theory as a useful first approximation. The situation is of course familiar in quantum particle mechanics — one does not find the properties of a complex atom or nucleus in the Born series; a useful first approximation is not the free many-body Schrödinger equation, rather it is the self-consistent Hartree–Fock equation.

I shall report to you results of the last year and a half, during which time a group of colleagues in Cambridge (UK and USA) developed approximation techniques which are useful for exhibiting collective phenomena in quantum field theory. We have analyzed several models by these methods and found new and unexpected results indicating a much richer particle spectrum and other structures than what is seen in the Born series. Although phenomenological applications have not yet been attempted, one may entertain the notion that some of the particles observed in nature correspond to these newly discovered states in quantum field theory.

It happens that in the first approximation we always solve the Euler–Lagrange equations classically. Our theory explains the role played by classical solutions in quantum mechanics and also gives a systematic prescription for computing the quantum corrections. Thus an alternate title for these lectures could be *Quantum Meaning of Classical Field Theory* [1].

2. *The quantum action*

A. Definitions

We first set the stage for encountering classical equations in a quantum theory. Consider some model involving the spinless quantum field Φ, governed by a Lagrangian density $\mathscr{L}$ and a Hamiltonian $H = \int d\vec{x}\left[\frac{\delta\mathscr{L}}{\delta\dot{\Phi}}\dot{\Phi}-\mathscr{L}\right]$. In conventional perturbation theory one focuses on a definite quantity, for example a Green's function or a scattering amplitude, and develops a series expansion which starts with the free-field value of that object. We do not wish to begin by selecting any one amplitude — we do not known a priori which amplitude will conveniently expose the phenomena we seek — rather we want to survey the entire field theory, all the Green's functions.

The usual way of doing this is to work with the generating functional $Z(J)$ which is defined by

$$Z(J) = \langle\Omega|T\exp\frac{i}{\hbar}\int dxJ(x)\Phi(x)|\Omega\rangle. \tag{2.1}$$

Here $|\Omega\rangle$ is the vacuum (ground) state of the theory and $J(x)$ is a c number function of space and time. (The symbols x, dx refer to space-time; the bold face symbols $\vec{x}$, $d\vec{x}$

refer only to space. However, when theories in one spatial dimension are considered, time will be denoted explicitly and x will refer to the single space dimension.) It is clear that repeated functional differentiation with respect to $iJ(x)$ produces at $J = 0$ all the Green's functions.

$$\hbar \left. \frac{\delta Z(J)}{\delta iJ(x)} \right|_{J=0} = \langle \Omega | \Phi(x) | \Omega \rangle,$$

$$\hbar^2 \left. \frac{\delta^2 Z(J)}{\delta iJ(x) \delta iJ(y)} \right|_{J=0} = \langle \Omega | T \Phi(x) \Phi(y) | \Omega \rangle. \tag{2.2}$$

Next the connected generating functional $W(J)$ is defined by

$$Z(J) = \exp \frac{i}{\hbar} W(J). \tag{2.3}$$

and one can show that, upon differentiation with respect to $J(x)$, $W(J)$ generates at $J = 0$ the connected Green's functions. In particular

$$\left. \frac{\delta W(J)}{\delta J(x)} \right|_{J=0} = \langle \Omega | \Phi(x) | \Omega \rangle. \tag{2.4}$$

Finally the quantum action Γ is obtained from $W(J)$ by a Legendre transform. We define the c number function $\varphi(x)$ by

$$\varphi(x) = \frac{\delta W(J)}{\delta J(x)}, \tag{2.5a}$$

eliminate $J(x)$ in favor of $\varphi(x)$, and

$$\Gamma(\varphi) = W(J) - \int dx \varphi(x) J(x). \tag{2.5b}$$

Since $\Gamma(\varphi)$ is the Legendre transform of $W(J)$, it is also true that

$$\frac{\delta \Gamma(\varphi)}{\delta \varphi(x)} = -J(x). \tag{2.6}$$

Upon comparing (2.4) with (2.5a), we see that the vacuum expectation of $\Phi(x)$ is given by that value of φ for which $\Gamma(\varphi)$ is stationary.

$$\frac{\delta \Gamma(\varphi)}{\delta \varphi(x)} = 0. \tag{2.7}$$

Moreover one can show that the n^{th} derivative of $\Gamma(\varphi)$, evaluated at the stationary value of $\varphi(x)$, gives the one-particle-irreducible n-point functions. For example

$$\frac{\delta^2 \Gamma(\varphi)}{\delta \varphi(x) \delta \varphi(y)} = i\hbar \Delta^{-1}(x, y), \tag{2.8}$$

where $\Delta(x, y)$ is the propagator. (The inverse is taken in the functional matrix sense.) All the Green's functions, and the complete field theory, can be reconstructed from $\Gamma(\varphi)$.

There are several advantages to working with $\Gamma(\varphi)$. First we see that it determines the vacuum value of $\Phi(x)$ by a variational principle. Second there are approximation schemes that one can give for computing $\Gamma(\varphi)$ which bypass the usual Born series [1]. Lastly let us take note of another property of $\Gamma(\varphi)$ which we shall use later. If $\varphi(x)$ is chosen to be time independent $\varphi(x) = \varphi(\vec{x})$, then $\Gamma(\varphi)$ acquires an infinite factor of time

$$\Gamma(\varphi)|_{\text{static}} = -E(\varphi) \int dt. \tag{2.9}$$

The coefficient $E(\varphi)$ has a simple interpretation: it is the expectation of H in a normalized state $|\Psi\rangle$ which minimizes $\langle\Psi|H|\Psi\rangle$ subject to the condition that $\langle\Psi|\Phi(x)|\Psi\rangle$ is held fixed at the value $\varphi(\vec{x})$. We may now understand the stability condition (2.7) as a variational principle for the energy: first find the minimum expectation of H subject to the subsidiary condition $\langle\Psi|\Phi(x)|\Psi\rangle = \varphi(\vec{x})$, then vary $\varphi(\vec{x})$ to find the minimum of $E(\varphi) = \langle\Psi|H|\Psi\rangle$[1].

(There is a useful generalization of the effective action: one may introduce sources for composite fields; that is in addition to $\int dx J(x)\Phi(x)$, one considers $\int dx dy K(x, y)\Phi(x)\Phi(y)$. A double Legendre transform then defines $\Gamma(\varphi, G)$, where $G(x, y)$ is conjugate to $K(x, y)$. The generalization sums all two-particle reducible graphs and can be used for studies of symmetry restoration at finite temperatures and for dynamical symmetry breaking [2]. Also one can give an interpretation for $\Gamma(\varphi)$ even when φ is time dependent, analogous to that for in the static case: $\Gamma(\varphi)$ is the time integral of the stationary matrix element of $\frac{i}{\hbar}\frac{\partial}{\partial t} - H$ between time dependent states for which the matrix element of $\Phi(x)$ is constrained to be $\varphi(x)$ [3]. I shall not use these results here.)

B. Approximate calculation

The approximation scheme that I shall use for calculating $\Gamma(\varphi)$ is the loop expansion. The first term is the sum of all graphs that do not contain any loops; the second involves just those graphs that have exactly one loop; the third has two loops; and so forth. Time is not available to derive this expansion [4], I merely quote the results for the first two terms.

$$\Gamma(\varphi) = I(\varphi) - \frac{i\hbar}{2} \ln \operatorname{Det} \frac{\delta^2 I(\varphi)}{\delta\varphi(x)\delta\varphi(y)}. \tag{2.10}$$

$I(\varphi)$ is the classical action

$$I(\varphi) = \int dx \mathscr{L}.$$

In the second term, Det is the Fredholm determinant of the (functional) matrix obtained by functionally differentiating $I(\varphi)$. For example if the Lagrange density has the form

$$\mathscr{L} = \tfrac{1}{2}\, \partial_\mu\varphi\partial^\mu\varphi - U(\varphi), \tag{2.11}$$

where $U(\varphi)$ contains the mass term as well as the interaction, then

$$\frac{\delta^2 I(\varphi)}{\delta\varphi(x)\delta\varphi(y)} = -\Box\delta(x-y) - U''(\varphi)\delta(x-y). \tag{2.12}$$

Note that the first term in (2.10) is entirely classical; there is no $\hbar$. (We assume that $\hbar$ occurs in the theory only through the definitions (2.1) and (2.3), and that it is not also present in $\mathscr{L}$.) The second term is the first quantum correction; it is proportional to $\hbar$. Moreover one can show that the loop expansion is precisely an expansion in powers of $\hbar$: the number of loops coincides with the power of $\hbar$.

If φ is taken to be time independent we obtain $E(\varphi)$, according to (2.9). In the time independent configuration, the time integrations inherent in the definition of the Fredholm determinant may be performed and we find [2]

$$E(\varphi) = E_c(\varphi) + \frac{\hbar}{4}\int d\vec{x}G^{-1}(\vec{x}, \vec{x}), \tag{2.13a}$$

where $E_c(\varphi)$ is the classical energy of a static field $\varphi(\vec{x})$

$$E_c(\varphi) = \int d\vec{x}[\tfrac{1}{2}(\nabla\varphi)^2 + U(\varphi)] \tag{2.13b}$$

and $G(\vec{x}, \vec{y})$ is defined by

$$\tfrac{1}{4}G^{-2}(\vec{x}, \vec{y}) = \frac{\delta^2 E_c(\varphi)}{\delta\varphi(\vec{x})\delta\varphi(\vec{y})} = [-\nabla^2 + U''(\varphi)]\delta(\vec{x}-\vec{y}). \tag{2.13c}$$

Let us for the moment ignore even the first quantum correction. Then in the time--dependent case the stability equations reduce to the classical Euler-Lagrange equations

$$\frac{\delta I(\varphi)}{\delta\varphi(x)} = 0,$$

$$-\Box\varphi(x) = U'(\varphi). \tag{2.14}$$

In the time-independent case one obtains the requirement that the classical energy be stationary

$$\frac{\delta E_c(\varphi)}{\delta\varphi(\vec{x})} = 0,$$

$$\nabla^2\varphi(\vec{x}) = U'(\varphi). \tag{2.15}$$

We see therefore that classical field equations are encountered in the first approximation to $\varphi(x)$, the expectation value of the quantum field.

Of course the development which I have here presented is familiar in the study of spontaneous symmetry breaking by the Nambu–Goldstone mechanism. Since in that context one is computing the expected vacuum value of $\Phi(x)$, one seeks constant solutions for $\langle\Omega|\Phi(x)|\Omega\rangle = \varphi$, because the vacuum is invariant under space-time translations

For constant φ, the equations (2.14) and (2.15) reduce simply to $U'(\varphi) = 0$. For example in a theory with

$$U(\varphi) = \frac{1}{2\lambda}(m^2 - \lambda\varphi^2)^2 \tag{2.16}$$

the minimum of $U(\varphi)$ is at

$$\varphi = \pm \frac{m}{\sqrt{\lambda}} \tag{2.17}$$

which is the lowest order approximation to the vacuum expectation of $\Phi(x)$. The fact that $\langle\Omega|\Phi(x)|\Omega\rangle$ is non-vanishing indicates spontaneous breakdown of the symmetry. $\varphi \leftrightarrow -\varphi$ which is present in (2.16). Note that the classical energy of this classical solution is zero

$$E_c(\varphi)|_{\varphi^2 = \frac{m^2}{\lambda}} = 0 \tag{2.18}$$

which is as it should be if we are dealing with the vacuum state.

Although a priori one expects only constant solutions to the stability equations, our formalism has allowed for a space-time dependence. Moreover, it is clear that since the lowest approximation coincides with the classical equations, non-constant solutions exist. Therefore the question arises whether or not these space-time dependent solutions have any significance for the quantum theory. One thing is clear — we cannot suppose that a non-constant solution has anything to do with the vacuum state, since we shall always insist that the vacuum is translationally invariant, and consequently $\langle\Omega|\Phi(x)|\Omega\rangle$ is constant. How then should we interpret the non-constant solutions? Our answer is that at least some of them have physical significance; they signal new and unexpected particle content of the theory. In the remainder of my lectures I shall explain how all this comes about. In the next Section we shall be concerned with time-independent but position-dependent solutions. In the following Section time-dependent solutions will be examined.

3. *Quantum meaning of static classical fields*

A. Stability criteria and the zero frequency mode

We want to examine those solutions of the classical static equation

$$\nabla^2\varphi(\vec{x}) = U'(\varphi) \tag{3.1}$$

that may have quantum mechanical significance [5]. A partial differential equation is under discussion; it possesses many solutions, and the first task is to delimit this vast variety. One requirement is that $E_c(\varphi)$, evaluated at a solution of (3.1), be finite. This is justified by the observation that E_c (φ) is the lowest approximation to the energy of the state which is being exposed, and certainly a physical state should have finite energy. A second requirement is that $E_c(\varphi)$ not only be stationary, but actually a minimum. Otherwise the classical solution is unstable, and correspondingly the quantum state is not a stable state of the theory. Amazingly this condition is severely restrictive, as is demonstrated by the following scaling argument. The static solutions of (3.1) are those fields $\varphi(\vec{x})$ which stationarize $E_c(\varphi) = E_T(\varphi)+E_V(\varphi); E_T(\varphi) = \int d\vec{x}\frac{1}{2}(\nabla\varphi)^2, E_V(\varphi) = \int d\vec{x}\, U(\varphi)$. If $\varphi_c(x)$ is such a solution, and $\varphi_a(\vec{x}) = \varphi_c(\vec{x}/a)$, then $E_c(\varphi_a)$ must be stationary at $a = 1$. A change of integration variable shows that

$$E_c(\varphi_a) = a^{d-2}E_T(\varphi_c)+a^d E_V(\varphi_c), \tag{3.2}$$

where d is the dimensionality of space in the models under consideration. From $\left.\frac{\partial E_c(\varphi_a)}{\partial a}\right|_{a=1} = 0$, we deduce a virial theorem.

$$E_V(\varphi_c) = \frac{2-d}{d} E_T(\varphi_c). \tag{3.3}$$

But now it follows that

$$\left.\frac{\partial^2 E_c(\varphi_a)}{\partial a^2}\right|_{a=1} = 2(2-d)E_T(\varphi_c). \tag{3.4}$$

Since $E_T(\varphi_c) = \int d\vec{x}\frac{1}{2}(\nabla\varphi_c)^2$ is positive for position dependent solutions, $E_c(\varphi_a)$ will be minimized only for $2-d > 0$, i.e. $d = 1$. Thus we see that only in the unphysical world of one spatial dimension will a static, finite energy solution of a scalar field theory be stable.

I do not at this stage abandon discussion of this entire subject, not because a one-dimensional world holds any particular interest, but because it is possible in three dimensions to construct classical solutions which lead to finite and minimized energy. What is required is particles with spin, and there exist several such examples in the literature involving Yang–Mills fields [6]. These physical examples are complicated, while the ideas that will be developed with the unphysical examples, involving a scalar field in one dimension, are completely applicable to the three-dimensional case.

Thus I confine the subsequent discussion to one-dimensional models, where the equation (3.1) simply becomes

$$\varphi''(x) = U'(\varphi). \tag{3.5a}$$

(Now x refers to the single spatial dimension.) This may be integrated once to give

$$\tfrac{1}{2}(\varphi'(x))^2 = U(\varphi). \tag{3.5b}$$

(Compare with (3.3) for $d = 1$.) The constant of integration which occurs when one passes from (3.5a) to (3.5b) is set to zero, otherwise the energy

$$E_c(\varphi) = \int dx[\tfrac{1}{2}(\varphi')^2+U(\varphi)] \tag{3.6}$$

would be infinite. Let us denote the solution of (3.5) by φ_c; then

$$E_c(\varphi_c) = \int dx(\varphi_c')^2. \tag{3.7}$$

Since $E_c(\varphi_c) > 0$, we again see that we are not discussing the vacuum state.

Even in one dimension, stability must be carefully investigated. Stability requires that the second variation of $E_c(\varphi)$ be non-negative at $\varphi = \varphi_c$.

The second variation is

$$\frac{\delta^2 E_c(\varphi)}{\delta\varphi(\vec{x})\delta\varphi(\vec{y})} = \left[-\frac{d^2}{dx^2} + U''(\varphi_c)\right]\delta(x-y). \tag{3.8a}$$

To study the spectrum of this operator, we solve a Schrödinger-like equation

$$\left[-\frac{d^2}{dx^2} + U''(\varphi_c)\right]\psi(x) = \omega^2\psi(x) \tag{3.8b}$$

and stability requires that $\omega^2 \geqslant 0$. This condition may be understood in another way. Consider the time-dependent classical equation of motion.

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right]\varphi(t, x) = U'(\varphi). \tag{3.9a}$$

If we try a solution of the form

$$\varphi(t, x) = \varphi_c(x) + e^{i\omega t}\delta\varphi(x) \tag{3.9b}$$

then to first order, $\delta\varphi$ satisfies (3.8b). The requirement that ω^2 be non-negative ensures that the frequencies are real and $\varphi_c(x)$ is stable against small oscillations in time.

A general property of the solutions of (3.8b) is that there is always a zero-frequency mode. It is a consequence of translation invariance: since $U(\varphi)$ does not depend explicitly on x, differentiating (3.5a) with respect to x gives

$$\left[-\frac{d^2}{dx^2} + U''(\varphi_c)\right]\varphi_c'(x) = 0. \tag{3.10}$$

For stability, the zero frequency mode, $\varphi_c'(x)$, must be the lowest mode. (In more than one dimension $\nabla\varphi(\vec{x})$ satisfies the analogous zero frequency Schrödinger equation. Since $\nabla\varphi(\vec{x})$ involves several components, it is a degenerate solution; but the lowest eigenstate of the ordinary Schrödinger equation is never degenerate, and there must exist a solution with negative ω^2. Thus we again see that in dimensions greater than one, stable solutions cannot be achieved with scalar fields. For a degenerate solution to be the lowest one, tensor forces must be present — in other words spin is required.)

One may understand the occurrence of the zero frequency mode in the following way. If we were computing the vacuum expectation value of $\Phi(t, x)$, then a position dependent solution would indicate spontaneous breaking of translational symmetry. Spontaneous breaking of a continuous symmetry leads to a zero frequency mode in the excitation spectrum, by Goldstone's theorem, and the zero frequency mode which is here encountered is an example of this. However, since we insist that we are not dealing with the vacuum state, the zero frequency mode will have to be interpreted in a different fashion, see below.

B. Explicit examples

There are many formulas for $U(\varphi)$ which lead to static stable solutions with finite energy. I shall discuss two examples explicitly; however our theory is independent of the specific form of $U(\varphi)$.

φ^4 theory:
$$U(\varphi) = \frac{m^4}{2\lambda}\left[1-\frac{\lambda\varphi^2}{m^2}\right]^2, \tag{3.11a}$$

Sine-Gordon (SG) theory:
$$U(\varphi) = \frac{m^4}{\lambda}\left[1-\cos\left(\frac{\sqrt{\lambda}}{m}\varphi\right)\right]. \tag{3.11b}$$

The stability values of φ for the vacuum state are $\varphi = \frac{m}{\sqrt{\lambda}}$ in the φ^4 theory, and $\varphi = 0$ in the SG theory [7]. The energy is zero for these solutions. We see that symmetry is broken spontaneously in both examples; in the first it is the $\varphi \leftrightarrow -\varphi$ symmetry; in the second the symmetry is $\varphi \leftrightarrow \pm\varphi + \frac{2\pi m}{\sqrt{\lambda}} n$, $n = \pm 1\ \pm 2, \ldots$. Expanding $U(\varphi)$ about the vacuum value of φ indicates that the mass of the "mesons" in the φ^4 theory is $2m$, while in the SG theory it is m.

Position dependent solutions to (3.5b) are the following [7]:

φ^4 theory:
$$\varphi_c(x) = \frac{m}{\sqrt{\lambda}}\tanh m(x-x_0), \tag{3.12a}$$

SG theory:
$$\varphi_c(x) = \frac{4m}{\sqrt{\lambda}}\tan^{-1} e^{\pm m(x-x_0)}. \tag{3.12b}$$

The occurrence of the parameter x_0 is a consequence of translation invariance; frequently we shall set it to zero. The classical energy of the solution is

φ^4 theory:
$$E_c(\varphi_c) = \tfrac{4}{3}\frac{m^3}{\lambda}, \tag{3.13a}$$

SG theory:
$$E_c(\varphi_c) = 8\frac{m^3}{\lambda}. \tag{3.13b}$$

The stability equations become

φ^4 theory:
$$\left[-\frac{d^2}{dx^2}+4m^2-\frac{6m^2}{\cosh^2 mx}\right]\psi(x) = \omega^2\psi(x), \tag{3.14a}$$

SG theory:
$$\left[-\frac{d^2}{dx^2}+m^2-\frac{2m^2}{\cosh^2 mx}\right]\psi(x) = \omega^2\psi(x). \tag{3.14b}$$

These Schrödinger equations can be completely solved. They are $L = 2$ and $L = 1$ cases of the class

$$\left[-\frac{d^2}{dz^2}+L^2-\frac{L(L+1)}{\cosh^2 z}\right]\psi(z) = \omega^2\psi(z), \tag{3.15}$$

with very simple properties. There is a continuous spectrum for $\omega^2 = k^2+L^2$, $k^2 \geqslant 0$ with $\psi(z) \sim e^{ikz}$ multiplied by a Jacobi polynomial of degree L in tanh z. (There is no reflection, only transmission.) In addition ω^2 takes the discrete values L^2-n^2, $n = L$, $L-1$, ..., 1. For (3.14) this means that in the φ^4 theory there is a zero frequency state φ_c', one discrete state, and a continuum beginning at $\omega^2 = (2m)^2$; in the SG theory the zero frequency state is again φ_c', and the continuum begins at $\omega^2 = m^2$. Note that in both cases the continuum begins at μ^2 where μ is the mass of the meson.

We conclude therefore that the solutions are stable, and have finite energy. Clearly they are some kind of approximation to the properties of a new state in the theory; the question is which state? We shall show presently it is a particle state of the theory, distinct from the meson states. However before establishing this we discuss the first quantum correction.

C. The first quantum correction

The energy functional, calculated to the one-loop approximation, so that it includes the first quantum correction, is

$$E(\varphi) = E_c(\varphi)+\frac{\hbar}{4}\int dx G^{-1}(x, x), \tag{3.16a}$$

$$E_c(\varphi) = \int dx[\tfrac{1}{2}(\varphi')^2+U(\varphi)], \tag{3.16b}$$

$$\tfrac{1}{4}G^{-2}(x, y) = \frac{\delta^2 E_c(\varphi)}{\delta\varphi(x)\delta\varphi(y)} = \left[-\frac{d^2}{dx^2}+U''(\varphi)\right]\delta(x-y). \tag{3.16c}$$

To compute $E(\varphi)$ to order $\hbar$, it is unnecessary to evaluate φ to this order, since $E_c(\varphi)$ is stationary at the classical value. It suffices therefore to determine $G(x, y)$ at $\varphi = \varphi_c$. This quantity is obtained from (3.16). We find the spectral decomposition of $G^{-2}(x, y)$ by again solving the Schrödinger equation

$$\left[-\frac{d^2}{dx^2}+U''(\varphi_c)\right]\psi_n(x) = \omega_n^2\psi_n(x) \tag{3.17a}$$

and

$$G^{-2}(x, y) = \sum_n \psi_n^*(x)4\omega_n^2\psi_n(y), \tag{3.17b}$$

$$G^{-1}(x, y) = \sum_n \psi_n^*(x)2\omega_n\psi_n(y). \tag{3.17c}$$

If the eigenfunctions were normalizable to one, then

$$E(\varphi_c) = E_c(\varphi_c) + \frac{\hbar}{2} \sum_n \omega_n. \tag{3.18}$$

Of course the continuum ones are not so normalizable, and the quantum correction possesses an infinity proportional to the volume of space. This infinity is removed by subtracting an analogous infinity occurring in the ground state energy, i.e. we compute the energy difference between our solution and the conventional vacuum solution. It turns out that even then the sum over frequencies is still logarithmically divergent. This logarithmic infinity is removed by renormalizing the mass parameter which occurs in $E_c(\varphi_c)$. The mass renormalization is entirely the conventional one, familiar from ordinary perturbation theory. (In our two examples, conventional estimates of divergences indicate that the mass, but not the coupling constant, must be renormalized.) We shall continue writing the energy as in (3.18); however it is understood that infinities have been removed [8].

The explicit results are [9, 10]

φ^4 theory:
$$E(\varphi_c) = \tfrac{4}{3}\frac{m^3}{\lambda} - \hbar m\left(\frac{3}{\pi} - \frac{1}{2\sqrt{3}}\right) \tag{3.19a}$$

SG theory:
$$E(\varphi_c) = \frac{8m^3}{\lambda} - \frac{\hbar m}{\pi}. \tag{3.19b}$$

We again encounter our stability criterion: if any $\omega_n^2 < 0$, then the energy is complex at the first quantum correction and quantum fluctuations destroy any physical interpretation that we might entertain concerning these solutions. The fact that for $\omega_n^2 \geqslant 0$ we can obtain a finite, real energy difference between the ground state and our new state, including the quantum correction, shows that quantum fluctuations are controllable in this model and that a sensible quantum theory can be discussed.

Note the emergence of a systematic coupling constant expansion: the classical term is $O(\lambda^{-1})$; the first correction is $O(\lambda^0)$. This will always happen if the potential is chosen to depend on the coupling constant according to rule

$$U(\varphi; \lambda) = \frac{1}{\lambda} U(\sqrt{\lambda}\,\varphi; 1) \tag{3.20a}$$

which is satisfied in our two examples. Then the classical action scales according to

$$I(\varphi; \lambda) = \frac{1}{\lambda} I(\sqrt{\lambda}\,\varphi; 1). \tag{3.20b}$$

Since in classical mechanics, the magnitude of the action is irrelevant, it follows that classically $\sqrt{\lambda}\varphi_c$ is independent of coupling constant, which is indeed the case (compare (3.11)). However in quantum mechanics the magnitude of the action does matter since it enters into the theory as $\frac{1}{\hbar} I(\varphi; \lambda) = \frac{1}{\hbar\lambda} I(\sqrt{\lambda}\varphi; 1)$. Thus we expect $\sqrt{\lambda}\varphi$ has an expansion in powers of λ, where the classical part is independent of λ; the first quantum correction proportional to λ; the second, to λ^2; etc. Therefore the energy starts with $O(\lambda^{-1})$; first quantum corrections are $O(\lambda^0)$; second, $O(\lambda^1)$, etc. A systematic coupling constant expansion is possible, and our results, which necessarily terminate after few terms because of computational difficulty, are accurate for weak coupling, even though the first term diverges as the coupling goes to zero. (In the subsequent we set $\hbar = 1$, since powers of $\hbar$ are correlated with powers of λ.)

D. Quantum reinterpretation

We now must confront the question of what is the physical, quantum mechanical meaning of all these computations. We suggest that in fact the exact $\Gamma(\varphi)$ does not have any stationary points for position dependent φ, and that our translation non-invariant solution is an artifact of the approximation. Nevertheless the approximate solution does expose, in an imperfect fashion, true properties of the theory. The situation is again familiar from ordinary quantum mechanics. For example the Hartree–Fock approximation to the nucleus violates translation invariance, which is an exact symmetry of the system. Nevertheless it is a good approximation, provided the nucleus is very heavy — in which circumstance translation invariance is not significant, disappearing entirely if the nucleus is infinitely heavy.

Evidence for the fact that the exact $\Gamma(\varphi)$ does not possess a position dependent stationary point appears if we try to compute the first quantum correction to φ. From (3.16) we see that to this order

$$0 = \frac{\delta E(\varphi)}{\delta\varphi(x)} = \frac{\delta E_c(\varphi)}{\delta\varphi(x)} + \tfrac{1}{4}\int dy\, \frac{\delta G^{-1}(y, y)}{\delta\varphi(x)}. \tag{3.21a}$$

The functional derivative of $\int dy G^{-1}(y, y)$ is computed as follows. From the identity

$$\frac{\delta G^{-2}(x, y)}{\delta\varphi(z)} = \int dx' \left[\frac{\delta G^{-1}(x, x')}{\delta\varphi(z)}\, G^{-1}(x', y) + G^{-1}(x, x')\, \frac{\delta G^{-1}(x', y)}{\delta\varphi(z)}\right]$$

it follows that

$$\int dx'\, \frac{\delta G^{-2}(x, x')}{\delta\varphi(z)}\, G(x', y)$$
$$= \frac{\delta G^{-1}(x', y)}{\delta\varphi(z)} + \int dx' dx'' G^{-1}(x, x')\, \frac{\delta G^{-1}(x', x'')}{\delta\varphi(z)}\, G(x'', y).$$

Setting x equal to y and integrating over y yields

$$\int dx dy\, \frac{\delta G^{-2}(x, y)}{\delta\varphi(z)}\, G(y, x) = 2\int dy\, \frac{\delta G^{-1}(y, y)}{\delta\varphi(z)}.$$

The left hand side is evaluated from (3.16c), and the final result for (3.21a) is

$$\varphi''(x) = U'(\varphi) + \tfrac{1}{2} G(x, x) U'''(\varphi). \tag{3.21b}$$

If we set $\varphi = \varphi_c + \delta\varphi$, then the first correction $\delta\varphi$ satisfies

$$\left[-\frac{d^2}{dx^2} + U''(\varphi_c)\right]\delta\varphi = -\tfrac{1}{2}\, G(x, x)U'''(\varphi_c), \tag{3.22a}$$

where $G(x, y)$ is evaluated at $\varphi = \varphi_c$; i.e. according to (3.17) it is given by

$$G(x, y) = \sum_n \varphi_n^*(x)\frac{1}{2\omega_n}\,\psi_n(y). \tag{3.22b}$$

However we now see that because of the zero frequency mode $G(x, x)$ does not exist since it involves $\psi_0^*(x)\dfrac{1}{2\omega_0}\,\psi_0(x)$ and $\omega_0 = 0$.

The singularity is not a consequence of any approximation; one can establish it exactly. If $\varphi(t, x) = \varphi_0(x)$ satisfies $\dfrac{\delta\Gamma(\varphi)}{\delta\varphi(t, x)} = 0$, then by translation invariance, so does $\varphi_0(x+\delta x)$ and

$$\int dt'dx' \left.\frac{\delta^2\Gamma(\varphi)}{\delta\varphi(t, x)\delta\varphi(t', x')}\right|_{\varphi=\varphi_0} \varphi_0'(x') = 0. \tag{3.23}$$

But $\dfrac{\delta^2\Gamma(\varphi)}{\delta\varphi(t, x)\delta\varphi(t', x')}$ is the inverse of the one-meson propagator, see (2.8), and according to (3.23) has an eigenvector at zero frequency with zero eigenvalue. The exact propagator therefore has an infinity exactly as in the approximation above.

The reason for the singularity is clear: $\Gamma(\varphi)$ is constructed for the purpose of studying translationally invariant, vacuum Green's functions; we have used it for a translationally non-invariant application. We now change our approach, taking into account the low order results, which show no inconsistencies, and modifying the perturbation theory so that the zero-frequency mode is removed. In order to insure translation invariance, we work in momentum space.

We postulate that in a theory where the classical field equations possess a stable, static solution of finite energy, there exist, in addition to the usual mesons, new particles which we call baryons. The states will be described by their momenta: $|k_1, \ldots, k_n\rangle$ for the multi-meson states, $|p\rangle$ for the one baryon state. Also there are baryon-multimeson states denoted by $|p; k_1, \ldots, k_n\rangle$ where p is the total momentum. When there are several distinct solutions to the classical theory, there are different baryons in the quantum theory. For example in the SG model there are two solutions, correspondingly there is a baryon and an antibaryon. The baryon is taken to be absolutely stable. This means that all matrix elements of the form $\langle k_1, \ldots, k_n| \prod_{i=1}^{N} \Phi(t_i, x_i)|p; k_1', \ldots, k_{n'}'\rangle$ are zero — no finite polynomial in the meson field can connect the baryon state with the meson states.

The single baryon state is a Lorentz covariant energy and momentum eigenstate

$$\begin{aligned} P|p\rangle &= p|p\rangle, \\ H|p\rangle &= E(p)|p\rangle, \\ E(p) &= \sqrt{p^2+M^2}. \end{aligned} \tag{3.24}$$

We further postulate that the baryon mass M has an expansion in powers of λ where the first term is $O(\lambda^{-1})$ and coincides with $E_c(\varphi_c)$, the classical energy of the classical static solution. It is now understood why the first order results appear to violate translation invariance. To lowest order, M dominates p and the baryon energy becomes static $E(p) \approx M$ — the baryon cannot move and becomes localized at some point. (This point is just x_0, the parameter in the classical solution.) Kinematic corrections arise only when the expansion of the energy is taken to the next term: $E(p) \approx M + \dfrac{p^2}{2M}$. Since M is $O(\lambda^{-1})$ only in $O(\lambda)$ does the energy exhibit a momentum dependence.

We now show that our postulates are consistent. First we demonstrate that $\varphi_c(x)$ is related to a lowest order approximation to the baryon field-form factor $\langle p|\Phi|p'\rangle = f(p, p')$. From the Heisenberg quantum equations of motion for the quantum field

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right)\Phi(t, x) = U'(\Phi) \tag{3.25}$$

it follows that [11]

$$[(E(p)-E(p'))^2-(p-p')^2+2m^2]f(p, p') = 2\lambda\langle p|\Phi^3|p'\rangle, \tag{3.26}$$

where for definiteness we have used the φ^4 theory as an example. Eq. (3.26) may be solved to lowest order; the following approximations are made. From Lorentz invariance, we know that $f(p, p')$ is a function of $(E(p)-E(p'))^2-(p-p')^2$. But to lowest order $E(p) = M$, hence the form-factor is a function only of momentum differences. Therefore the left hand side of (3.26) becomes in lowest order $[-(p-p')^2+2m^2]f(p-p')$. On the right hand side we saturate with intermediate states. Since the baryon is stable, only one-baryon multi-meson states contribute. In lowest order we keep only the baryon states, and get $2\lambda \int \dfrac{dp''dp'''}{(2\pi)^2} f(p-p'')f(p''-p''')f(p'''-p')$. Upon introducing the Fourier transform of the form-factor

$$f(p-p') = \int dx e^{i(p-p')x}\varphi(x) \tag{3.27}$$

Eq. (3.26) is recognized as the static classical equation.

$$\varphi''(x) = -2m^2\varphi(x)+2\lambda\varphi^3(x) = U'(\varphi). \tag{3.28}$$

Hence $\varphi_c(x)$ is the Fourier transform of the field form-factor in lowest order. (The arbitrariness of the origin of the coordinate system now becomes an arbitrary phase of $\langle p|\Phi|p'\rangle$.)

In the same approximation we may calculate the energy.

$$\langle p|H|p'\rangle = E(p)\,(2\pi)\delta(p-p'), \tag{3.29a}$$

$$H = \int dx[\tfrac{1}{2}\,\dot{\Phi}^2+\tfrac{1}{2}\,(\Phi')^2+U(\Phi)], \tag{3.29b}$$

$$E(p) = \langle p|[\tfrac{1}{2}\,\dot{\Phi}^2+\tfrac{1}{2}\,(\Phi')^2+U(\Phi)]|p\rangle. \tag{3.29c}$$

The matrix element is evaluated by saturating with intermediate states, retaining to lowest order only the baryon state. Matrix elements of $\dot{\Phi}$ do not contribute, since they involve baryon energy differences, which vanish to lowest order. It is then easy to show that

$$\begin{aligned} E(p) &= \int dx[\tfrac{1}{2}(\varphi')^2+U(\varphi)] \\ &= \int dx(\varphi_c')^2 = M. \end{aligned} \tag{3.30}$$

Thus, consistent with our postulate, the lowest order energy coincides with the classical energy and is identified with the $O(\lambda^{-1})$ expression for the baryon mass. Therefore the lowest order results are entirely consistent with our interpretation. We also see that if a wave packet state is constructed $|\psi\rangle = \int dp\varrho(p)|p\rangle$ corresponding to a baryon localized at x_0, then in the limit as the baryon gets very heavy, $\langle\psi|\Phi(t, x)|\psi\rangle = \varphi_c(x-x_0)$.

To compute to first order, we may still keep $E(p)$ independent of p, since the kinematical dependence enters in $O(\lambda)$, two orders beyond the lowest $O(\lambda^{-1})$. However in the saturation by intermediate states we must keep the baryon, one-meson state $|p; k\rangle$; an expression for $\langle p|\Phi|p'; k\rangle = f_k(p, p')$ is needed. The exact equation for that quantity is

$$[(E(p)-E_k(p'))^2-(p-p')^2+2m^2]f_k(p, p') = 2\lambda\langle p|\Phi^3|p'; k\rangle. \tag{3.31}$$

$E_k(p')$ is the energy of the baryon one-meson state; p' is the total momentum; k is the meson momentum. To lowest order we take $E_k(p)$ to be $M+\omega(k)$. In saturating the right-hand side we keep the no-meson, and the one-meson states, thus encountering the following matrix elements $\langle p|\Phi|p'\rangle$, $\langle p|\Phi|p'; k\rangle$ and $\langle p; k|\Phi|p'; k'\rangle$. The first is known to lowest order; the second is being calculated. The third we decompose into a connected and disconnected piece.

$$\begin{aligned} \langle p;k|\Phi|p'; k'\rangle &= (2\pi)\delta(k-k')\langle p|\Phi|p'\rangle \\ &\quad + \langle p; k|\Phi|p'; k'\rangle_c. \end{aligned}$$

To lowest order only the disconnected piece is kept. Also we take $f_k(p, p')$ to be, in lowest order, a function of $p-p'$, the total momentum difference. With these steps (3.31) becomes

$$\begin{aligned} \omega^2(k)f_k(p-p') &= [(p-p')^2-2m^2]f_k(p-p') \\ &+2\lambda\int\frac{dp''dp'''}{(2\pi)^2}f(p-p'')f(p''-p''')f_k(p'''-p'). \end{aligned} \tag{3.32}$$

Upon introducing the Fourier transform

$$f_k(p-p') = \int dx e^{i(p-p')x}f(k; x) \tag{3.33}$$

(3.32) is recognized as the Schrödinger equation which we have repeatedly encountered.

$$\left[-\frac{d^2}{dx^2}-2m^2+6\lambda\varphi_c^2(x)\right]f(k;x)=\omega^2(k)f(k;x),$$

$$\left[-\frac{d^2}{dx^2}+U''(\varphi_c)\right]f(k;x)=\omega^2(k)f(k;x). \tag{3.34}$$

Now we have a clear physical interpretation for the solutions of this equation. The continuum solutions, which begin at $\omega(k)=\sqrt{k^2+\mu^2}$, where μ is the meson mass, are interpreted as meson-baryon scattering states. If there are discrete states, other than the zero frequency state, (as in the φ^4 theory) they are excited states of the baryon. The zero frequency solution of (3.34) is not associated with any state. For later convenience let us set $f(k;x)$ equal to $\frac{1}{\sqrt{2\omega(k)}}\psi_k(x)$ for all states with the exception of the zero frequency state. Note also that the normalized zero frequency state is $\varphi_c'(x)/\sqrt{M}$, since $M=\int dx(\varphi_c')^2$.

Is it consistent to exclude the zero-frequency mode; i.e. are the physical states complete even though we are excluding one of the functions which contribute to a set of mathematically complete functions? Note also that (3.34) does not determine the normalization of $\psi_k(x)$. To settle both these points, we consider matrix elements of the canonical commutator between baryon states

$$\langle p|[\Phi(0,x),\dot\Phi(0,y)]|p'\rangle=i\delta(x-y)(2\pi)\delta(p-p'). \tag{3.35}$$

Saturate with no-meson states and one-meson states. The contribution of the one-meson states can be shown to be

$$i\sum_k{}'\,\omega(k)\left\{\int dz e^{i(p'-p)z}\frac{\psi_k^*(x-z)}{\sqrt{2\omega(k)}}\frac{\psi_k(y-z)}{\sqrt{2\omega(k)}}+x\leftrightarrow y\right\}. \tag{3.36a}$$

The prime on the sum indicates that the zero-frequency state is excluded. If we take the ψ_k's to be properly continuum normalized, then the sum can be evaluated by completeness. A delta function does not emerge, since the zero frequency mode is excluded; rather we get

$$i\delta(x-y)\,(2\pi)\delta(p-p')-i\int dz e^{i(p'-p)z}\frac{\varphi_c'(x-z)}{\sqrt{M}}\frac{\varphi_c'(y-z)}{\sqrt{M}}. \tag{3.36b}$$

Next the no-meson contribution is evaluated; here are encountered contributions of the form

$$\int\frac{dq}{(2\pi)}\langle p|\Phi(0,x)|q\rangle\langle q|\dot\Phi(0,y)|p'\rangle$$

$$=i\int\frac{dq}{(2\pi)}[E(q)-E(p')]\langle p|\Phi(0,x)|q\rangle\langle q|\Phi(0,y)|p'\rangle. \tag{3.36c}$$

The matrix elements are each $O(\lambda^{-1/2})$, consequently we must retain $E(q)-E(p')$ to order λ, since we are computing the commutator which is seen from the right-hand side of (3.35) to be $O(\lambda^2)$. The energy difference is taken to be $\frac{q^2-p'^2}{2M'}$, where we have put a prime to distinguish the mass that occurs in the kinetic term from the rest mass; we shall prove that in fact $M = M'$. Evaluating the relevant integrals gives the no-meson contribution to $O(\lambda^0)$

$$\frac{i}{M'}\int dz e^{i(p'-p)z}\varphi_c'(x-z)\varphi_c'(y-z). \tag{3.36d}$$

Thus when $M = M'$, (3.36d) cancels the second term in (3.36b).

By this exercise we have learned three things. First, the properly normalized matrix element is $\langle p|\Phi|p';k\rangle = \frac{1}{\sqrt{2\omega(k)}}\int dx e^{i(p-p')x}\psi_k(x)$, where $\psi_k(x)$ is a normalized solution of the Schrödinger equation. Second, the zero frequency solution is not a state of the theory, rather it describes the first correction to the motion of the baryon. Third, to $O(\lambda^{-1})$ the theory is Lorentz invariant since the rest mass coincides with the kinetic mass.

From the scattering solutions of (3.34) the meson-baryon S matrix can be found. For the φ^4 and SG theories there is no reflection, only transmission. The transmission amplitude T is a pure phase by unitarity

$$T = e^{2i\delta(k)},$$

$$\tan\delta(k) = -\sum_{n=1}^{L}\tan^{-1}\frac{k}{nm}+\frac{L\pi}{2},$$

$$\varphi^4 \text{ theory: } L = 2,$$

$$\text{SG theory: } L = 1. \tag{3.37}$$

Note that phase shift is independent of λ.

With the one-meson matrix element determined, the first order correction to the energy and baryon form-factor can be computed. Returning to (3.26) and retaining the one-meson states in the saturation of the right hand side, we find that the equation satisfied by $\varphi(x)$ is

$$\frac{d^2}{dx^2}\varphi(x) = U'(\varphi)+\tfrac{1}{2}\tilde{G}(x,x)U'''(\varphi),$$

$$\tilde{G}(x,y) = \sum_k{}' \psi_k^*(x)\frac{1}{2\omega(k)}\psi_k(x). \tag{3.38}$$

This is analogous to the equation derived from the effective action, (3.21), with a crucial difference: $\tilde{G}(x, y)$ is no longer infinite, since the zero frequency mode is excluded. Similarly one can calculate the energy to order $O(\lambda^0)$. Keeping the one-meson intermediate states in (3.29) gives agreement with (3.18).

$$E(p) = E_c(\varphi_c)+\tfrac{1}{2}\sum_k \omega(k). \tag{3.39}$$

To $O(\lambda^0)$ there is no kinetic term; and we identify the second term in (3.39) with the first quantum correction to the baryon mass.

Let us note the important fact that again a systematic coupling constant expansion has emerged: $\langle p|\Phi|p'\rangle$ is $O(\lambda^{-1/2})$, $\langle p|\Phi|p'; k\rangle$ is $O(\lambda^0)$ and one can show that the connected part of $\langle p; k_1, \ldots, k_n|\Phi|p'; k'_1, \ldots, k'_{n'}\rangle$ is $O\left(\lambda^{\frac{n+n'-1}{2}}\right)$. By keeping track of powers of λ, one can perform calculations in the one-baryon sector to arbitrary order of λ.

Finally we turn to the question of the baryon's stability: if it is heavy, why does it not decay into ordinary mesons? Stability is usually associated with an absolutely conserved quantum number. To see the existence of a conservation law in our models, observe that

$$J^\mu = \varepsilon^{\mu\nu}\partial_\nu\Phi \tag{3.40}$$

is a conserved current, not because it arises by Noether's theorem from a symmetry of the theory, rather because it is trivially conserved since it is a divergence of an antisymmetric tensor. The charge associated with this current is

$$Q = \int dxJ^0 = \int dx\Phi' = \Phi|_{x=\infty}-\Phi|_{x=-\infty}. \tag{3.41}$$

In the meson sector the field tends to the same value as $x \to \pm\infty$, and Q vanishes. In the baryon sector, the field tends to different values, Q is non-zero and its conservation renders the baryon stable. (In higher dimensions, a totally antisymmetric tensor, whose divergence is a conserved current, can be constructed with the help of the spin degrees of freedom.)

E. Canonical quantization

The perturbation theory which was outlined above involves expanding matrix elements of the field in powers of λ. One may exhibit a more conventional perturbation theory by shifting the quantum field Φ by the classical solution φ_c. (Recall that in the vacuum sector, conventional perturbation theory is obtained by first shifting the quantum field by its vacuum value.) However since φ_c is position dependent, translation covariance would be lost if we merely subtract φ_c from Φ. Rather we proceed as follows. A new dynamical quantum variable, the center of mass $X(t)$, is introduced by the definition

$$\Phi(t, x) = \chi(t, x-X(t))+\varphi_c(x-X(t)). \tag{3.42}$$

Here χ is a new quantum field; in order that the total number of degrees of freedom not be changed it must satisfy a subsidiary condition, which for convenience we select to be

$$\int dx\varphi'_c(x)\chi(t, x) = 0. \tag{3.43}$$

We view (3.42) as a canonical point transformation from the dynamical variable Φ to the dynamical variables χ and X [12]. The corresponding transformation law for canonical momenta is complicated

$$\pi(t, x) = \pi_\chi(t, x-X(t)) - \tfrac{1}{2}\left\{\frac{\varphi_c'(x-X(t))}{M+\xi(t)},\ P(t)+\int dx\chi'(t, x)\pi_\chi(t, x)\right\}_+, \tag{3.44}$$

where $M = \int dx(\varphi_c')^2$ and $\xi(t) = \int dx\chi'(t, x)\varphi_c'(x)$. Again a new variable, $P(t)$, has been introduced; it is the momentum conjugate to $X(t)$. To preserve the total number of degrees of freedom, π_χ is required to satisfy a subsidiary condition

$$\int dx\varphi_c'(x)\pi_\chi(t, x) = 0. \tag{3.45}$$

If the P, X variables are chosen to commute with the π_χ, χ variables, and if the nonvanishing commutators are taken to be

$$i[P(t), X(t)] = 1,$$

$$i[\pi_\chi(t, x), \chi(t, y)] = \delta(x-y) - \frac{\varphi_c'(x)\varphi_c'(y)}{M}. \tag{3.46}$$

one can show that the transformation is canonical [12].

The significance of $P(t)$ is seen by computing the total momentum operator. In terms of the old variables it is $-\int dx\pi(t, x)\Phi'(t, x)$; in terms of the new it is just P. (The t dependence is irrelevant since P is a constant of motion.) The Hamiltonian operator in terms of the new variables is obtained by using (3.42) and (3.44) in (3.29), and shifting the x integration from x to $x-X$. The result is

$$H = M + \frac{\bar{P}^2(t)}{2M} + H_\chi(t) - \tfrac{1}{8}\frac{\int dx(\varphi_c'')^2}{(M+\xi(t))^2},$$

$$\bar{P}(t) = \tfrac{1}{2}\left[P + \int dx\pi_\chi(t, x)\chi'(t, x)\right]\frac{1}{1+\xi(t)/M} + \text{h.c.},$$

$$H_\chi(t) = \int dx\left[\tfrac{1}{2}\pi_\chi^2 + \tfrac{1}{2}(\chi')^2 + U(\chi, \varphi_c)\right],$$

$$U(\chi, \varphi_c) = U(\chi+\varphi_c) - \chi U'(\varphi_c) - U(\varphi_c). \tag{3.47}$$

In spite of its awkward, non-covariant appearance, the theory is Lorentz invariant, as is established by constructing the Lorentz generators and verifying their algebra. One may separate H into a "free" part H_0 and an "interacting" part H_I

$$H_0 = M + \tfrac{1}{2}\int dx[\pi_\chi^2 + (\chi')^2 + U''(\varphi_c)\chi^2],$$

$$H_I = H - H_0. \tag{3.48}$$

Thus it is seen that in the "free" theory the χ field provides for meson excitations, superimposed on a static baryon of mass M. Corrections are computed in the standard way by expanding H_I in powers of λ. In this manner one regains all the results of the previous section. In particular the subsidiary conditions (3.43) and (3.45) insure that the zero frequency mode is absent from the spectrum.

F. Additional aspects of the theory in one dimension

The reasoning has been quite general so far; now we mention some especially interesting properties of baryon solutions in the two explicit examples that we have examined: φ^4 and SG theories. Let us observe that in the φ^4 theory the explicit solution (3.12a) is an antisymmetric function of its argument. Consequently $\langle p|\Phi|p'\rangle$ is antisymmetric in $p \leftrightarrow p'$. Since there is only one kind of baryon in this model, we may also consider $\langle p|\Phi|p'\rangle$ to be the analytic continuation of the two-baryon, vacuum matrix element of Φ. Evidently the latter is antisymmetric in $p \leftrightarrow p'$, hence the baryons satisfy Fermi statistics [5]. (There is of course no spin in one dimension.)

For the SG theory the above argument cannot be used since there are two kinds of baryons: particle and antiparticle corresponding to the two distinct classical solutions (3.12b). Therefore $\langle p|\Phi|p'\rangle$, though antisymmetric, is the analytic continuation of a baryon-antibaryon, vacuum matrix element, and no constraint on the statistics of identical particles is found.

However one can show that the SG theory is equivalent to the massive Thirring model with the Lagrange density

$$\mathscr{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi - \frac{g}{2}\bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma_\mu\psi, \tag{3.49}$$

and the identifications [13]

$$\frac{4\pi m^2}{\lambda} = 1 + g/\pi,$$
$$m^{-1}\lambda^{1/2}\varepsilon^{\mu\nu}\partial_\nu\Phi = 2\pi\bar{\psi}\gamma^\mu\psi. \tag{3.50}$$

Thus the fermion number current is just our "trivial" current, and the fermion field of Thirring model is associated with the baryon solutions to the SG theory.

It appears that there are two equivalent description of the physical content. One may use the boson formalism of the SG theory, in which the mesons are elementary and the fermions are coherent solutions of the theory. Alternatively one may use the fermion formalism of the massive Thirring model where the baryons are elementary and the mesons are ordinary bound states. It is not known at the present time if this fascinating duality occurs in other models in one space dimension, nor is it clear whether any such phenomena is present in physical models in three space dimensions.

4. *Quantum meaning of time dependent classical fields*

A. Soliton solutions of classical field equations

The previous section was devoted to an exhaustive discussion of the quantum mechanical significance of static solutions to classical field theory. An interpretation was given in terms of a new state in the theory — the baryon, and the one-baryon sector of the

Hilbert space was analyzed. It is plausible to suppose that the time-dependent solutions of the classical field equations have something to do with multi-baryon states. As yet a complete and systematic perturbation theory for the multi-baryon states has not been developed. However using semi-classical methods for field theory some results have been obtained. In these semi-classical calculations, a direct quantum mechanical role, analogous to the interpretation of the static solution as the baryon field-form factor, is not assigned to the classical solutions. Rather information about the quantum theory is extracted from them by WKB techniques. We first examine the nature of time dependent classical solutions.

Whenever there exists a static, time-independent solution $\varphi_c(x-x_0)$, then by Lorentz invariance there is also a time-dependent solution $\varphi_c(\gamma[x-vt-x_0])$, $\gamma = (1-v^2)^{-1/2}$, $v^2 \leqslant 1$, with energy $M\gamma$. This is sometimes called a stationary wave solution; for the SG equation it is the one-soliton solution. It may be associated with the quantized baryon state in a moving reference frame, and need not concern us any further.

For the SG theory there exist other time dependent solutions whose form can be explicitly given, and in the remainder of my lecture, I shall deal exclusively with this model. These are the famous soliton solutions which have been discussed extensively in the mathematical literature [14]. They have the following properties: the N soliton solution depends on $2N$ parameters. As $t \to -\infty$, the solution becomes a superposition of N one soliton solutions and $2N$ parameters correspond to the asymptotic velocities $v^{(i)}$ and positions $x_0^{(i)}$ of the N solitons. As $t \to +\infty$, the solution again decomposes into a superposition of N one soliton solutions. The asymptotic final velocities are the same as the initial ones, the asymptotic positions differ from the initial ones by an amount that can be ascribed to a time-delay in the multi-soliton collision. By translation invariance, two constants of motion can be arbitrarily set to zero, and a third can also be made to vanish if the calculation is performed in the center-of-mass frame. For example the 2 soliton solution depends on 1 constant, u, the relative velocity of the two solitons. The explicit form of the 2 soliton solutions is [14]

soliton-soliton:
$$\varphi_{ss} = \frac{4m}{\sqrt{\lambda}} \tan^{-1} \frac{u \sinh m\gamma x}{\cosh m\gamma ut}, \tag{4.1a}$$

soliton-antisoliton:
$$\varphi_{s\bar{s}} = \frac{4m}{\sqrt{\lambda}} \tan^{-1} \frac{1}{u} \frac{\sinh m\gamma ut}{\cosh m\gamma x}. \tag{4.1b}$$

$$\gamma = (1-u^2)^{-1/2}.$$

The total momentum of each solution is zero; the energy is $2M\gamma$, $M = 3m^3/\lambda$. Examination of the asymptotic forms of the two solutions shows that in both cases there is time delay

$$\Delta t(u) = \frac{2}{mu\gamma} \ln u. \tag{4.2}$$

There is another solution, the soliton-antisoliton bound state or "breather" which is periodic in time. It is obtained from (4.1b) by taking $u = ia$, a real.

$$\varphi_B = \frac{4m}{\sqrt{\lambda}} \tan^{-1} \frac{1}{a} \frac{\sin m\gamma a t}{\cosh m\gamma x}, \tag{4.3}$$

$$\gamma = (1+a^2)^{-1/2}.$$

The energy is $2M\gamma$. There is no soliton-soliton bound state.

B. WKB method for field theoretic bound states

The periodic solution (4.3) may be quantized by the semiclassical method. Let us recall that in one-dimensional quantum particle mechanics, the semiclassical quantization condition, valid for large quantum numbers, is in lowest order

$$\int_{x_1}^{x_2} p(q)dq = n\pi. \tag{4.4}$$

Here $p(q)$ is the local momentum and x_1, x_2 are the turning points. Eq. (4.4) may be equivalently written as

$$\int p\dot{q}dt = \int (L+H)dt = I_T(E)+ET = n\pi, \tag{4.5}$$

where the integration is over a semi-period. $I_T(E)$ is the classical action of a periodic solution with semi-period T and energy E. In the next approximation n is replaced by $n+\gamma$ where γ depends on the shape of the potential.

For field theory, the lowest order quantization condition is again (4.5)

$$I_T(E)+ET = \int dt \int dx \pi(t,x)\dot{\Phi}(t,x) = n\pi, \tag{4.6}$$

where π is the canonical momentum. The next approximation has also been derived, but we shall not use it here [15]. Therefore in order to quantize the breather mode, we evaluate $\int_{-\pi/2m\gamma a}^{\pi/2m\gamma a} dt \int_{-\infty}^{\infty} dx \dot{\varphi}_B^2$ and set it equal to $n\pi$, thus quantizing a. Since the energy is expressible in terms of a, this is equivalent to energy quantization. An elementary calculation gives

$$E_n = 2M \sin \frac{m}{2M} n, \; n = 1, 2, \ldots, \leqslant \frac{8\pi m^2}{\lambda}. \tag{4.7}$$

(Eq. (4.7) is valid also in the second approximation, provided the value (3.19b) is used for M [10].)

This result indicates that a soliton and an anti-soliton (baryon, anti-baryon) bind and form mesons. Note that for weak coupling and small excitation $E \approx mn$, which makes it plausible that these bound states may be equivalently described as n-meson bound states; this is an example of the duality mentioned previously.

C. WKB method for field theoretic scattering states

It is well known that in the semiclassical approximation, the S matrix for one dimensional "scattering" is given by

$$S(E) = e^{2i\delta(E)},$$

$$\delta(E) = \delta(E_{th}) + \tfrac{1}{2} \int_{E_{th}}^{E} dE' \Delta t(E'). \tag{4.8}$$

Here E_{th} is the threshold energy and $\Delta t(E)$ is the time delay as a function of energy. Eq. (4.8) merely states that the time delay is twice the energy derivative of the phase-shift [16].

The constant of integration may be evaluated as follows. Consider the classical action $I_T(E)$ for a solution to the equations of motion, with energy E, which passes from an initial configuration to a final configuration in time T. Since $\dfrac{dI_T(E)}{dT} = -E$, it follows that

$$\frac{d}{dE}(I_T(E)+ET) = \left(\frac{dI_T(E)}{dT}+E\right)\frac{dT}{dE}+T = T. \tag{4.9a}$$

That is, the time of flight can be expressed as an energy derivative. Total time delay is equal to the time of flight in the presence of forces, less the time of flight in the absence of forces, in the limit as T goes to infinity. But the time of flight in the absence of forces is given by (4.9a), where the term in parentheses on the left hand side may be written as $p(E)[x_f(T)-x_i]$, with $p(E)$ being the relative momentum of the particles and x_i, x_f the initial, final position. Thus the total time delay is

$$\Delta t(E) = \lim \frac{d}{dE}(I_T(E)+ET-p(E)\,[x_f(T)-x_i]) \tag{4.9b}$$

and the phase shift can be taken to be

$$2\delta(E) = \lim\,(I_T(E)+ET-p(E)\,[x_f(T)-x_i]). \tag{4.10}$$

At threshold $p(E_{th}) = 0$,

$$2\delta(E_{th}) = \lim\,(I_T(E_{th})+E_{th}T). \tag{4.11}$$

Next let us consider the quantization condition (4.6). The total number of bound states is given by n_B, the maximum value of n, which occurs for E just below E_{th}, since at E_{th} the semi-period becomes infinite. Hence it is true that

$$n_B\pi = \lim\,(I_T(E)+ET). \tag{4.12}$$

Comparing (4.12) and (4.11) we find

$$\delta(E_{th}) = \frac{\pi}{2}\,n_B, \tag{4.13}$$

which we call the semi-classical Levinson's theorem. (The exact Levinson's theorem is $\delta(E_{th})-\delta(\infty) = \dfrac{\pi}{2}n_B$, where the factor of 1/2 is peculiar to one-dimensional motion.)

Therefore the semi-classical phase-shift is given by

$$\delta(E) = \tfrac{1}{2}\,n_B\pi+\tfrac{1}{2}\int_{E_{th}}^{E} dE'\Delta t(E'). \tag{4.14}$$

Using (4.2), as well as the fact that $n_B = 0$ for the soliton-soliton channel and $n_B = 8\pi m^2/\lambda$ for the soliton-antisoliton channel, we find the following phase-shifts [16]

$$\delta_{ss}(E) = \frac{16m^2}{\lambda} \int_0^u dx \, \frac{\ln x}{1-x^2}, \tag{4.15a}$$

$$\delta_{s\bar{s}}(E) = \frac{4\pi^2 m^2}{\lambda} + \frac{16m^2}{\lambda} \int_0^u dx \, \frac{\ln x}{1-x^2}, \tag{4.15b}$$

$$E = \frac{8m^3}{\lambda} (1-u^2)^{-1/2}.$$

The phase-shifts come out to be proportional to λ^{-1}; we suspect that they are the first terms of an expansion in powers of λ. We expect therefore that they are accurate for weak coupling, except that as always with WKB approximations, the analytic behavior near threshold is unreliable. Away from threshold they are analytic functions, and it can be shown that the soliton-soliton phase-shift is obtained from the soliton-antisoliton phase-shift by the crossing relation [17]. Note that when λ is small, the phase-shifts are large, indicating strong forces, even though the coupling is weak. At present we are studying the $O(\lambda^0)$ corrections to (4.15) and we hope to give a systematic coupling constant expansion for scattering in the multi-baryon sector.

5. *Prospects for the theory*

The structures that have been exposed in the examined models are very fascinating. I find the following aspects especially interesting. As it has been long suspected [18], there exist stable states in quantum field theory, which are not bound states of a definite number of mesons; rather they are coherent superpositions of an infinite number of mesons. The stability is not a consequence of a conventional conservation law of the Noether variety; rather a topological distinction between different kinds of solutions prevents transitions between them. Finally the emergence of large effects and strong forces, for weak coupling, due to a singularity in the coupling constant, suggests a unification of strong and weak interactions. It is certain that the above features persist for three-dimensional models. Equally intriguing are the phenomena found in the one-dimensional examples; fermions form bosons, with an attendant duality which eliminates the distinction between elementary and composite particles. It is not known to what extent this carriers over to three dimensions.

In my opinion, the most important area for further investigation concerns the multi--baryon sectors. Techniques must be developed for analyzing these states in a way which does not depend on the soliton property of classical differential equations. With the exception of the unphysical SG theory, where the equivalence with the massive Thirring model provides us with a local field for the solitons, we do not as yet know how to allow for baryon creation and annihilation.

The practical utility of these ideas for realistic physical theory is unclear, though it is very tempting to speculate that some of the "fundamental" particles occurring in nature can be associated with our mathematical baryons, and that some absolute conservation laws are topological in character.

Editorial note. This article was proofread by the editors only, not by the author.

REFERENCES

[1] For the previous summaries, see R. Jackiw, *Quantum Mechanical Approximations in Quantum Field Theory*, to be published in the proceedings of Orbis Scientiae II, Coral Gables Conference, January 1975; and R. Rajaraman, *Some Non-Perturbative Semi-Classical Methods in Quantum Field Theory*, IAP preprint.

[2] J. Cornwall, R. Jackiw, E. Tomboulis, *Phys. Rev.* **D10**, 2428 (1974).

[3] R. Jackiw, A. Kerman, in preparation.

[4] R. Jackiw, *Phys. Rev.* **D9**, 1686 (1974).

[5] The development here follows J. Goldstone, R. Jackiw, *Phys. Rev.* **D11**, 1486 (1975).

[6] G. 't Hooft, *Nucl. Phys.* **B79**, 276 (1974); A. M. Polyakov, Landau Institute preprint.

[7] We take a definite sign for the square root. The solutions with the opposite sign lead to an identical copy of the theory.

[8] That conventional renormalizations render the effective action, and hence $E(\varphi)$, finite was shown by S. Coleman, R. Jackiw, H. D. Politzer, *Phys. Rev.* **D10**, 2491 (1974). The explicit evaluations in the φ^4 and S G theories were given by the Princeton group, Ref. [9] and [10].

[9] R. Dashen, B. Hasslacher, A. Neveu, *Phys. Rev.* **D10**, 4130 (1974).

[10] R. Dashen, B. Hasslacher, A. Neveu, *Phys. Rev.* **D**, in press.

[11] The study of a field theory through the equations satisfied by the matrix elements of the quantum field was pioneered by A. Kerman, A. Klein, *Phys. Rev.* **132**, 1326 (1963).

[12] This canonical transformation was given by E. Tomboulis, *Phys. Rev.* **D**, in press. Previously J. L. Gervais, B. Sakita, *Phys. Rev.* **D**, in press, used a similar transformation for change of variable in a functional integral. Subsequent developments are found in J. L. Gervais, A. Jericki, B. Sakita, CCNY preprint; C. Callan, D. Gross, Princeton University preprint; L. Faddeev, unpublished; N. Christ, T. D. Lee, Columbia University preprint; E. Tomboulis, G. Woo, MIT preprint.

[13] S. Coleman, *Phys. Rev.* **D11**, 2088 (1975); B. Schroer, Freie Universität Berlin preprint; S. Mandelstam, Berkeley preprint.

[14] G. Whitham, *Linear and Non-linear Waves*, Wiley 1974; A. Scott, F. Chu, D. Mc Laughlin, *Proc. IEEE* **61**, 1443 (1973).

[15] R. Dashen, B. Hasslacher, A. Neveu, *Phys. Rev.* **D10**, 4114 (1974).

[16] The calculation of this Section are due to R. Jackiw, G. Woo, MIT preprint.

[17] S. Coleman, Harvard University preprint.

[18] T. Skyrme, *Proc. R. Soc.* **A262**, 237 (1961).