METHODS OF DERIVING EXACT SOLUTIONS OF SPHERICAL SYMMETRY IN THE EINSTEIN-CARTAN THEORY FOR A PERFECT FLUID WITH A "CLASSICAL DESCRIPTION" OF SPIN

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(Received April 22, 1974; Revised version received September 20, 1974)

The Einstein-Cartan theory of gravitation allows for avoiding singularities. This idea is supported by giving exact solutions of cosmological equations which are extensions of the Friedmannian cosmology. Various singularity-free models of dust and radiative universes are constructed. Two general relations are given which allow for a transition from dust-filled to radiation-filled universes with the same value of the scale factor. These relations are characteristic of the Einstein-Cartan theory, and cannot be brought over into general relativity. It is shown how exact solutions of the Einstein-Cartan theory can be derived from known solutions of general relativity. This procedure which may be applied to material spheres, is simple enough to generate a large set of useful solutions. Two examples are given.

1. Introduction

Exact solutions of the field equations are of much value and use in any theory of gravitation, as they provide simple models both of the universe and of individual celestial bodies. A lot of such solutions exist in Einstein's theory both for empty space and for regions filled with matter. Recently, a very simple and natural extension of Einstein's theory has been proposed by incorporating spin from the beginning as a dynamical quantity. This is the Einstein-Cartan theory (Hehl 1966, 1970, 1973; Trautman 1972a, b, c, 1973a) which has the useful feature that its equations in empty space are exactly the same as those of Einstein's theory. Exact solutions for cosmological applications have been derived in this theory very recently (Kopczyński 1972, 1973; Tafel 1973). It has been shown that it is possible to avoid the usual initial singularity, and this increases our interest in the Einstein-Cartan theory (Trautman 1973b). We shall make therefore a systematic search for exact solutions, first for the most simple case that is usually studied, namely for the case of spherical symmetry of the distribution of matter under study.

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It is not necessary to consider here separately the exact solution for the Einstein-Cartan theory in empty space; this is just the well known external Schwarzschild solution of general relativity. The difference with general relativity exists for the system of underlying equations in regions filled with matter; here it is our task to solve the equations. It will be shown that, in fact, we can construct exact solutions of the Einstein-Cartan theory from exact solutions of general relativity for individual celestial bodies (as approximated by spherical distributions of matter). The essential task ab initio has to be performed for cosmological models as we are interested here in giving explicit models without singularity which are characteristic of the Einstein-Cartan theory. We are going to present here extensions of the well known Friedmann models, comprising Kopczyński's model (Kopczyński 1972) as the specific case for dust matter and spin conservation relation. Then we are to consider the case of relativistic stars: we show how exact solutions can be constructed in the static and non-static case. The question of gravitational collapse is only slightly touched at the end of the paper as it needs a separate treatment; it is indicated that it is possible to avoid a singularity much in the same way as in cosmology.

2. *Fundamental geometric and physical assumptions*

In our notation we follow rather closely Trautman's papers (Trautman 1972a, b, c; 1973a), with some minor differences (as e. g. the sign difference in R_i^j). We use the following form of the Einstein-Cartan equations:

$$R^j{}_i - \tfrac{1}{2}\,\delta^j{}_i R = -8\pi G t^j{}_i, \tag{2.1}$$

$$Q^k{}_{ij} - \delta^k{}_i Q^l{}_{lj} - \delta^k{}_j Q^l{}_{il} = 8\pi G s^k{}_{ij}, \tag{2.2}$$

where $R_i^j = R_{ik}^{kj}$, $R = R_k^k$, Q_{ij}^k is the torsion tensor, t_i^j is the "canonical", asymmetric energy-momentum tensor, s_{ij}^k is the spin tensor, and the light velocity in vacuum $c = 1$. The following set of basis orthogonal 1-forms corresponds to the most general line element of spherical symmetry:

$$\theta^1 = e^{\lambda/2}dr, \quad \theta^3 = re^{\sigma/2}\sin\theta d\varphi,$$

$$\theta^2 = re^{\sigma/2}d\theta, \quad \theta^4 = e^{\nu/2}dt, \tag{2.3}$$

where the three functions λ, ν and σ may depend on r and t. The connection 1-forms corresponding to the basis 1-forms θ^i are given in Appendix I.

Now it is time to make assumptions of a physical character, as it is impossible to derive without them any further expressions. Let us state explicitly the assumptions (which are the same as in Kopczyński's 1972 paper):

A. We use comoving coordinates for matter, which means that u^4 is the only non-vanishing component of the four-velocity of matter.

B. The symmetric energy-momentum tensor $\tilde{\mathscr{T}}_i^j$ of matter is that of a perfect fluid: (with p — pressure, and ϱ — energy density):

$$\tilde{\mathscr{T}}^j{}_i = (p+\varrho)u_i u^j - p\delta_i{}^j. \tag{2.4}$$

The canonical 3-form t_i of energy-momentum is related to the symmetric 4-form T_i^j by Trautman's identity:

$$T^j{}_i = \theta^j \wedge t_i - \tfrac{1}{2} Ds_i{}^j, \tag{2.5}$$

where Ds_i^j denotes the covariant exterior derivative of the spin 3-form $s_{ij} = \eta_k s_{ij}^k$, the canonical 3-form t_i is expressed in terms of the canonical tensor t_i^j: $t_i = \eta_j t_i^j$, and the 4-form T_i^j is equal to $\eta \mathscr{T}_i^j$. The η-forms, which are the duals to the basis 1-forms θ^i and to their exterior products, are defined precisely in Trautman's paper (Trautman 1973a).

C. We use the "classical description" of spin, i. e. we assume

$$s^k{}_{ij} = S_{ij} u^k, \qquad \text{with} \qquad S_{ij} u^j = 0, \tag{2.6}$$

where S_{ij} is the antisymmetric tensor of the density of spin, and u_i are the 4-velocity components. In case of spherical symmetry, the tensor S_{ij} has the only non-vanishing component $S_{23} = -S_{32}$, and the set of the Einstein-Cartan equations (2.2) reduces to:

$$Q^4_{23} \overset{\text{def}}{\equiv} Qu^4 = 8\pi G S_{23} u^4 \overset{\text{def}}{\equiv} 8\pi G S u^4. \tag{2.7}$$

The only non-vanishing component of the torsion tensor: Q^4_{23} ($\equiv -Q^4_{32}$) which is denoted briefly by the only letter Q, may depend, in general, on r and on t. It enters some of the expressions for the connection 1-forms ω^i_k which are derived from the first equations of structure giving the torsion 2-form Θ^i in terms of the exterior differentials of the basis 1-forms θ^i and of the connection 1-forms ω^i_k:

$$\Theta^i = d\theta^i + \omega^i_k \wedge \theta^k. \tag{2.8}$$

3. *The Einstein–Cartan equations for the case of spherical symmetry*

With the assumptions listed above, we are able to use the connection 1-forms ω^i_k to calculate the curvature 2-form Ω^i_j from the "second equations of structure":

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i_k \wedge \omega^k_j. \tag{3.1}$$

Now that the curvature 2-form Ω^i_j is associated with the curvature 0-form (Riemann tensor) R^i_{jmn}:

$$\Omega^i{}_j = \tfrac{1}{2} R^i{}_{jmn} \theta^m \wedge \theta^n, \tag{3.2}$$

we find the non-vanishing frame components of the Riemann tensor (see Appendix II), and then we can calculate the Ricci tensor R_i^j, and Einstein's geometric tensor $G_i^j \overset{\text{def}}{=} R_i^j - \tfrac{1}{2}\delta_i^j R$. The latter has the same non-vanishing components as in general relativity (diagonal components and $R_4^1 = -R_1^4$) plus the component $R_3^2 = -R_2^3$ (see Appendix II).

In order to obtain the right-hand side of Eq. (2.1), we have to calculate first the covariant exterior derivative of the spin 3-form s_i^j:

$$Ds_i{}^j = (Dg^{kj}) \wedge \eta_l s^l{}_{ik} + g^{kj}(D\eta_l) s^l{}_{ik} - g^{kj}\eta_l \wedge (Ds^l{}_{ik}). \tag{3.3}$$

By applying the two relations (Trautman 1972a):

$$Dg^{kj} = 0, \qquad D\eta_l = Q^j{}_{lj}\eta, \tag{3.4}$$

and the classical description of spin, which leads to the sum $Q^j_{lj} = 0$, we find that in Eq. (3.3) only the third term contributes, and since s^l_{ik} is a tensor-valued 0-form, the covariant exterior derivative D is reduced here to the usual covariant derivative $\theta^k\nabla_k$. With the resulting expression for Ds^j_i, and with the symmetric energy-momentum 4-form $T^j_i = \eta\tilde{\mathscr{T}}^j_i$ where $\tilde{\mathscr{T}}^j_i$ is given by Eq. (2.4), we get from the identity (2.5) the following expression for the canonical energy-momentum tensor t^j_i:

$$t^j{}_i = \tilde{\mathscr{T}}^j{}_i + \frac{1}{16\pi G}(g^{3j}\delta^2_i - g^{2j}\delta^3_i)\nabla_l(Qu^l). \tag{3.5}$$

With this energy-momentum tensor, the set of Einstein-Cartan equations (2.1) has the following form:

$$-8\pi G\tilde{\mathscr{T}}^1_1 = 8\pi Gp = \left[\frac{1}{4}\sigma'(\sigma'+2\nu')+\frac{\sigma'+\nu'}{r}+\frac{1}{r^2}\right]e^{-\lambda}$$
$$+\left[-\ddot{\sigma}-\frac{3}{4}\dot{\sigma}^2+\frac{\dot{\nu}\dot{\sigma}}{2}\right]e^{-\nu}-\frac{e^{-\sigma}}{r^2}+\frac{Q^2}{4}, \tag{3.6}$$

$$-8\pi G\tilde{\mathscr{T}}^2_2 = -8\pi G\tilde{\mathscr{T}}^3_3 = 8\pi Gp = \left[\frac{\sigma''+\nu''}{2}+\frac{(\sigma')^2+(\nu')^2+\nu'(\sigma'-\lambda')-\lambda'\sigma'}{4}\right.$$
$$\left.+\frac{2\sigma'+\nu'-\lambda'}{2r}\right]e^{-\lambda}+\left[-\frac{\ddot{\sigma}+\ddot{\lambda}}{2}-\frac{\dot{\sigma}^2+\dot{\lambda}^2}{4}+\frac{\dot{\nu}(\dot{\sigma}+\dot{\lambda})-\dot{\sigma}\dot{\lambda}}{4}\right]e^{-\nu}+\frac{1}{4}Q^2, \tag{3.7}$$

$$8\pi G\tilde{\mathscr{T}}^4_4 = 8\pi G\varrho = \left[-\sigma''-\frac{3}{4}(\sigma')^2+\frac{\lambda'\sigma'}{2}\right.$$
$$\left.+\frac{\lambda'-3\sigma'}{r}-\frac{1}{r^2}\right]e^{-\lambda}+\left[\frac{1}{2}\dot{\lambda}\dot{\sigma}+\frac{\dot{\sigma}^2}{4}\right]e^{-\nu}+\frac{e^{-\sigma}}{r^2}+\frac{1}{4}Q^2, \tag{3.8}$$

$$\left[\dot{\sigma}'+(\dot{\sigma}-\dot{\lambda})\left(\frac{1}{r}+\frac{\sigma'}{2}\right)-\frac{\dot{\sigma}\nu'}{2}\right]e^{-\frac{\lambda+\nu}{2}} = 0. \tag{3.9}$$

It is interesting to notice that the first three equations for the diagonal elements of the perfect fluid tensor differ from the corresponding expressions in general relativity only by the additive contributions 1/4 Q^2 to the right-hand side. Equation (3.9) is just the condition R^1_4, the same as in general relativity. And the equation $R^2_3 = -8\pi G t^2_3$ turns into an identity, and gives no new condition. This may be easily seen from an inspection of Eq. (3.5) and the corresponding formula for R^2_3 at the end of Appendix II; this equation, being the only equation involving the antisymmetric component of t_{ij}, is essentially identical to Trautman identity (2.5).

It may be useful to give at the same time the contracted Bianchi identities for the spherically symmetric case. From their general form (Trautman 1973):[1]

$$Dt_j = Q^k{}_{jl}\theta^l \wedge t_k - \tfrac{1}{2} R^{kl}{}_{jm}\theta^m \wedge s_{kl},$$

$$Ds_{kl} = \theta_l \wedge t_k - \theta_k \wedge t_l, \tag{3.10}$$

we get in our notation:

$$8\pi G \nabla_l[(p+\varrho)u^l u_j - p\delta^l_j] = \tfrac{1}{2} Q\nabla_l(Qu^l)u_j + QR^{23}{}_{lj}u^l. \tag{3.11}$$

This is equivalent to the following two relations (for $j = 1$, and $j = 4$):

$$8\pi G\left[p' + \frac{v'}{2}(p+\varrho)\right] = \frac{1}{2} Q\left(Q' + \frac{v'}{2} Q\right), \tag{3.12}$$

$$8\pi G\left[\dot{\varrho} + \left(\frac{\dot{\lambda}}{2} + \dot{\sigma}\right)(p+\varrho)\right] = \frac{1}{2} Q\left[\dot{Q} + \left(\frac{\dot{\lambda}}{2} + \dot{\sigma}\right) Q\right]. \tag{3.13}$$

The terms of the right-hand sides of these identities make the essential difference with the corresponding identities in general relativity. It is possible to demand, following Kopczyński (1972) that the theory under study differs as little as possible from the known theory in a Riemannian spacetime. Thus we had to demand that the left and right-hand sides of the contracted Bianchi identities should be separately equal to zero. Thus we should have the standard continuity equations of general relativity:

$$p' = -\frac{v'}{2}(p+\varrho) \quad \text{and} \quad \dot{\varrho} = -\left(\frac{\dot{\lambda}}{2} + \dot{\sigma}\right)(p+\varrho), \tag{3.14}$$

and in addition, we get the two continuity equations for spin (which are written in terms of torsion Q but this is directly proportional to the non-vanishing spin component S_{23}):

$$Q' + \frac{v'}{2} Q = 0, \tag{3.15a}$$

$$\dot{Q} + \left(\frac{\dot{\lambda}}{2} + \dot{\sigma}\right) Q = 0. \tag{3.15b}$$

Eq. (3.15b) was already used by Kopczyński (1972) and by Trautman (1973b) in cosmology where it well deserves the name of "spin conservation" equation. It constitutes an additional constraint upon the distribution of spin, and it is not necessary to assume it, as well as its counterpart — Eq. (3.15a). It will be shown even here that we are forbidden, in general, to assume the validity of any of these equations for Einstein–Cartan spheres. One may

[1] The second identity is of no use for our aims here, and we give it only for completeness sake.

try to use Eq. (3.15a, b) as a condition to be imposed upon the function Q in the search for a solution of the system of equations (3.6)–(3.9), as we have now an additional degree of freedom when comparing the situation with general relativity. And this is all one can say about this constraint.

4. Solutions of cosmological interest

As it appears that the most important field of application of the Einstein–Cartan theory of gravitation may be the cosmology, let us start with it. Even now, when sophisticated cosmological models with many parameters, inhomogeneity, anisotropy, magnetic fields etc. are available within the framework of general relativity, the simple class of Friedmann models forms the basis of cosmology. These models, as useful as they prove for today research, do exhibit, however, a feature which appears unesthetic as it points to the uselessness of the theory in certain situations. This is the singularity: one is unable to avoid at least one moment in the history of the universe with an infinite density of matter. One knows about this fact since almost half a century. A long time one did not worry about it since it appeared first that the singularity might be a consequence of the high symmetries of the models involved, and of the idealizations in choosing the energy-momentum tensor of matter. A systematic search for cosmological models with less symmetries revealed that one cannot get rid so easily of the singularities. Finally, global techniques have been applied to the study of singularities, and it has been shown that under very general assumptions one is unable to prevent the occurrence of singularities in general relativity (Penrose 1965, Geroch 1966; Hawking 1966a, b; Hawking and Penrose 1970). Among these assumptions there is the following inequality that should be valid at every point of the manifold, and for any timelike vector u^{ν}: $-G_{\mu\nu}u^{\mu}u^{\nu} \geqslant 0$, where $G_{\mu\nu}$ is Einstein's geometric tensor (defined in the preceding section). For a perfect fluid this leads to the inequality: $G(p+\varrho) \geqslant 0$ which appears to be rather trivial. But the form of the inequality which is the necessary assumption to guarantee the occurrence of a singularity, is able to suggest the way that should be followed in order to avert the singularity. It is useful to violate the "strong energy condition", as the inequality above is called, and to try to find whether a singularity occurs under such modified conditions. Generally speaking, this is the search for the physical consequences of the introduction of repulsive forces into the theory. This may possibly be taken into account by introducing the bulk viscosity term into the frame of Friedmannian cosmology. It has been shown (Klimek 1971, Heller et al. 1973) that under such conditions, the singularities are not excluded automatically from the theory, they may be removed, however, in some cases. An alternative way to avoid singularities was suggested by Trautman (1973a) according to whom the necessary repulsion might originate in the spin-spin interaction of matter. Now, such a self--interaction is a natural consequence of the Einstein–Cartan theory (Hehl 1973; Hehl and von der Heyde 1973; Kuchowicz 1973). In Kopczyński's (1972) dust model of the universe, this repulsive spin-spin interaction prevented any singularity. Trautman (1973b) estimated a minimum radius of ca 1 cm, and a maximum density of ca 10^{55} g cm^{-3} for a flat universe of 10^{80} neutrons. As unrealistically high as these numbers may be, they may be further reduced e. g. with the help of the two-tensor f–g theory (Isham et al. 1973); what matters, is the fact that by a slight modification of general relativity we are able to prevent the singularity. Therefore it is meaningful to study further generalizations of the Friedmann models in the Einstein–Cartan theory.

Now, it is important to state that these models, in the framework of the Einstein–Cartan theory, cannot be labelled by the adjective "homogeneous". The homogeneity postulate is fulfilled by the metric tensor but not by the torsion tensor; the point $r = 0$ in space is distinguished. May be, this is due to the unphysical assumption of a spherically symmetric spin distribution. At any case, this constitutes a minor trouble when compared with the standard singularities in general relativity. The models to be considered in this section may be thus regarded only as a first approximation to a further study on more realistic, axially symmetric universes.

In dealing with cosmological models, usually we assume $\nu' = 0$. Let us restrict to the system of isotropic coordinates, i. e. $\sigma = \lambda$. Then from comparing the two expressions (3.6) and (3.7) which must equal each other because of the isotropy of pressure we get:

$$\lambda'' = \frac{(\lambda')^2}{2} + \frac{\lambda'}{r}, \tag{4.1}$$

which may be easily solved so that we obtain the Robertson–Walker metric:

$$e^{\lambda} = e^{\sigma} = \frac{R^2(t)}{(1+\frac{1}{4}\,kr^2)^2}, \quad \text{with} \quad k = 0, \ \pm 1. \tag{4.2}$$

Eq. (3.9) is now an identity $0 = 0$, whereas the rest of the Einstein–Cartan equations is now reduced to:

$$8\pi G\varrho = \frac{3}{R^2}(\dot{R}^2+k)+\frac{1}{4}Q^2 = \frac{3}{R^2}(\dot{R}^2+k)+(4\pi GS)^2, \tag{4.3}$$

$$8\pi Gp = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2+k}{R^2} + \frac{1}{4}Q^2 = -\frac{2\ddot{R}}{R} - \frac{\dot{R}^2+k}{R^2} + (4\pi GS)^2. \tag{4.4}$$

We should stress that in numerical estimations we have actually to replace G by G/c^4 which points to the exceptionally small value of the spin-induced term in the cosmological equations under normal conditions. But just this term enables us to get rid of the initial singularity. As usually, dots denote differentiation with respect to the cosmic time t.

Directly from the two equations (4.3) and (4.4) we may deduce the relation:

$$\frac{d}{dt}[(\varrho-2\pi GS)^2R^3]+(p-2\pi GS^2)\frac{dR^3}{dt} = 0 \tag{4.5}$$

which for vanishing spin (and torsion) reduces to the well-known "energy conservation" equation of Friedmannian cosmology. This is just a form of the contracted Bianchi identity for our system; it does not introduce any new information, and points only to a standard interpretation of the constitutive terms in parentheses.

By combining Eq. (4.3) and (4.4), we get the following result:

$$8\pi G\left(p+\frac{\varrho}{3}\right) = -\frac{2\ddot{R}}{R}+\frac{1}{3}(8\pi GS)^2. \tag{4.6}$$

The left-hand side has to be positively definite. In general relativity, where we have no spin-induced term on the right-hand side of this equation, it is easy to conclude that the second time derivative of the scale factor R should be negative, i. e. the expansion of the universe occurs with a deceleration. In our cosmology with spin, we can only say that at sufficiently late evolutionary stages of the universe, when the spin term drops faster to zero than density and pressure, $\ddot{R}$ has to be negative.

4.1. General formulae for universe models filled with either dust or radiation

The formulae (4.3) and (4.4) simplify for two limiting cases often used in cosmology:

A. For dust models, with the condition $p = 0$, we get the following formulae for the square of spin density and for energy density:

$$(4\pi GS)^2 = \frac{2\ddot{R}}{R}+\frac{\dot{R}^2+k}{R^2}, \tag{4.7}$$

$$8\pi G\varrho = \frac{2\ddot{R}}{R}+4\frac{\dot{R}^2+k}{R^2}. \tag{4.8}$$

B. For radiation universes, with the equation of state: $p = 1/3\,\varrho$, we get the analogous set of equations:

$$(4\pi GS)^2 = \frac{3\ddot{R}}{R}+3\frac{\dot{R}^2+k}{R^2}, \tag{4.9}$$

$$8\pi G\varrho = \frac{3\ddot{R}}{R}+6\frac{\dot{R}^2+k}{R^2}. \tag{4.10}$$

As we insert the expressions for dust and for radiative models into Eq. (4.6), we find that in both cases the following inequality for the scale factor should be fulfilled:

$$-R\ddot{R} < 2(\dot{R}^2+k). \tag{4.11}$$

For closed and flat models it follows hence that on longer it is necessary to restrict $\ddot{R}$ to negative values only. For open models with $\ddot{R} < 0$, a lower limit to the square of expansion velocity is found to exist.

Let us compare now the universe models filled with dust with radiative models possessing the same scale factor $R(t)$. Indexes "d" and "r" are used to denote quantities related to dust, or to radiative models, respectively, at the same instant of cosmic time. Then from an inspection of Eq. (4.8) and (4.10) we obtain the remarkable result:

$$\frac{\varrho_r}{\varrho_d} = \frac{3}{2}. \tag{4.12}$$

This result may be interpreted in the following way: Provided we have at a given time t a certain scale factor $R(t)$ which corresponds to a certain energy density ϱ_d in a dust universe, the corresponding energy density ϱ_r for a radiative universe with the same $R(t)$ is equal to 3/2 ϱ_d. For the spin density squares, one obtains in an analogous manner the following relation:

$$\frac{S_r^2}{S_d^2} = \frac{3}{2} + \frac{\frac{3}{2}(\dot{R}^2+k)}{2R\ddot{R}+\dot{R}^2+k}. \tag{4.13}$$

This relation may be helpful in obtaining the spin density of a radiation-filled universe from the spin density of a dust universe with the same scale factor $R(t)$. Relation (4.13) is not of such a simple form as Eq. (4.12) but it may also be useful, especially while needing to construct a radiative model from a given dust universe. Some additional information may be obtained when we restrict ourselves to flat or closed models ($k = 0$ or $+1$); now, depending on the sign of the denominator $D = 2R\ddot{R}+\dot{R}^2+k$ of the second term, we have: $(S_r/S_d)^2 > 1$ for $D > 0$, and $(S_r/S_d)^2 < 1$ for $D < 0$.

It should be remarked here that the two relations (4.12) and (4.13) are specific of the situation in the Einstein–Cartan theory, and have no analogy in the standard general relativistic cosmology. In general relativity, it is impossible for us to have a freedom in the choice of $R(t)$ after we have imposed the equation of state. We have in this case a given differential equation for the function $R(t)$; the form of this equation is different for different equations of state. In the Einstein–Cartan theory, the equation of state was used to express ϱ and S^2 in terms of R, and its derivatives, and R remains, till now, arbitrary. We are able thus to consider dust universes and radiation universes (and possibly, in future, also universes containing a mixture of dust and radiation) with the same scale factor. This was principially impossible in general relativity where the functional dependence of R on t was uniquely related to the equation of state.

4.2. Some kinds of dust-filled universe models

It is a relatively easy task to construct cosmological models corresponding to different assumptions on the relation between matter density and spin density.

4.2.1. $S = C\varrho$, C — const.

With this assumption we insert Eq. (4.8) into (4.7), and we substitute:

$$W = \frac{\dot{R}^2+k}{4R^2}\,C^2, \tag{4.14}$$

wherein W is treated as a function of the new independent variable $x = 4R/C^2$. We get the following differential equation which is at the same time a quadratic equation in the quantity $(x\, dW/dx)$:

$$\left(x\frac{dW}{dx}\right)^2 + (12W-1)x\frac{dW}{dx} + 36W^2 - 3W = 0, \tag{4.15}$$

from which we can express the product $x dW/dx$ as a function of W. This gives after an integration:

$$W = \frac{1}{12}\left\{1 - \left[\left(\frac{x_0}{x}\right)^3 \mp 1\right]^2\right\}, \tag{4.16}$$

where x_0 is an integration constant. As we go now back to the old variables R and $\dot{R}$, we arrive at the following first-order differential equation:

$$R(\dot{R}^2 + k) + \frac{AR_0^3}{R^3} = 2A, \tag{4.17}$$

with the new constants: $A = 1/3\ x_0^3 C$, $R_0 = x_0^3 C^2/4$. This equation may be solved exactly, in general, in terms of elliptic integrals. Only for $k = 0$ (flat space), the integration is straightforward and gives:

$$R = \sqrt[3]{\tfrac{1}{2} R_0^3 + \tfrac{9}{2} A t^2}\,. \tag{4.18}$$

This form of the flat expanding metric guarantees us that there is no initial singularity. Not only in the flat case, but for all other cases too, our model universe cannot shrink to a point. This is evident from Eq. (4.17) where for $R \to 0$ the left-hand side should go to infinity while the right-hand side remains constant. There should thus exist a minimum value of R below which it is impossible to compress the universe. Our exact solution (4.18) is that used by Trautman (1973b) while the differential equation (4.17) is actually Eq. (3) of Kopczyński (1972) who derived it in another way, starting from the spin continuity equation (3.15b) which yielded the following dependence of the spin density on the scale factor:

$$S = \frac{S_0}{R^3}, \quad S_0 = \text{const.} \tag{4.19}$$

We see thus that our assumption of a constant ratio of the spin density to matter density corresponds exactly to the assumption of "spin conservation". All the other models to be presented for dust universes do not fulfil this assumption (which is, of course, one of the many possibilities that could be assumed for the behaviour of S).

4.2.2. $S = C\varrho$, $C = C(t)$, $k = 0$

Let us now derive two cosmological models under the assumption that C depends explicitly on time. It is easiest to perform calculations in a flat space ($k = 0$). From Eq. (4.7) and (4.8) we get, with the substitution $X(t) = \dot{R}/R$:

$$-C^4\dot{X}^2 + (2 - 6C^2X^2)C^2\dot{X} + 3C^2X^2 - 9C^4X^4 = 0 \tag{4.20}$$

which may be treated as an algebraic equation in $C^2\dot{X}$, so we have:

$$C^2\dot{X} = 1 - 3C^2X^2 \pm \sqrt{1 - 3C^2X^2}\,. \tag{4.21}$$

In order to perform easily the integration, let us introduce such an auxiliary function $f(t)$ that we have:

$$\sqrt{1-3C^2X^2} = f(t) \quad \text{i.e.} \quad C^2 = \frac{1-f^2(t)}{3X^2}\,. \tag{4.22}$$

For a given functional form of $f(t)$, we obtain the following general solution of Eq. (4.21):

$$\frac{1}{X} = \frac{1}{X_0} + 3\int \frac{f\,dt}{f\mp 1}\,, \tag{4.23}$$

where X_0 is an integration constant. Results for two specific forms of the function $f(t)$ are given in Table I. The constraints upon the parameters follow from a discussion of the physical behaviour of S^2, ϱ and R. It is reasonable to demand that the quantities S^2, and ϱ should decrease with time, while the expansion rate $\dot{R}/R$ should be positive. These conditions give the constraints from Table I. An additional constraint: $\alpha > 1$ is derived for solution II when we wish that the expansion of the universe should decelerate ($\ddot{R} < 0$).

TABLE I

Two flat models of dust universe

	Model I	Model II
"Ansatz"	$f = \dfrac{1}{1+e^{\alpha t}}$	$f = \dfrac{1}{1-\alpha}$
Scale factor $R(t)$	$R_0 e^{X_0 t}\left(\dfrac{1}{X_0} + \dfrac{3}{\alpha} e^{-\alpha t}\right)^{\frac{X_0}{\alpha}}$	$R_0 \left\lvert \alpha t + \dfrac{1}{X_0}\right\rvert^{\alpha}$
$8\pi G\varrho$	$\dfrac{6(1+e^{-\alpha t})}{\left(\dfrac{1}{X_0} + \dfrac{3}{\alpha} e^{-\alpha t}\right)^2}$	$\dfrac{2(3-\alpha)}{\left(\alpha_0 t + \dfrac{1}{X_0}\right)^2}$
$(4\pi GS)^2$	$\dfrac{3(1+2e^{-\alpha t})}{\left(\dfrac{1}{X_0} + \dfrac{3}{\alpha} e^{-\alpha t}\right)^2}$	$\dfrac{3-2\alpha}{\left(\alpha_0 t + \dfrac{1}{X_0}\right)^2}$
Constraints on parameters	$X_0 > 0$, $\alpha > 9X_0$	$X_0 > 0$, $3/2 > \alpha > 0$

A fault of model I is the permanently positive sign of $\ddot{R}$; in this model we may introduce the additional condition that the spin density should go faster to zero than the matter density. This is guaranteed by a throughout negative sign of the time derivative of $C^2(t)$ which gives the additional restriction upon one of the parameters involved: $\alpha < 15\,X_0$.

The procedure of deriving exact solutions may be repeated for other assumptions concerning the arbitrary function $f(t)$. It is necessary, of course, to look where the quantities S^2 and ϱ are positively definite, diminishing functions of the cosmic time t, while the conditions upon the scale factor are: $R > 0$, $\dot{R} > 0$, and for most of the time, $\ddot{R} < 0$.

4.2.3. $S^2 = C\varrho$

Now, from the proportionality of the expressions (4.7) and (4.8), we get the simple equation:

$$\alpha(\dot{R}^2+k)+R\ddot{R} = 0, \quad \text{with} \quad \alpha \overset{\text{def}}{=} \frac{1-8\pi GC}{2(1-2\pi GC)}, \tag{4.24}$$

where α is constant for a constant C, and depends on t when C is a function of t. A first integral of this equation is:

$$\dot{R}^2+k = R_0^2 \exp\left[-2\int\frac{\alpha dR}{R}\right], \tag{4.25}$$

where R_0 is a constant. Let us consider first the case when C is constant (but such that neither the nominator nor the denominator of α are zero; otherwise we have uninteresting cases of an empty universe or a universe with a zero value of deceleration). Eq. (4.25) is simplified to:

$$\dot{R}^2+k = \left(\frac{R_0}{R}\right)^{2\alpha}, \tag{4.25a}$$

and may be solved exactly for arbitrary α in a flat spacetime. For $C = 0$ (i.e. when the spin density is zero, and we are back in general relativity) we obtain the well-known flat Friedmann dust metric: $R = (\frac{3}{2})^{2/3}R^{1/3}t^{2/3}$, while in the limiting case $C \to \infty$ the dust metric has the following time dependence: $R = \sqrt[3]{3R_0^2}\, t^{1/3}$, i. e. the spin-spin interaction is responsible for a faster expansion of the universe (when compared with the spinless case) in earlier phases, and a relatively slower expansion up from a certain moment. In both cases (and also for other intermediate values of C), the initial singularity remains.

The form of Eq. (4.25) suggests us to regard α as an implicit function of R, and to make a choice of some simple dependence of α on R in order to carry out the integrations. We give below the results for two assumptions on α in a flat spacetime:

I. With $\alpha = \alpha_0 R+1$ we get from integrating Eq. (4.25):

$$e^{\alpha_0 R}\left(\frac{R}{\alpha_0}-\frac{1}{\alpha_0^2}\right) = R_0 t. \tag{4.26}$$

It follows from this relation between R and t that we must have always $\alpha_0 > 0$ (in order to deal with an expanding universe), and that the minimum radius is α_0^{-1}. We have thus a flat expanding universe without singularity. The proportionality function $C(t)$ diminishes from an infinite value at the initial moment of time to an asymptotic value of $1/2\pi G$ for infinite time.

II. With $\alpha = R/(R+\delta)-R/(R+\eta)$ $(\delta \neq \eta)$ we get from an integration of Eq. (4.25):

$$(R+\eta)^{\delta-\eta}e^R = e^{R_0 t}. \tag{4.27}$$

This relation between R and t shows us that we have no initial singularity provided $\delta > \eta > 0$.

4.3. Some models of radiation filled universes

In view of the relations (4.12) and (4.13) it does not seem to be necessary to consider separately and in detail the methods of deriving exact solutions for models of universes filled with radiation (or with ultrarelativistic particles). Provided we have a given function $R = R(t)$ for a dust universe, it is always possible for us to interpret this as the scale factor for a radiative universe, and to calculate the energy and spin densities for this universe from Eq. (4.12) and (4.13). Therefore we restrict ourselves here to considering the case when the continuity equation for spin holds, and to presenting a general method of deriving exact solutions which might equally be used also in the dust case.

4.3.1. A model universe with the spin continuity equation

The model considered here is a direct analogue to Kopczyński's non-singular dust universe (Kopczyński 1972). The spin continuity equation (3.15b) gives us the result (4.19) which is now inserted into the equation of state $p = (1/3)\varrho$, in which the left-hand side is given by Eq. (4.4), and the right-hand side — by Eq. (4.10). Thus, we obtain the following differential equation:

$$\ddot{R}R+\dot{R}^2+k-\frac{(4\pi GS_0)^2}{3R^4} = 0. \tag{4.28}$$

A first integral of this equation has the form:

$$R^2(\dot{R}^2+k)+\frac{(4\pi GS_0)^2}{3R^2} = A \tag{4.29}$$

which may be compared with the corresponding Eq. (4.17) for the dust case. The difference with the Friedmann equation of general relativity and of Newtonian cosmology is again, as in the case of dust, in the spin-induced term $(4\pi GS_0)^2/3R^2$. The interpretation of the "centrifugal potential energy" (Kopczyński 1972) remains, we find only a slightly different dependence of this term on R. But singularity is avoided, again.

Like in the case of dust, this equation is easily integrable in a flat space. The explicit solution then has the form:

$$[\sqrt{AR^2-(8\pi GS_0)^2}+\sqrt{A}\,R]^{(8\pi GS_0)^2}\exp\,[\sqrt{AR^2-(8\pi GS_0)^2}\,\sqrt{A}\,R] = e^{2t}, \tag{4.30}$$

which is much more complicated than the corresponding dust solution (4.18). In the limit of $S_0 = 0$, i.e. when spin and torsion vanish, we get the well-known Friedmannian behaviour: $R = R_0 t^{1/2}$.

4.3.2. The method of the auxiliary function

We may make use directly of (4.9) and (4.10)[2]. The ambiguity in chooosing one of the functions involved is shifted upon a choice of some function $H(t)$ which is defined to be equal to:

$$H(t) = \dot{R}^2+k. \tag{4.31}$$

[2] Or alternatively, of Eq. (4.7) and (4.8), if we like to derive a dust model.

We choose this function so that it is easy to integrate Eq. (4.22). Then we calculate the second derivative of R, and we insert our expressions into Eq. (4.9) and (4.10). We have to consider, finally, the behaviour of the physical quantities involved. Though a majority of solutions generated in this way will have some faults, it is a very straightforward and simple method of producing new exact solutions. Let us give an example. With the simple "Ansatz":

$$H = \left(\frac{\alpha+\beta t}{\gamma+\delta t}\right)^2, \tag{4.32}$$

we get for a flat universe the following time dependence of the scale factor:

$$R = R_0 + \frac{\beta}{\delta^2}(\gamma+\delta t) + \frac{\alpha\delta-\beta\gamma}{\delta^2}\ln(\gamma+\delta t), \tag{4.33}$$

and the expressions for the physical quantities:

$$8\pi G\varrho = \frac{3}{R^2(\gamma+\delta t)^2}\left\{(\beta\gamma-\alpha\delta)\left[R_0+\frac{\beta}{\delta^2}(\gamma+\delta t)\right]+2(\alpha+\beta t)^2-\frac{(\beta\gamma-\alpha\delta)^2}{\delta^2}\ln(\gamma+\delta t)\right\} \tag{4.34}$$

$$(4\pi GS)^2 = \frac{3}{R^2(\gamma+\delta t)^2}\left\{(\beta\gamma-\alpha\delta)\left[R_0+\frac{\beta}{\delta^2}(\gamma+\delta t)\right]+(\alpha+\beta t)^2-\frac{(\beta\gamma-\alpha\delta)^2}{\delta^2}\ln(\gamma+\delta t)\right\}. \tag{4.35}$$

A positive sign of $\dot{R}$ and negative sign of $\ddot{R}$ are achieved with the following condition imposed on the four positive parameters α, β, γ and δ:

$$\beta\gamma-\alpha\delta < 0. \tag{4.36}$$

5. Solutions of interest for relativistic astrophysics, and the correspondence theorem

It was necessary for us to derive explicitly several cosmological solutions, because we wanted to show that they can be free of singularity in the Einstein-Cartan theory, and it was thus meaningless to try to produce these solutions in some way from known solutions of general relativity. Now, as we go over to consider solutions for individual celestial bodies (static or pulsating, but always of dimensions exceeding their Schwarzschild radius!), situation changes. Why to start a search for new, exact solutions of the Einstein-Cartan equations, when there is at hand a sufficient amount of suitable exact solutions of general relativity, and one should only think over about a way how to adapt them to the conditions of the Riemann-Cartan geometry.

In the case of the Einstein-Cartan theory we should think only of physically meaningful internal solutions, because the theory reduces to general relativity for empty space, and from Birkhoff's theorem we know that in the spherically symmetric case the only meaningful solution for vacuum is the well-known external Schwarzschild solution. Thus there remains only the task of a search for internal solutions. This task is facilitated when we turn to write down the pressure isotropy condition, i.e. the equality of the expressions given by the right-hand sides of Eq. (3.6) and (3.7). We get the following relation:

$$\left[\nu''+\sigma''+\frac{\nu'}{2}(\nu'-\lambda')-\frac{\sigma'}{2}(\nu'+\lambda')-\frac{\nu'+\lambda'}{r}-\frac{2}{r^2}\right]e^{-\lambda}$$
$$+\left[\ddot{\sigma}-\ddot{\lambda}+\dot{\sigma}^2+\frac{\dot{\lambda}}{2}(\dot{\nu}-\dot{\lambda})-\frac{\dot{\sigma}}{2}(\dot{\nu}+\dot{\lambda})\right]e^{-\nu}+\frac{2}{r^2}e^{-\sigma}=0, \tag{5.1}$$

which is exactly the same as in general relativity where it was used to derive new exact solutions (Kuchowicz 1970, 1971). This is a relation between the three metric functions: λ, ν and σ, and their derivatives. One may call Eq. (5.1) the fundamental self-consistence relation for a metric corresponding to a perfect fluid configuration. The metric functions must fulfil identically this relation.

Now we may take a sufficiently useful (i.e. non-singular, smooth, etc.) metric of general relativity. We are allowed to regard this metric as describing the manifold of the Einstein-Cartan theory (in case of spherical symmetry, of course!), because the fundamental relation (5.1) is fulfilled in general relativity, and in the Einstein-Cartan theory for the classical description of spin[3]. When ϱ_E and p_E now denote the expressions for energy density and pressure in general relativity — given in terms of the three mentioned metric functions and their derivatives, then the corresponding expressions for the same physical quantities in the Einstein-Cartan theory are obtained from these expressions with the additive spin-induced terms:

$$8\pi G\varrho = 8\pi G\varrho_E+\tfrac{1}{4}Q^2,$$
$$8\pi Gp = 8\pi Gp_E+\tfrac{1}{4}Q^2. \tag{5.2}$$

This is evident when we look upon Eq. (3.6)...(3.8). The additional equation (3.9), which arises in the non-static case, is exactly the same in general relativity and in the Einstein--Cartan theory. Thus we are able to find that we may take over a given metric from general relativity into the Einstein-Cartan theory, where we consider in addition the torsion Q (as a new degree of freedom), and where we get the additional terms $(1/4)Q^2$ in the expressions for energy density and for pressure.

Now, what to say about possible constraints on the torsion? It is obvious to demand that the whole pressure p vanishes at a boundary $r = r_b$ of the spherical matter distribution. But we know that already the general-relativistic contribution p_E vanishes here. So we have to demand separately the torsion-induced term $(1/4)Q^2$ to vanish also here, which leads to the condition upon the hitherto arbitrary function Q:

$$Q(r_b) = 0. \tag{5.3}$$

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[3] It will be shown later, that when we resign from this description and use the most general form of the torsion tensor compatible with spherical symmetry, the generalized pressure isotropy condition contains some new terms in addition to those already present in Eq. (5.1).

This condition may be supplemented even by a demand that also the normal derivative of Q with respect to the boundary of the sphere is equal to zero, which would guarantee us a sufficiently smooth behaviour of Q in this region. At the same time, the two geometries, the Riemannian geometry of the external Schwarzschild solution, and the non-Riemannian geometry inside the sphere, are joined together in a quite continuous manner. The initial torsion-free solution is extended into a whole family of solutions for spinning matter, characterized by the functional form of Q (which, in the internal region under study, may depend in some way on r and on t, but should fulfil Eq. (5.3) at the boundary — also when r_b is a function of time). The ambiguity stemming from introducing the new function Q is related to the degree of alignment of spins of matter inside the sphere; this seems to be in accord with the new degrees of freedom introduced into the standard general relativistic approach by the concept of spin and torsion.

When we use the boundary condition (5.3) for torsion, it is evident that the boundary conditions for the metric functions and their derivatives have to be essentially the same as in general relativity. Now, the situation might change if we would like to impose some additional restrictions for spin (or torsion), e.g. if we would like to extend Kopczyński's (1972) treatment with the spin continuity equation (3.15b) from cosmology to the study of individual celestial bodies. In this case we had to take into account also the analogous equation (3.15a). From an integration of the two equations we get the two expressions for the function Q:

$$Q = R(r)e^{-\left(\sigma + \frac{\lambda}{2}\right)} \quad \text{and} \quad Q = T(t)e^{-\nu/2}, \tag{5.4}$$

where $R(r)$ is some arbitrary function of r, while $T(t)$ is some arbitrary function of t. Since the two expressions above should give the same physical quantity Q, we obtain the following restriction for the form of the dependence of the metric functions on r and on t:

$$e^{\sigma + \frac{\lambda - \nu}{2}} = \frac{R(r)}{T(t)}. \tag{5.5}$$

This condition may be very restrictive in an adaptation of general relativistic solutions to our theory. The factorization condition (5.5) is very inhibitive for almost all known metrics corresponding to non-static spheres, it chooses a too narrow subclass for an extension into the Einstein-Cartan theory. It was not so bad in cosmology (Kopczyński 1972) where only the first of the two equations (5.4) was valid — and this was just the result (4.19). But in the case of the Einstein-Cartan spheres, we have difficulties even with static configurations, when only the second of the two equations (5.4) remains, and the function $T(t)$ is reduced to a constant: $Q = Q_0 e^{-\nu/2}$. We are unable to fulfil Eq. (5.3), because it appears to be impossible to demand that the function $e^{-\nu}$ goes to zero at the surface of the sphere. Thus a jump of the torsion at the surface follows, and this leads not only to a destruction of the smooth continuity of geometry but also to difficulties with the behaviour of pressure. It is difficult for us to reconcile with a jump of pressure at the surface, and the only apparent possibility for us would be to demand that the whole expression for pressure including the Q^2 term (Eq. (3.6)) should vanish at the surface, without having

$Q(r_b)$ equal to zero. This contradicts the standard Lichnerowicz's boundary conditions, and the latter should have to be relaxed. Such a procedure with modified boundary conditions was recently applied by Prasanna (1973) in his studies on introducing torsion into some known Tolman (1939) solutions. It appears therefore that we are not allowed to apply the "spin conservation equations" (3.15a and b) for finite spheres as this might lead to inconsistencies and contradictions with well established boundary conditions. The equations (3.15a) and (3.15b) are, besides, no necessary conditions to be imposed on torsion (or spin).

Our results may be summarized in the following

Correspondence theorem:

Every spherically symmetric metric of a Riemannian spacetime of general relativity may be regarded also as the metric of a family of spacetimes with torsion, with the set of boundary or asymptotic conditions of general relativity being replaced by the corresponding set of conditions for the Einstein-Cartan theory. Exact expressions for the pressure and density corresponding to this metric in the Einstein-Cartan theory are obtained from the respective expressions in general relativity theory by the substitution (5.2).

This enables us to apply practically every spherically symmetric exact solution of the Einstein equations in the Einstein-Cartan theory. We did not use this correspondence theorem for cosmological solutions, as we wished to derive exact, singularity-free solutions, essentially different from the Friedmann solutions in their behaviour near the singularity. But this may be applied to any other solutions, even to those with an infinite radius. Let us give first the example of such a solution. This is a relativistic fluid sphere resembling a classical Emden polytrope with an infinite radius but still finite mass $M = 2A/G\sqrt{B}$, of index 5. In isotropic coordinates ($\lambda = \sigma$) we have the solution (Buchdahl 1964):

$$e^{\lambda} = (1+f)^4, \quad e^{\nu} = \left(\frac{1-f}{1+f}\right)^2, \quad \text{where} \quad f = A(1+Br^2)^{-1/2}, \tag{5.6}$$

with the constants A and B. We have no boundary conditions, and the value of mass results from an asymptotic condition: Very far from the origin, the solution behaves as an external Schwarzschild solution corresponding to a central mass M. The density and pressure in the Einstein-Cartan theory are:

$$8\pi G\varrho = 12\frac{B}{A^4}\left(\frac{f}{1+f}\right)^5 + \frac{1}{4}Q^2, \quad 8\pi Gp = \frac{48f^6}{(1-f)(1+f)^5} + \frac{1}{4}Q^2, \tag{5.7}$$

where Q may be completely arbitrary, as the condition (5.3) does not apply for an infinite sphere. In case we restrict ourselves by the "spin conservation equation", the hitherto arbitrary function Q is to be specified:

$$Q = Q_0 e^{-\nu/2} = Q_0\left(\frac{1+f}{1-f}\right). \tag{5.8}$$

We see that the solution corresponding to the Kopczyński-Prasanna treatment is not the most general one that is possible in this case.

The role of boundary conditions can be studied for the example of the internal Schwarzschild solution which is given in the following form after Tolman (1939):

$$e^{-\lambda} = 1 - \frac{r^2}{R^2}, \quad e^{\nu} = \left[A - B\sqrt{1 - \frac{r^2}{R^2}}\right]^2. \tag{5.9}$$

With the same form of the metric in a spacetime with torsion, we get the following expressions for density and pressure:

$$8\pi G\varrho = \frac{3}{R^2} + \frac{1}{4}Q^2, \tag{5.10}$$

$$8\pi Gp = \frac{1}{R^2}\,\frac{3B\sqrt{1-\dfrac{r^2}{R^2}} - A}{A - B\sqrt{1-\dfrac{r^2}{R^2}}} + \frac{1}{4}Q^2. \tag{5.11}$$

It is interesting to find that the density no longer is constant. We have the standard set of boundary conditions:

$$e^{\nu(r_b)} = e^{-\lambda(r_b)} = 1 - a, \quad e^{\nu(r_b)}\left.\frac{d\nu}{dr}\right|_{r=r_b} = a, \quad Q(r_b) = 0, \tag{5.12}$$

where the parameter a denotes mass concentration (defined as the ratio of the Schwarzschild radius to the geometric radius r_b of the sphere in cannonical coordinates ($\sigma = 0$)). From these conditions we obtain the standard values of the integration constants:

$$A = \frac{3}{2}\sqrt{1-a}, \quad B = \frac{1}{2}, \quad R^2 = \frac{r_b^2}{a}, \tag{5.13}$$

and q is expressed in terms of a completely arbitrary function q and its boundary value:

$$Q(r) = q(r) - q(r_b). \tag{5.14}$$

The family of solutions which in the Einstein-Cartan theory correspond to the internal Schwarzschild solution of general relativity is now characterized by the inherent ambiguity in $q(r)$; estimations on q can be made only on the basis of a study on the alignment of spins inside the sphere, and the like.

Now, a reduction in the ambiguity occurs as we use Eq. (3.15a). Now Q is specified:

$$Q = Q_0\left[A - B\sqrt{1 - \frac{r^2}{R^2}}\right]^{-1}, \tag{5.15}$$

but it cannot vanish for $r = r_b$. Now the last two boundary conditions (5.12) should be replaced by $p(r_b) = 0$, and this gives three of the integration constants in terms of the fourth one (Q_0):

$$R^2 = \frac{r_b^2}{a}, \quad A = \left[\frac{3}{2} - \frac{Q_0^2 r_b^2}{8a(1-a)}\right]\sqrt{1-a}, \quad B = \frac{1}{2} - \frac{Q_0^2 r_b^2}{8a(1-a)}. \tag{5.16}$$

In this case the quantity Q_0 remains as a parameter characterizing the degree of alignment of spins inside the sphere. But ν' is now no longer continuous across the surface, and this is hardly justifiable. It appears that the torsion distribution given by Eq. (5.15) is incompatible with the metric of the sphere. We see that the assumption of the "spin conservation equation" is too hard a restriction even for a static sphere.

6. *Final remarks*

It seems that we have pointed in a sufficient way to the possibilities which arise in the Einstein-Cartan theory for cosmology. The models considered in the text are examples of what can be achieved in the latter theory, and they may be regarded as an introduction to more complicated, anisotropic, especially axially symmetric models. It is shown here that the initial singularity can be avoided in many ways for dust, for radiation, and with and without spin conservation.

While many new elements arise in cosmology, in the case of relativistic spheres with a finite radius we have only to adapt known exact solutions from general relativity into our theory. This is straightforward, and a characteristic correspondence theorem for generating exact solutions of the Einstein-Cartan theory from those of general relativity is presented, which makes it practically possible for any solution of Einstein's equations to be taken over into our theory in case of spherical symmetry. Though the effect of torsion may be vanishingly small even for such compact objects like the neutron stars, it appears appropriate that we are able to obtain exact solutions for configurations of a spinning perfect fluid in a spacetime with torsion.

We did not consider here the effect of torsion on collapse, because it seems that this should be studied in detail separately. But let us mention only that from the results for cosmology it is now clear to us that singularity can be avoided also in collapse. The difference with general relativity may arise only inside the region occupied by the collapsing matter configuration. An external observer, outside the Schwarzschild radius of the configuration, does not find any difference because the external Schwarzschild solution is valid in empty space in the two theories. We have found already that an Einstein-Cartan sphere has no spin hair, and this applies equally well to collapsed Einstein-Cartan spheres[4]. It is possible to consider the collapse of a dust distribution of uniform density, by matching the internal geometry of a Friedmannian dust model to the external Schwarzschild geometry. This match holds during the whole evolution of the collapsing configuration. In general relativity, a contraction of the interior up to infinite density occurs. It is easy to see that in the Einstein-Cartan theory we have now in the collapsing region the situation envisaged in Section 4.2: the collapse may be stopped at some sufficiently high density, thanks to the spin-spin repulsive interaction. There occurs the same phenomenon of a minimum radius and maximum density, as in cosmology.

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[4] May this be treated as an analogue to the well-known property of black holes: "A black hole has no hair".

Further generalizations of the approach presented here — to axial symmetry (which is natural for spin), to more components of torsion, and to various equations of state are in preparation.

It is a pleasure to acknowledge the stimulus to these studies from reading the papers of Professors A. Trautman and F. Hehl, and from the nice opportunity to lecture about cosmological applications of the Einstein-Cartan theory at the Erice School of Cosmology and Gravitation.

APPENDIX I

Frame components of the connection 1-forms ω^i_k and the curvature 2-forms Ω^i_k

Below we give the components with respect to the set of basis 1-forms θ^i. As usually, differentiation with respect to t is denoted by a dot, while differentiation with respect to r is denoted by a prime.

A. Connection 1-forms:

$$\omega^1_2 = -\omega^2_1 = -\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda/2}\theta^2,$$

$$\omega^1_3 = -\omega^3_1 = -\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda/2}\theta^3,$$

$$\omega^1_4 = \omega^4_1 = \frac{\dot\lambda}{2}e^{-\nu/2}\theta^1+\frac{\nu'}{2}e^{-\lambda/2}\theta^4,$$

$$\omega^2_3 = -\omega^3_2 = -\frac{1}{r}e^{-\sigma/2}\,\mathrm{ctg}\,\theta\theta^3-\frac{Q}{2}\theta^4,$$

$$\omega^2_4 = \omega^4_2 = \frac{\dot\sigma}{2}e^{-\nu/2}\theta^2-\frac{Q}{2}\theta^3,$$

$$\omega^3_4 = \omega^4_3 = \frac{Q}{2}\theta^2+\frac{\dot\sigma}{2}e^{-\nu/2}\theta^3.$$

B. Curvature 2-forms:

$$\Omega^1_2 = -\Omega^1_2 = \left[\left(-\frac{\sigma''}{2}-\frac{(\sigma')^2}{4}-\frac{\sigma'}{r}+\frac{\lambda'}{2r}+\frac{\lambda'\sigma'}{4}\right)e^{-\lambda}+\frac{\dot\lambda\dot\sigma}{4}e^{-\nu}\right]\theta^1\wedge\theta^2$$

$$-\frac{Q}{4}\dot\lambda e^{-\nu/2}\theta^1\wedge\theta^3+\left[\frac{\dot\sigma'}{2}+\frac{\dot\sigma-\dot\lambda}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right)-\frac{\dot\sigma\nu'}{4}\right]e^{-\frac{\lambda+\nu}{2}}\theta^2\wedge\theta^4$$

$$+\frac{Q}{2}\left(\frac{\nu'-\sigma}{2}-\frac{1}{r}\right)e^{-\lambda/2}\theta^3\wedge\theta^4,$$

$$\Omega^1_3=-\Omega^3_1=\frac{Q}{4}\dot{\lambda}e^{-\nu/2}\theta^1\wedge\theta^2+\left\{\left[\frac{1}{r^2}-\frac{\sigma''}{2}+\left(\frac{\lambda'-\sigma'}{2}-\frac{1}{r}\right)\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\right]e^{-\lambda}\right.$$

$$\left.+\frac{\dot{\lambda}\dot{\sigma}}{4}e^{-\nu}\right\}\theta^1\wedge\theta^3+\frac{Q}{2}\left(\frac{1}{r}+\frac{\sigma'-\nu'}{2}\right)e^{-\lambda/2}\theta^2\wedge\theta^4$$

$$+\left[\frac{\dot{\sigma}'}{2}-\frac{\nu'\dot{\sigma}}{4}+\left(\frac{1}{r}+\frac{\sigma'}{2}\right)\frac{\dot{\sigma}-\dot{\lambda}}{2}\right]e^{-\frac{\lambda+\nu}{2}}\theta^3\wedge\theta^4,$$

$$\Omega^1_4=\Omega^4_1=\left[\left(\frac{\dot{\lambda}\dot{\nu}}{4}-\frac{\ddot{\lambda}}{2}-\frac{\dot{\lambda}^2}{4}\right)e^{-\nu}+\left(\frac{\nu''}{2}-\frac{\nu'\lambda'}{4}+\frac{(\nu')^2}{4}\right)e^{-\lambda}\right]\theta^1\wedge\theta^4$$

$$+Q\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda/2}\theta^2\wedge\theta^3,$$

$$\Omega^2_3=-\Omega^3_2=-\left(\frac{Q\nu'}{4}+\frac{Q'}{2}\right)e^{-\lambda/2}\theta^1\wedge\theta^4$$

$$+\left[\frac{1}{r}e^{-\sigma}+\frac{\dot{\sigma}^2}{4}e^{-\nu}-\left(\frac{1}{r}+\frac{\sigma'}{2}\right)^2e^{-\lambda}+\frac{Q^2}{4}\right]\theta^2\wedge\theta^3,$$

$$\Omega^2_4=\Omega^4_2=\left(\frac{\dot{\sigma}'}{2}+\frac{\dot{\sigma}(\sigma'-\nu')}{4}-\frac{\dot{\lambda}\sigma'}{4}+\frac{\dot{\sigma}-\dot{\lambda}}{2r}\right)e^{-\frac{\lambda+\nu}{2}}\theta^1\wedge\theta^2$$

$$-\frac{1}{2}\left[Q\left(\frac{1}{r}+\frac{\sigma'}{2}\right)+Q'\right]e^{-\lambda/2}\theta^1\wedge\theta^3+\left[\left(-\frac{\ddot{\sigma}}{2}+\frac{\dot{\sigma}\dot{\nu}}{4}-\frac{\dot{\sigma}^2}{4}\right)e^{-\nu}\right.$$

$$\left.+\frac{\nu'}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda}+\frac{Q^2}{4}\right]\theta^2\wedge\theta^4+\frac{1}{2}(Q\dot{\sigma}+\dot{Q})e^{-\nu/2}\theta^3\wedge\theta^4,$$

$$\Omega^3_4=\Omega^4_3=\frac{1}{2}\left[Q\left(\frac{1}{r}+\frac{\sigma'}{2}\right)+Q'\right]e^{-\lambda/2}\theta^1\wedge\theta^2$$

$$+\left[\frac{\dot{\sigma}'}{2}+\frac{\dot{\sigma}(\sigma'-\nu')}{4}-\frac{\dot{\lambda}\sigma'}{4}+\frac{\dot{\sigma}-\dot{\lambda}}{2r}\right]e^{-\frac{\lambda+\nu}{2}}\theta^1\wedge\theta^3-\frac{1}{2}(Q\dot{\sigma}+\dot{Q})e^{-\nu/2}\theta^2\wedge\theta^4$$

$$+\left[\left(-\frac{\ddot{\sigma}}{2}+\frac{\dot{\sigma}\dot{\nu}}{4}-\frac{\dot{\sigma}^2}{4}\right)e^{-\nu}+\frac{\nu'}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda}+\frac{Q^2}{4}\right]\theta^3\wedge\theta^4.$$

APPENDIX II

Riemann tensor, Ricci tensor and curvature scalar for the Riemann-Cartan geometry in case of spherical symmetry

I. Components of the Riemann tensor with respect to the 1-forms θ^i:

$$R^1{}_{212} = -R^1{}_{221} = -R^2{}_{112} = R^1{}_{313} = -R^1{}_{331} = -R^3{}_{113}$$

$$= \left(-\frac{\sigma''}{2} + \frac{\sigma'(\lambda'-\sigma')}{4} + \frac{\lambda'-2\sigma'}{2r}\right)e^{-\lambda} + \frac{\dot\lambda\dot\sigma}{4}e^{-\nu},$$

$$R^1{}_{213} = -R^1{}_{231} = -R^2{}_{113} = -R^1{}_{312} = R^1{}_{321} = R^3{}_{112} = -\tfrac{1}{4}Q\dot\lambda e^{-\nu/2},$$

$$R^1{}_{224} = -R^1{}_{242} = -R^2{}_{124} = R^1{}_{334} = -R^1{}_{343} = -R^3{}_{134} = R^2{}_{412} = -R^2{}_{421}$$

$$= R^4{}_{212} = \left[\frac{\dot\sigma'}{2} + \frac{\dot\sigma-\dot\lambda}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right) - \frac{\dot\sigma\nu'}{4}\right]e^{-\frac{\lambda+\nu}{2}} = R^3{}_{413} = -R^3{}_{431} = R^4{}_{313},$$

$$R^1{}_{234} = -R^1{}_{243} = -R^2{}_{134} = -R^1{}_{324} = R^1{}_{342} = R^3{}_{124} = \frac{1}{2}Q\left[\frac{\nu'-\sigma'}{2} - \frac{1}{r}\right]e^{-\lambda/2},$$

$$R^1{}_{414} = -R^1{}_{441} = R^4{}_{114} = \left(\frac{\nu''}{2} - \frac{\nu'\lambda'}{4} + \frac{(\nu')^2}{4}\right)e^{-\lambda} + \left(\frac{\dot\lambda\dot\nu}{4} - \frac{\ddot\lambda}{2} - \frac{\dot\lambda^2}{4}\right)e^{-\nu},$$

$$R^1{}_{423} = -R^1{}_{432} = R^4{}_{123} = Q\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda/2},$$

$$R^2{}_{314} = -R^2{}_{341} = -R^3{}_{214} = -\frac{1}{2}\left(\frac{Q\nu'}{2} + Q'\right)e^{-\lambda/2},$$

$$R^2{}_{323} = -R^2{}_{332} = -R^3{}_{223} = \frac{e^{-\sigma}}{r^2} - \left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda} + \frac{\dot\sigma^2}{4}e^{-\nu} + \frac{Q^2}{4},$$

$$R^2{}_{413} = -R^2{}_{431} = R^4{}_{213} = R^3{}_{421} = -R^3{}_{412} = -R^4{}_{312}$$

$$= -\frac{1}{2}\left[Q\left(\frac{1}{r}+\frac{\sigma'}{2}\right) + Q'\right]e^{-\lambda/2},$$

$$R^2{}_{424} = -R^2{}_{442} = R^4{}_{224} = R^3{}_{434} = -R^3{}_{443} = R^4{}_{334} = \frac{\nu'}{2}\left(\frac{1}{r}+\frac{\sigma'}{2}\right)e^{-\lambda}$$

$$+ \left(-\frac{\ddot\sigma}{2} + \frac{\dot\sigma(\dot\nu-\dot\sigma)}{4}\right)e^{-\nu} + \frac{Q^2}{4},$$

$$R^2{}_{434} = -R^2{}_{443} = R^4{}_{234} = -R^3{}_{424} = R^3{}_{442} = -R^4{}_{324} = \tfrac{1}{2}(Q\dot\sigma + \dot Q)e^{-\nu/2}.$$

II. Components of the mixed Ricci tensor:

$$R^1{}_1 = R^1{}_{212}+R^1{}_{313}-R^1{}_{414}$$

$$= -\left[\sigma''+\frac{\nu''}{2}+\frac{(\sigma')^2}{2}+\frac{(\nu')^2}{4}+\frac{2\sigma'-\lambda'}{r}-\frac{\lambda'(\nu'+2\sigma')}{4}\right]e^{-\lambda}+\left[\frac{\ddot\lambda}{2}+\frac{(\dot\lambda+2\dot\sigma-\dot\nu)\dot\lambda}{4}\right]e^{-\nu},$$

$$R^2{}_2 = R^2{}_{121}+R^2{}_{323}-R^2{}_{424}$$

$$= -\left[\frac{\sigma''}{2}+\frac{(\sigma')^2}{2}+\frac{\sigma'(\nu'-\lambda')}{4}+\frac{4\sigma'+\nu'-\lambda'}{2r}+\frac{1}{r^2}\right]e^{-\lambda}$$

$$+\left[\frac{\ddot\sigma}{2}+\frac{\dot\sigma(\dot\lambda-\dot\nu)}{4}+\frac{\dot\sigma^2}{2}\right]e^{-\nu}+\frac{e^{-\sigma}}{r^2},$$

$$R^3{}_3 \equiv R^2{}_2,$$

$$R^4{}_4 = R^4{}_{141}+R^4{}_{242}+R^4{}_{343}$$

$$= -\left[\frac{\nu''}{2}+\frac{\nu'(2\sigma'+\nu'-\lambda')}{4}+\frac{\nu'}{r}\right]e^{-\lambda}+\left[\ddot\sigma+\frac{\ddot\lambda}{2}+\frac{\dot\lambda^2}{4}+\frac{\dot\sigma^2}{2}-\frac{(\dot\lambda+2\dot\sigma)\dot\nu}{4}\right]e^{-\nu}-\frac{Q^2}{2},$$

$$R = R^1{}_1+R^2{}_2+R^3{}_3+R^4{}_4$$

$$=\left[-2\sigma''-\nu''-\frac{3}{2}(\sigma')^2-\frac{1}{2}(\nu')^2+\sigma'(\lambda'-\nu')+\frac{\nu'\lambda'}{2}+\frac{2\lambda'-2\nu'-6\sigma'}{r}-\frac{2}{r^2}\right]e^{-\lambda}$$

$$+\left[2\ddot\sigma+\ddot\lambda+\frac{3}{2}\dot\sigma^2+\frac{\dot\lambda^2}{2}+(\dot\lambda-\dot\nu)\dot\sigma-\frac{\dot\nu\dot\lambda}{2}\right]e^{-\nu}+\frac{2e^{-\sigma}}{r^2}-\frac{Q^2}{2}.$$

While the expressions above could be referred either to the 1-forms θ^i or to coordinates, the expressions below are to be understood as referring to the 1-forms only:

$$R^1{}_4 = R^4{}_1 = -2R^1{}_{224} = -\left[\dot\sigma'+(\dot\sigma-\dot\lambda)\left(\frac{1}{r}+\frac{\sigma'}{2}\right)-\frac{\dot\sigma\nu'}{2}\right]e^{-\frac{\lambda+\nu}{2}},$$

$$R^2{}_3 = -R^3{}_2 = R^1{}_{321}+R^4{}_{324} = -\frac{1}{2}\left[\dot Q+Q\left(\dot\sigma+\frac{\dot\lambda}{2}\right)\right]e^{-\frac{\nu}{2}} = -\frac{1}{2}\nabla_l(Qu^l).$$

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