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THE ELECTROMAGNETIC FIELD OF AN OSCILLATING DIPOLE IN SCHWARZSCHILD SPACE-TIME IN LINEAR APPROXIMATION

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The electromagnetic field of an arbitrary source distribution in Schwarzschild space time is given in linear approximation, using the technique of Green's functions. For an oscillating dipole at infinity, the result is the generalized plane wave. This wave gives the usual deflection of light, and is the starting point for the exact solution of the problem of the diffraction of a plane wave at a Schwarzschild Black Hole.

1. Introduction

Our goal is to get a first approximation (linear in the gravitational constant) of a plane electromagnetic wave scattered by a Schwarzschild field. We attack this problem by solving the Maxwell equation for an oscillating dipole situated at infinity.

This paper starts with a short account of notations, the technique of the bitensor Green functions for the vector wave equation in an external gravitational field and the explicit expressions for these bitensor Green functions in Schwarzschild space-time in linear approximation.

The main results are presented in the following order: In Section 2 the spectral shift in Schwarzschild space-time for arbitrary positions and velocities of source and observer, in Section 3 the electromagnetic field (3.14) of an oscillating dipole in the Schwarzschild space-time (the generalized plane wave), and in Section 4 the well-known formula for the deflection of light rays not only in geometrical optics limit, but also for an arbitrary wavelength.

The exact solution of the diffraction of a plane electromagnetic wave at a Schwarzschild Black Hole is given in [5] and [6].

2. Bitensor Green's functions for the vector wave equation in Schwarzschild space-time

To solve Maxwell equations in an external gravitational field

$$B^{mn};n = \frac{1}{c} j^m; \quad B_{\langle mn;k\rangle} = 0, \tag{2.1}$$

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we introduce the four-potentials A_n by $B_{mn} = A_{n;m} - A_{m;n}$. Using the Lorentz condition $A^n{}_{;n} = 0$ we get the vector wave-equation

$$g^{ij}A^k{}_{;i;j} + R^{kl}A_l = -\frac{1}{c}j^k. \tag{2.2}$$

De Witt and Brehme [1] solved the equation by the retarded potentials

$$A_\mu^{\mathrm{ret}} = \frac{1}{c}\int G_{\mu\tilde{\nu}}^{\mathrm{ret}}(x, \tilde{x})j^{\tilde{\nu}}(\tilde{x})d^4\tilde{x}. \tag{2.3}$$

Here the retarded bitensor Green's functions has the structure

$$G_{\mu\tilde{\nu}}^{\mathrm{ret}}(x, \tilde{x}) = \frac{1}{4\pi}\,\Theta[\Sigma(x), \tilde{x}]\,\{\Delta^{1/2}g_{\mu\tilde{\nu}}\delta(\Omega) - v_{\mu\tilde{\nu}}\Theta[-\Omega]\}, \tag{2.4}$$

where $\Omega(x, \tilde{x})$ is one half of the square of the geodesic distance between the two world points x and $\tilde{x}$; δ is the Dirac delta function, and Θ is the Heaviside step function, $\Theta[\Sigma(x),\tilde{x}]$ having the value one for all world points $\tilde{x}$ in the past of the hyper-surface $\Sigma(x)$, and vanishing otherwise. $g_{\mu\tilde{\nu}}$ and $\Delta(x, \tilde{x})$ are the bitensor of parallel displacement and the biscalar of De Witt and Brehme, respectively. The so-called tail-term bitensor $v_{\mu\tilde{\nu}}(x, \tilde{x})$ describes the back-scattering of electromagnetic waves due to the curvature of the space-time. The geometrical meaning of all these bitensors was investigated by Synge [2].

In the linear approximation the metric of Schwarzschild space-time in isotropic coordinates reads

$$ds^2 = \left(1 + \frac{\alpha}{r}\right)(dx^2 + dy^2 + dz^2) - \left(1 - \frac{\alpha}{r}\right)(dx^4)^2, \tag{2.5}$$

and the constitutive parts of Green's functions are found to be [7], [8]

$$\Omega = \frac{1}{2}[a^2 - (x^4 - \tilde{x}^4)^2] + \frac{\alpha}{2}\ln\left[\frac{r + \tilde{r} + a}{r + \tilde{r} - a}\right]\left\{\frac{(x^4 - \tilde{x}^4)^2}{a} + a\right\}, \tag{2.6}$$

$$g_{b\tilde{b}} = 1 + \frac{\alpha}{2}\left(\frac{1}{r} + \frac{1}{\tilde{r}}\right), \quad g_{4\tilde{4}} = -\left[1 - \frac{\alpha}{2}\left(\frac{1}{r} + \frac{1}{\tilde{r}}\right)\right], \tag{2.7}$$

(no summation) $b = \tilde{b} = 1, 2, 3,$

$$g_{b\tilde{4}} = -g_{4\tilde{b}} = -\alpha(x^4 - \tilde{x}^4)\frac{x^b\tilde{r} + \tilde{x}^b r}{2r\tilde{r}[r\tilde{r} + x\tilde{x} + y\tilde{y} + z\tilde{z}]},$$

$$g_{c\tilde{b}} = -\alpha\frac{(r + \tilde{r})\,(x^b\tilde{x}^c - x^c\tilde{x}^b)}{2r\tilde{r}[r\tilde{r} + x\tilde{x} + y\tilde{y} + z\tilde{z}]}, \quad c \neq \tilde{b}, \quad c, \tilde{b} = 1, 2, 3,$$

$$v_{\mu\tilde{\nu}}(x, \tilde{x}) = 0, \quad \Delta = 1, \tag{2.8}$$

with the abbreviations

$$r = [x^2+y^2+z^2]^{1/2}, \qquad \tilde{r} = [\tilde{x}^2+\tilde{y}^2+\tilde{z}^2]^{1/2}, \tag{2.9}$$

$$a = [(x-\tilde{x})^2+(y-\tilde{y})^2+(z-\tilde{z})^2]^{1/2}. \tag{2.10}$$

In all these expressions (2.5)–(2.10) terms of order α^2 or higher are neglected. Formula (2.8) shows that the back-scattering of electromagnetic waves due to the curvature of space-time is a higher order effect. Following Synge [2], we can use (2.6) to get the spectral shift

$$\frac{\tilde{\nu}-\nu}{\tilde{\nu}} = \frac{\Omega_{,\tilde{i}}u^{\tilde{i}}+\Omega_{,i}u^{i}}{\Omega_{,\tilde{i}}u^{\tilde{i}}} \tag{2.11}$$

between the frequencies $\tilde{\nu}$ and ν of emission and absorption, valid for arbitrary positions $(\tilde{w}, w)$ and velocities $(\tilde{\mathfrak{w}}, \mathfrak{w})$ of source and observer

$$\frac{\tilde{\nu}-\nu}{\tilde{\nu}} = 1 - \frac{\sqrt{1-\frac{\alpha}{\tilde{r}}-\left(1+\frac{\alpha}{\tilde{r}}\right)\frac{\tilde{\mathfrak{w}}^2}{c^2}}}{\sqrt{1-\frac{\alpha}{r}-\left(1+\frac{\alpha}{r}\right)\frac{\mathfrak{w}^2}{c^2}}} \cdot \frac{\frac{\mathfrak{w}}{c}\left[w-\tilde{w}+\frac{\alpha}{r}\,\frac{(w-\tilde{w})(r-\tilde{r})r-(w-\tilde{w})^2 w}{r\tilde{r}+w\tilde{w}}\right]-|w-\tilde{w}|}{\frac{\tilde{\mathfrak{w}}}{c}\left(w-\tilde{w}+\frac{\alpha}{\tilde{r}}\,\frac{(w-\tilde{w})(r+\tilde{r})\tilde{r}+(w-\tilde{w})^2\tilde{w}}{r\tilde{r}+w\tilde{w}}\right)-|w-\tilde{w}|}. \tag{2.12}$$

For the two trivial cases $\mathfrak{w} = \tilde{\mathfrak{w}} = 0$ (source and observer both at rest in the Schwarzschild field) and $\alpha = 0$, $w = \mathfrak{w} = 0$ (observer at rest at the flat space-time) we get from (2.12) the text-book formula for gravitational redshift in Schwarzschild space-time, and Doppler shift in flat space-time, respectively. Another way to calculate the spectral shift is given by Jordan, Ehlers and Kundt [3].

3. *The electromagnetic field of an oscillating dipole in Schwarzschild space-time*

We investigate the field of an oscillating charge situated at

$$x^{\tilde{\alpha}} = [\varepsilon \sin\langle\omega\{\tilde{t}+\varphi\}\rangle, 0, \tilde{z} = \text{const.}\ c\tilde{t}]. \tag{3.1}$$

Later on we will choose the arbitrary function φ in a suitable manner.

From the four-potentials for a moving charge in a given gravitational field (I)

$$A_\mu^{\text{ret}} = \frac{e}{4\pi}\{\Delta^{1/2} g_{\mu\tilde{\nu}}\dot{x}^{\tilde{\nu}}(\Omega_{,\tilde{\beta}}\dot{x}^{\tilde{\beta}})^{-1}\}_{\tilde{\tau}=\tilde{\tau}_{\text{ret}}} + \frac{e}{4\pi}\int_{\tau_{\text{ret}}}^{-\infty} v_{\mu\tilde{\nu}}\dot{x}^{\tilde{\nu}}\,d\tilde{\tau}, \tag{3.2}$$

we get with (2.5), (2.6), (2.7) and (2.8) from (3.2)

$$\frac{4\pi}{e}A_\mu^{\text{ret}} = g_{\mu\tilde{\nu}}\dot{x}^{\tilde{\nu}}(\Omega_{,\tilde{\beta}}\dot{x}^{\tilde{\beta}})^{-1}\Big|_{x^{\tilde{4}}=x^{\tilde{4}}_{\text{ret}}}, \tag{3.3}$$

and the retardation condition

$$\Omega(x^\mu, \tilde{x}^\mu(\tilde{\tau}_{ret})) = 0,\ x^4 - \tilde{x}^4_{ret} > 0, \tag{3.4}$$

is equivalent to

$$x^4 - \tilde{x}^4_{ret} = a + \alpha \ln \frac{r+\tilde{r}+a}{r+\tilde{r}-a}. \tag{3.5}$$

We get the electromagnetic field of an oscillating charge situated at infinity (the generalized plane electromagnetic wave) from (3.3) by the following limiting process:

$$\begin{aligned} e &\to \infty, & \frac{e}{\tilde{z}} = -C = \text{const.} \\ \tilde{z} &\to -\infty, & \end{aligned} \tag{3.6}$$

Because of the back-scattering of electromagnetic waves we have to limit ourselves to the far field

$$\frac{\alpha}{r} \ll 1, \tag{3.7}$$

and to the region outside the geometrical shadow of the Black Hole

$$r\tilde{r} + x\tilde{x} + z\tilde{z} = O(\tilde{z}). \tag{3.8}$$

Using (2.6) and (3.5) we get

$$\tilde{t}_{ret} = t - \frac{z}{c} + \frac{\alpha}{c} \ln [r-z] + \frac{\tilde{z}}{c} - \frac{\alpha}{c} \ln(-2\tilde{z}) + O\left(\frac{1}{\tilde{z}}\right), \tag{3.9}$$

$$\Omega_{,\tilde{4}} = -\tilde{z} + z + O\left(\frac{1}{\tilde{z}}\right), \tag{3.10}$$

$$\Omega_{,\tilde{1}} = -(x-\tilde{x}) - \frac{\alpha x}{r-z} + O\left(\frac{1}{\tilde{z}}\right). \tag{3.11}$$

To get finite phases after the limiting process we have to specialize the arbitrary function φ in (3.1)

$$\varphi = -\frac{\tilde{z}}{c} + \frac{\alpha}{c} \ln(-2\tilde{z}). \tag{3.12}$$

Inserting all this into (3.3) and performing a gauge transformation $\hat{A}^\mu = A^\mu - \chi'^\mu$ with $\chi = -\frac{\alpha C}{8\pi} \ln [r-z]$ we finally get

$$A_1 = \left(1 + \frac{\alpha}{2r}\right) A \cdot \cos\left\{\omega \left\langle t - \frac{z}{c} + \frac{\alpha}{c} \ln [r-z] \right\rangle\right\},$$

$$A_2 = 0$$

$$A_3 = \frac{\alpha}{2} A \cdot \frac{x}{r(r-z)} \cos\left\{\omega\left\langle t - \frac{z}{c} + \frac{\alpha}{c}\ln[r-z]\right\rangle\right\},$$

$$A_4 = \frac{\alpha}{2} A \cdot \frac{x}{r(r-z)} \cos\left\{\omega\left\langle t - \frac{z}{c} + \frac{\alpha}{c}\ln[r-z]\right\rangle\right\} - C\left(1 - \frac{\alpha}{2r}\right),$$

$$\left(A = \frac{\varepsilon\omega C}{4\pi}\right). \tag{3.13}$$

as the four-potential of an oscillating charge at infinity.

The field of an oscillating dipole at infinity, i.e. the generalized plane electromagnetic wave in Schwarzschild space-time is the superposition of (3.13), and the field of a charge $-e$ situated at $\tilde{x}^\alpha = [-\varepsilon \sin\{\omega\langle t+\varphi\rangle\},\ 0,\ \tilde{z} = \text{const},\ c\tilde{t}]$. The result is

$$A_1 = \frac{B}{\omega}\left(1 + \frac{\alpha}{2r}\right) e^{i\omega\Delta},$$

$$A_2 = 0, \qquad \Delta = t - \frac{z}{c} + \frac{\alpha}{c}\ln(r-z),$$

$$A_3 = A_4 = \frac{B}{2\omega}\,\frac{\alpha x}{r(r-z)}\, e^{i\omega\Delta}. \tag{3.14}$$

In the case $\alpha = 0$ we get the plane electromagnetic wave in flat space-time in z-direction.

These expressions (3.14) for the potentials of a plane wave show that the gravitational field alters the phase of the incoming plane electromagnetic wave even at infinity.

The result (3.14) is valid for $\lambda \ll \alpha$ in far field of the star in regions, where only on light ray connects source and observer (where $\Omega(x, \tilde{x})$ is a single valued function). Fffects due to interfereuce of two rays are discussed in [9].

Mo and Papas [4] gave a similar formula, but the phases in the plane wave by Mo and Papas differ from the phases in (3.14) by an additional term — $2\alpha \ln[x^2+y^2]$ which is singular on the whole z-axis.

4. *Light deflection for arbitrary wavelength*

To determine the angle β of deflection of light rays (see Fig. 1) we need the components S^x and S^z of the Poynting vector $\vec{S}$.

$$S^i = T^{ik}\xi_k, \qquad \xi^k = (0, 0, 0, 1). \tag{4.1}$$

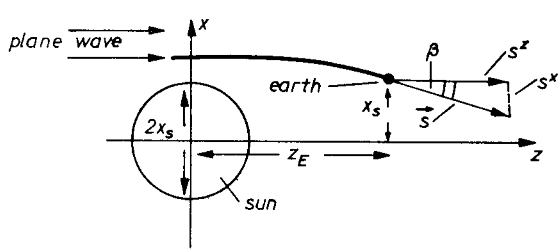

Fig. 1. Light deflection and Poynting vector

T^{ik} is the energy-momentum tensor of the electromagnetic field. We use the isotropic coordinate system (2.5).
After a short calculation we get

$$S^x = \alpha \ln (r-z)_{,x} \cdot \sin^2 \omega\Delta; \qquad S^z = -\left(1+\frac{\alpha}{2r}\right) \sin^2 \omega\Delta. \tag{4.2}$$

Therefore the deflection angle β is given by

$$\tan \beta = -\left.\frac{S^x}{S^z}\right|_{\text{Earth}} = \frac{\alpha}{\sqrt{x_S^2+z_E^2}} \frac{x_S}{\sqrt{x_S^2+z_E^2}-z_E} \tag{4.3}$$

x_S being the radius of the sun and z_E the distance between the sun and the earth. For $x_{S/z_E} \ll 1$ we get the classical deflection formula of geometrical optics

$$\beta = \frac{2\alpha}{x_S}. \tag{4.4}$$

To get (4.4), no short-wavelength approximation was used: This wave-theoretical treatment does not produce additional effects to (4.4) with the exception of second and higher order terms in gravitational constant (see [1]).

5. *Concluding remarks*

We have found the solution of the Maxwell equations in the linearized Schwarzschild space-time for an incident plane electromagnetic wave. This solution includes the classical light deflection formula for arbitrary wavelength. The solution is valid for all domains of space-time with the exception of the neighbourhood and the geometrical shadow of the Black Hole (see [2]) and domains of space time where $\Omega(x, \tilde{x})$ is uot a single valued function. Only in these domains one should expect measurable differences from the first approximation; these problems are investigated in [5], [6], and [9].

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