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SOLUTIONS OF THE EINSTEIN FIELD EQUATIONS FOR A ROTATING PERFECT FLUID. PART 2. PROPERTIES OF THE FLOW-STATIONARY AND VORTEX-HOMOGENEOUS SOLUTIONS*

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The first family of the flow-stationary and vortex-homogeneous solutions presented in Part 1 is investigated. Their symmetry group is shown to be 3-parametric, abelian and acting transitively on timelike hypersurfaces. Exterior solutions of the same symmetry are found and matched to the interior ones. The conformal curvature tensors are of Petrov type I, with one exception which is of Petrov type II. Matter filling the spacetime is shown to consist of co-axial cylinders rotating with different angular velocities. The redshift is found to be strongly anisotropic. The equation of state appears to result from the field equations.

Introduction

In this paper the flow-stationary and vortex-homogeneous solutions presented in Part 1 [1] are investigated. Since the second and third family were discussed by many other authors, the present investigation is concentrated on the first family, references being given for the corresponding properties of the other families.

The symmetry group of the first family solutions is recognized as a 3-parametric abelian group acting simply transitively on timelike hypersurfaces. Invariant properties of this group are then used to construct a general metric tensor with these symmetries, which is substituted in the empty-space field equations. Four types of solutions are found, one of which is static while the other ones are stationary-nonstatic. In the case $\Lambda = 0$ all the solutions are identified with metrics found long ago by many other writers. Their generalization to $\Lambda \neq 0$ is a new result. It is shown that with appropriate values of constant

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parameters they are the exterior metrics to the first family solutions. It turns out that the type of an exterior solution is limited or even fixed by the type of the interior solution. Empty-space metrics with $\Lambda \neq 0$ are shown to be the formal limitting cases $\kappa \to 0$ of the interior ones. The conformal curvature of the first family solutions is shown to be of Petrov type I, with one exception which is of type II. The geometrical structure of the space-time appears to be that of a family of co-axial cylinders of matter rotating with different angular velocities around an axis. The red-shift of light emitted from one particle and received by the neigbouring one is shown to be strongly anisotropic and thus unrealistic. Also it is shown that the equation of state is, in a sense, a consequence of the field equations.

Most of the discussion is carried out so that all solutions may be treated simultaneously, without differentiating between the types. The numeration of sections, formulas and tables is continued from Part 1.

8. *Symmetries of the solutions*

a) The first family of solutions

It is seen from the very definition of the symmetry group that symmetries constitute a subgroup of *every* group of admissible transformations[1]. We have finished (see remarks after (5.7) and (5.8) in [1]) with a very simple group of admissible transformations given by (2.18) with $\gamma = 0$. It is easy to see that it is a symmetry group only when $\alpha = 1$. Thus for all the first family solutions the symmetry transformations are as follows:

$$x^0 = x^{0'} + t_0,$$

$$x^1 = x^{1'} + t_1,$$

$$x^2 = x^{2'},$$

$$x^3 = x^{3'} + t_3, \qquad t_0, t_1, t_3 = \text{const.} \tag{8.1}$$

The corresponding Killing vectors are $\underset{(i)}{k}^\mu = \delta^\mu_i$, $i = 0, 1, 3$. We see that (8.1) is an abelian group. Notice that the group (8.1) acts simply transitively on the timelike hypersurfaces $x^2 = \text{const}$. Space-times with a symmetry group acting simply transitively on 3-dimensional *spacelike* hypersurfaces were classified by Bianchi (see e.g. [2], [3] and Appendix C in [4]) into 9 types. No specific signature of the metric on the hypersurface is assumed in that classification. Therefore spacetimes with a congruence of homogeneous *timelike* hypersurfaces may be conformed to the same classification. Since the group (8.1) is abelian, we see that it is of Bianchi type I.

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[1] For the definition of admissible transformations see [1], Sec. 2c.

The group (8.1) is completely characterized by the following statements:

a) There exist 3 commuting Killing vectors $\underset{(0)}{k^\mu}$, $\underset{(1)}{k^\mu}$, $\underset{(3)}{k^\mu}$, whose integral lines are the x^0, x^1, x^3 — coordinate lines. The x^0-line is timelike.

b) The x^2-line is orthogonal to the other ones.

c) $g_{\mu\nu}\underset{(0)(3)}{k^\mu k^\nu} = g_{\mu\nu}\underset{(1)(3)}{k^\mu k^\nu} = 0.$

d) $g_{\mu\nu}\underset{(0)(1)}{k^\mu k^\nu} \neq 0.$ (8.2)

b) The second family of solutions

Gödel suggests in his paper [5] that the solution obtained by him has a symmetry group of only four parameters. This statement has been repeated by Wright [6]. Three of the parameters are connected with the group (8.1), while the fourth one belongs to the symmetry transformation:

$$x^0 = x^{0'}, \quad x^1 = e^{-t_2}x^{1'}, \quad x^2 = e^{t_2}x^{2'}, \quad x^3 = x^{3'} \tag{8.3}$$

(in the coordinate system of (6.9) in [1]).

In fact, the solution (6.9) has a five-parametric symmetry group, as pointed out by Ellis [7] and Wainwright [8]. The transformation connected with the fifth parameter is:

$$x^0 = x^{0'} + \frac{2\sqrt{2}}{K}\operatorname{arc\,tg}\frac{\sqrt{2}\,t_4}{Kx^{2'}(1-t_4x^{1'})}, \qquad x^1 = \frac{K^2x^{1'}(x^{2'})^2(1-t_4x^{1'})-2t_4}{2t_4^2+K^2(x^{2'})^2(1-t_4x^{1'})^2},$$

$$x^2 = (1-t_4x^{1'})^2x^{2'}+2t_4^2/K^2x^{2'}, \quad x^3 = x^{3'}, \tag{8.4}$$

where t_4 is the group parameter and $K \overset{\text{def}}{=} (\kappa\varrho H^{-1})^{1/2}$. We see that when $t_4 = 0$ the transformation reduces to the identity.

c) The third family of solutions

Wright [6] and Ellis [7] have shown that (8.1) is the entire symmetry group for this solution.

9. Exterior solutions

a) Statement of the problem

It is reasonable to look for exterior solutions with the same symmetry group as the interior ones, to which they are to be matched. Taking the properties (8.2a)–(8.2d) as axioms we easily arrive at the following metric form:

$$ds^2 = (\alpha dx^0+\beta dx^1)^2-(\gamma dx^1)^2-(\delta dx^2)^2-(\varepsilon dx^3)^2, \tag{9.1}$$

where α, β, γ, δ, ε are arbitrary functions of one variable x^2. Now two cases must be distinguished:

$$\text{I.} > (\beta/\alpha)_{,2} = 0,$$

$$\text{II.} > (\beta/\alpha)_{,2} \neq 0. \tag{9.2}$$

In the first case $\beta = S\alpha$ where S = const, and the transformation $x^0 = x^{0'} - Sx^{1'}$, $x^i = x^{i'}$, $i = 1, 2, 3$ shows that the form (9.1) is static. In the second case the metric (9.1) is stationary but possibly nonstatic.

It is easy to see that (9.2.I) may be considered as the special case of (9.2.II), except when $(\varepsilon/\alpha)_{,2} = (\gamma/\alpha)_{,2} = (\beta/\alpha)_{,2} = 0$. If $(\beta/\alpha)_{,2} = 0$, but $(\gamma/\alpha)_{,2} \neq 0$ then the transformation $x^1 = x^{1'} + \text{const} \cdot x^0$ produces a new β' with the property $(\beta'/\alpha)_{,2} \neq 0$. If $(\gamma/\alpha)_{,2} = (\beta/\alpha)_{,2} = 0$, but $(\varepsilon/\alpha)_{,2} \neq 0$ then the same result is yielded by $x^1 = x^{3'}$, $x^3 = x^{1'} + \text{const} \cdot x^0$. If $(\varepsilon/\alpha)_{,2} = (\gamma/\alpha)_{,2} = (\beta/\alpha)_{,2} = 0$ then the direct integration of the field equations yields:

$$ds^2 = w^{-2/3}[(dx^0)^2 - (dx^1)^2] - w^{-2}(dx^2)^2 - w^{-2/3}(dx^3)^2, \tag{9.3}$$

where:

$$w = Px^2 + Q, \tag{9.4}$$

$$\Lambda = \tfrac{1}{3}P^2, \tag{9.5}$$

P and Q being arbitrary constants.

This metric might be called a Λ-version of the flat metric since for $\Lambda = 0$ it becomes flat. It is not interesting for us.

b) Algebraic form of the metric tensor

We consider now the case II of (9.2). Then we can introduce a new coordinate:

$$x^{2'} \stackrel{\text{def}}{=} \beta(x^2)/\alpha(x^2). \tag{9.6}$$

We also introduce the following notation:

$$\alpha = f^{-1}(x^{2'}), \quad \beta^2 = g(x^{2'}), \quad \delta^2 = K(x^{2'}), \quad \varepsilon^2 = -R^{-1}(x^{2'})f^3(x^{2'}). \tag{9.7}$$

Now f, g, K, R are the new unknown functions, and (9.1) assumes the form (we drop the primes)

$$ds^2 = f^{-2}(dx^0 + x^2 dx^1)^2 - g(dx^1)^2 - K(dx^2)^2 + R^{-1}f^3(dx^3)^2. \tag{9.8}$$

This is closely analogous to (3.4) from [1] under the following identifications:

$$f \leftrightarrow H, \quad g \leftrightarrow h, \quad K \leftrightarrow k, \quad R \leftrightarrow \varrho/G. \tag{9.9}$$

The equation $R^0_1 = 0$ (in the scalar components R^i_j, see section 3b in [1]) is easily integrated to give:

$$gKRf^3 = \text{const} \stackrel{\text{def}}{=} -J^2. \tag{9.10}$$

This is an analog of (3.3) from [1].

We consider the field equations $R^i_j = \Lambda\delta^i_j$. The procedure of integration goes exactly the same way as in Sec. 5b of [1]. The results are as follows:

$$g = V(x^2)/f^3, \tag{9.11}$$

$$V(x^2) \overset{\text{def}}{=} (x^2)^2 + px^2 + q, \tag{9.12}$$

$$K = -J^2/(RVf), \tag{9.13}$$

$$R = s\frac{f^5}{V}\exp\left(\int \frac{x^2}{V}\,dx^2\right), \tag{9.14}$$

$$f = v^{1/3}, \tag{9.15}$$

where $s = \text{const} > 0 \neq J = \text{const}$.

The function v must obey the following equation:

$$v_{,22} - \frac{V_{,2} - x^2}{V}\,v_{,2} + \frac{3}{4}\left(-\frac{V_{,22}}{V} + \frac{V_{,2}^2}{V^2} - \frac{x^2 V_{,2}}{V^2} + \frac{1}{V}\right)v = 0. \tag{9.16}$$

Formulas (9.11)–(9.16) were obtained by subtracting the field equations from one another. Substituting those expressions in any of the diagonal components of R^i_j we obtain the definition of Λ by means of the other constants:

$$\Lambda = -\frac{1}{2J^2}sf^6\left[\exp\left(\int\frac{x^2}{V}\,dx^2\right)\right](V^{-1} - 2V^{-1}V_{,2}f^{-1}f_{,2} + 2f^{-2}f_{,2}^2 - 2f^{-1}f_{,22}) = 0. \tag{9.17}$$

So we have found that the stationary-nonstatic empty metrics have the following algebraic form:

$$ds^2 = f^{-2}\{(dx^0)^2 + 2x^2dx^0dx^1 + [(x^2)^2 - V]\,(dx^1)^2\} - \left[\exp\left(-\int\frac{x^2}{V}\,dx^2\right)\right]\cdot\left[\frac{J^2}{sf^6}(dx^2)^2 + \frac{V}{sf^2}(dx^3)^2\right]. \tag{9.18}$$

The reader is asked to compare (9.18) with (5.12) from [1] to see the high similarity of those two metrics.

Here Eq. (9.16) has also various types of singularities (analogously to (5.11) from [1]) and various solutions depending on the type and number of roots of V.

The reader should have (9.18) in mind when looking into the tables below. The constants P and Q appearing there are both real and never vanish simultaneously.

c) Type A solutions

$$p^2 - 4q < 0. \tag{9.19}$$

V has two complex roots $x^2 = p_0$ and $x^2 = q_0 = p_0^*$, and may be represented as:

$$V = (x^2 - p_0)(x^2 - q_0). \tag{9.20}$$

For definiteness we assume $\text{Im}(p_0) < 0$. Here $V > 0$ for all values of x^2.

TABLE VII

Type A solutions

$$f = [Pv_1 + Qv_2]^{1/3}$$

$$v_1 = \left(\frac{x^2 - p_0}{q_0 - L}\right)^{\mu} \left(\frac{x^2 - q_0}{p_0 - L}\right)^{\nu}$$

$$v_2 = \left(\frac{x^2 - p_0}{q_0 - L}\right)^{\mu'} \left(\frac{x^2 - q_0}{p_0 - L}\right)^{\nu'}, \quad L = \text{const}$$

$$\left.\begin{matrix}\mu \\ \mu'\end{matrix}\right\} = \frac{1}{2(p_0 - q_0)} [p_0 - 2q_0 \pm (p_0^2 - p_0 q_0 + q_0^2)^{1/2}]$$

$$\left.\begin{matrix}\nu = \mu^* \\ \nu' = \mu'^*\end{matrix}\right\} = \frac{1}{2(p_0 - q_0)} [2p_0 - q_0 \mp (p_0^2 - p_0 q_0 + q_0^2)^{1/2}]$$

$$\Lambda = (s/3J^2) PQ (p_0^2 - p_0 q_0 + q_0^2)(p_0 - L)^{-q_0/(p_0 - q_0)} (q_0 - L)^{p_0/(p_0 - q_0}$$

$$\int \frac{x^2}{V} dx^2 = \frac{1}{2} \ln [(x^2 - \operatorname{Re} p_0)^2 + (\operatorname{Im} p_0)^2] + \frac{\operatorname{Re} p_0}{\operatorname{Im} p_0} \operatorname{arc\,tg} \frac{x^2 - \operatorname{Re} p_0}{\operatorname{Im} p_0}.$$

Both linearly independent functions v_1 and v_2 are real so they could be represented without using complex numbers but our notation is simpler. When $Q = 0$ (and consequently $\Lambda = 0$) this solution is identical, with one of Lewis' solutions [9] (formulas (4.6)'–(4.9)' in his paper). In both cases $P = 0$ and $Q = 0$ the type A solutions are of the form (18) from the paper of Dautcourt, Papapetrou and Treder [10]. When $\Lambda \neq 0$ these solutions are new.

If p_0 and q_0 are such that $p_0^2 - p_0 q_0 + q_0^2 = 0$ then $\mu = \mu'$, $\nu = \nu'$ and v_1 with v_2 become linearly dependent. This case is called "type A'" and must be treated separately.

d) Type A′ solutions

$$\left.\begin{matrix}p_0 \\ \\ q_0\end{matrix}\right\} = q \exp\left(\mp i \frac{\pi}{6}\right), \tag{9.21}$$

where $q > 0$ is an arbitrary constant.

In the case $Q = 0$ (what means $\Lambda = 0$) this solution is still of the form (18) from the paper of Dautcourt, Papapetrou and Treder [10], while for $\Lambda \neq 0$ it is new.

TABLE VIII

Type A solutions

$$V = (x^2 - qe^{-i\pi/6})(x^2 - qe^{+i\pi/6})$$

$$f = [Pv_1 + Qv_2]^{1/3}$$

$$v_1 = (x^2 - qe^{-i\pi/6})^{(3 - i\sqrt{3})/4} (x^2 - qe^{+i\pi/6})^{(3 + i\sqrt{3})/4}$$

$$v^2 = v_1 \ln\left(\frac{x^2-qe^{-i\pi/6}}{x^2-qe^{+i\pi/6}}\right)$$

$$\Lambda = (s/3J^2)Q^2q^2$$

$$\int \frac{x^2}{V}\, dx^2 = \frac{1}{2} \ln\left[(x^2)^2-\sqrt{3}\, qx^2+q^2\right]+\sqrt{3}\operatorname{arc\,tg}(2x^2-\sqrt{3}).$$

e) Type B solutions

$$p^2-4q > 0. \tag{9.22}$$

V has now two distinct real roots $x^2 = p_0$ and $x^2 = q_0 > p_0$, and may be represented as

$$V = (x^2-p_0)(x^2-q_0). \tag{9.23}$$

In order to assure the proper signature of (9.18) we have to demand $V > 0$. This is fulfilled only for $x^2 < p_0$ and $x^2 > q_0$.

TABLE IX

Type B solutions

All the formulas, except for the last one, are identical with those of Table VII. This time p_0 and q_0 are just independent real constants, $p_0 \neq L \neq q_0$, so (μ, ν) and (μ', ν') are also pairs of distinct real constants and no analogue of the equations $\nu = \mu^*$, $\nu' = \mu'^*$ holds. The solution has the proper signature in the non-connected region $\{x^2 < p_0$ or $x^2 > q_0\}$. The points $x^2 = p_0$ and $x^2 = q_0$ are singular points of the solution.

$$\int \frac{x^2}{V}\, dx^2 = \frac{1}{p_0-q_0}\left[p_0 \ln(x^2-p_0)-q_0 \ln(x^2-q_0)\right].$$

It is just the type B solution which appears to be static, and by appropriate choice of coordinates may be changed to the form (9.2.I). In the case $\Lambda \neq 0$ the solution is new. In the case $\Lambda = 0$ it has been known since long ago and appeared several times in various papers. In the cases $P = 0$ and $Q = 0$ it is identical (exact to coordinate transformations) with the metrics found by Levi-Civita [11], Kasner [12], [13], Lewis [9], Marder [14], Dautcourt, Papapetrou and Treder (formula (18) in [10]), Gautreau and Hoffman [15]. It is also identical with Mitter's solution [16] if only Mitter's constant m is equal to (-2) instead of the crazy value given by the author; otherwise the field equations are violated. Moreover, it is the special, cylindrically symmetric case of Weyl's axially symmetric metrics [17], [18]. Mukherji's solution [19] in the case of vanishing electromagnetic field is the special case $P = 0 = p_0$ of the type B solution.

f) Type C solutions

$$p^2-4q = 0, \qquad q \neq 0. \tag{9.24}$$

Then V has a double real root $x^2 = p_0 \neq 0$ and is of the form:

$$V = (x^2-p_0)^2. \tag{9.25}$$

Now $V > 0$ everywhere except for $x^2 = p_0$.

TABLE X

Type C solutions

$$f = |x^2 - p_0|^{1/2}[P + Qe^{p_0/(x^2-p_0)}]^{1/3}$$

$$\Lambda = (s/3J^2)PQp_0^2$$

$$\int \frac{x^2}{V}\,dx^2 = \ln|x^2-p_0| - \frac{p_0}{x^2-p_0}$$

For $x^2 = p_0$ the solution has a singularity.

When $Q = 0$ this solution is the degenerate case $p_0 \to q_0$ of the type A and B solutions. Then it is also the special (independent of z) case of Hoffman's solution [20]. When $P = 0$ the solution is contained in the class of metrics considered by Lewis [9], although it is not given in his paper (one can find it in van Stockum's paper [21]).It is then the degenerate case of the type A solution with $Q = 0$.

In both cases $P = 0$ and $Q = 0$ ($\Lambda = 0$) this solution is of the form (21) in the paper of Dautcourt, Papapetrou and Treder [10]. With $\Lambda \neq 0$ this is a new solution.

g) Type D solutions

$$p = q = 0. \tag{9.26}$$

Then V is of the form:

$$V = (x^2)^2. \tag{9.27}$$

$V > 0$ everywhere except for $x^2 = 0$.

TABLE XI

Type D solutions

$$f = (P|x^2|^{3/2} + Q|x^2|^{1/2})^{1/3}$$

$$\Lambda = -(s/3J^2)Q^2$$

$$\int \frac{x^2}{V}\,dx^2 = \ln x^2$$

$x^2 = 0$ is a singular point of the solution.

In the case $Q = 0$ this is again a special Hoffman's solution [20], the special case $p_0 = 0$ of the type C solution, and at the same time it is of the form (24b) from the paper of Dautcourt, Papapetrou and Treder [10]. With $Q \neq 0$ this is a new solution.

h) Exterior solutions matched to the first family interior solutions

We use (5.12) from [1] and (9.18) as the representations for the interior and exterior metrics, respectively. Then the equations of continuity of $g_{\alpha\beta}$ and the second fundamental form on the hypersurface $x^2 = r_0 = \text{const}$ are equivalent to:

$$J = 1/G, \tag{9.28}$$

$$\begin{cases} V(r_0) = G^{-1}W(r_0), \\ V_{,2}(r_0) = G^{-1}W_{,2}(r_0), \end{cases} \tag{9.29}$$

$$-\frac{1}{s}\exp\left(-\int \frac{x^2}{V}\,dx^2\right)\bigg|_{x^2=r_0} = \frac{G^2}{D}\exp\left(-\int \frac{Gx^2}{W}\,dx^2\right)\bigg|_{x^2=r_0}, \tag{9.30}$$

$$\begin{cases} f(r_0) = H(r_0), \\ f_{,2}(r_0) = H_{,2}(r_0). \end{cases} \tag{9.31}$$

Eqs (9.29) define the constants appearing in V in terms of G, B, E appearing in W. After (9.29) are solved, Eq. (9.39) defines s in terms of D, G, B and E. Then Eqs (9.31) define the constants P and Q in f in terms of M and N appearing in H.

Now we are going to establish which types of exterior solutions can be matched to an interior solution of a given type. The ranges of values of r_0 appearing in subsequent sections result from the investigation of Eqs (9.29).

i) Exterior solutions for type I

Here Eqs (9.29) do not lead to a contradiction only when V is of type A. Then:

$$\left.\begin{matrix} p_0 \\ q_0 \end{matrix}\right\} = \frac{1}{2a}\{2(a-1)r_0+b+c'$$

$$\mp[(b+c')^2-4abc'+4(a-1)(b+c')r_0-4(a-1)r_0^2]^{1/2}\}. \tag{9.32}$$

Eqs (9.31) are equivalent to:

$$(Pv_1+Qv_2)|_{x^2=r_0} = [M(u+u^*)-iN(u-u^*)]|_{x^2=r_0},$$

$$(Pv_{1,2}+Qv_{2,2})|_{x^2=r_0} = [M(u+u^*)_{,2}-iN(u-u^*)_{,2}]|_{x^2=r_0}. \tag{9.33}$$

They can be solved for P and Q if the determinant of the left-hand side is different from 0. As v_1 and v_2 are linearly independent, this condition may be broken at most in some single points. Therefore, if our first choice of r_0 were unfortunate, so that the determinant of (9.33) would be 0, then in an arbitrarily small neighbourhood of r_0 there would exist such points in which this determinant is different from 0.

The equations analogous to (9.32)–(9.33) and the remark above are true for all the other types discussed below and we will not repeat them.

j) Exterior solutions for type II, case $a < 0$

Since these solutions have the proper signature in the region $b < x^2 < c'$, one can match exterior solutions in both regions outside the segment $[b, c']$. However, it cannot be the same metric for both regions. Once p_0 and q_0 are given, r_0 is unique.

Let us define:

$$\left.\begin{matrix} r_0^{\mathrm{I}} \\ r_0^{\mathrm{II}} \end{matrix}\right\} \stackrel{\mathrm{def}}{=} \frac{1}{2}\left[b+c'\mp\left(\frac{a}{a-1}\right)^{1/2}(c'-b)\right]. \tag{9.34}$$

Since $a < 0$ and $b < c'$, one verifies that:

$$b < r_0^{\mathrm{I}} < r_0^{\mathrm{II}} < c'. \tag{9.35}$$

Eqs (9.29) imply now:

If $r_0^{\mathrm{I}} < r_0 < r_0^{\mathrm{II}}$, then the exterior metric is of type A (or A′, if a, b and c' are suitably chosen).

If $r_0 = r_0^{\mathrm{I}}$ or $r_0 = r_0^{\mathrm{II}}$, then the exterior metric is of type C.

If $b < r_0 < r_0^{\mathrm{I}}$ or $r_0^{\mathrm{II}} < r_0 < c'$, then the exterior metric is of type B, and thus static, with p_0 and q_0 given by (9.32), and $p_0 < q_0 < b$ or $c' < p_0 < q_0$, respectively.

k) Exterior solutions for type II, case $a > 1$

Here the interior solution has the proper signature in the disconnected region $x^2 < b$ and $x^2 > c'$. Notice first that it is not possible to match one empty-space "bridge" to both these regions because again, once p_0 and q_0 were determined, r_0 is unique. Therefore these two regions must be considered as two different solutions.

We define r_0^{I} and r_0^{II} as in (9.34). This time, however, $a > 1$ and:

$$r_0^{\mathrm{I}} < b < c' < r_0^{\mathrm{II}}. \tag{9.36}$$

If $r_0 < r_0^{\mathrm{I}}$ or $r_0 > r_0^{\mathrm{II}}$, then the exterior metric is of type A.

If $r_0 = r_0^{\mathrm{I}}$ or $r_0 = r_0^{\mathrm{II}}$, then the exterior metric is of type C.

If $r_0^{\mathrm{I}} < r_0 < b$ or $c' < r_0 < r_0^{\mathrm{II}}$, then the exterior solution is of type B with p_0 and q_0 given by (9.32), and $b < p_0 < q_0$ or $p_0 < q_0 < c'$, respectively.

l) Exterior solutions for type III

Here (9.29) imply that V is necessarily of type A.

m) Exterior solutions for type IV

Again (9.29) point at type A.

n) Exterior solutions for type V

Here Eqs (9.29) yield:

$$\left.\begin{matrix} p_0 \\ q_0 \end{matrix}\right\} = r_0 + \tfrac{1}{2} \mp (r_0 + E_0 + \tfrac{1}{4})^{1/2}. \tag{9.37}$$

If $r_0 < -E_0 - \frac{1}{4}$, then the exterior solution is of type A.

If $r_0 = -E_0 - \frac{1}{4}$, then the exterior solution is of type C.

If $r_0 > -E_0 - \frac{1}{4}$, then the exterior solution is of type B, and thus staic. Since always $r_0 < -E_0$, we have $r_0 < p_0 < q_0$.

o) Exterior solutions for type VI

Here they are necessarily of type A with:

$$\left.\begin{matrix} p_0 \\ q_0 \end{matrix}\right\} = r_0 \mp i(-E/\kappa)^{1/2} \tag{9.38}$$

p) Exterior solutions for Raval-Vaidya's [22] and Gödel's [5] metrics

Since they are the limitting cases of type IV from the first family, their exterior metrics are of type A.

r) Exterior solutions for Lanczos' metric [23]

This problem has been solved by van Stockum [21]: the exterior solutions are those of Lewis [9].

s) Some remarks concerning exterior solutions

Notice that the field equations inside matter reproduce the empty-space field equations in the formal limit $\kappa = 8\pi k/c^2 \to 0$. Therefore we expect that if some interior solution still exists when $\kappa = 0$, it should then reduce to an empty-space solution. It happens that our first family solutions reduce in this limit precisely to the metrics considered here. The reader is asked to consult section 5 from [1]. If $\kappa = 0$, then (5.18) implies $a = 1$. Then we easily see that type I reduces to type A, type II to type B, type III to type C and type IV to type D. The types V and VI do not exist in the limit $\kappa \to 0$.

10. The type of conformal curvature

a) The first family

A special solution of type IV corresponding to $B = E = 0$, $G = -(\kappa/2)(\sqrt{2}+1)$ in (5.5) (see [1]) is of Petrov type II. It has the following form:

$$ds^2 = N^{-2/3}(x^2)^{1-\sqrt{2}}[(dx^0)^2+2x^2dx^0dx^1+2(\sqrt{2}-1)(x^2)^2(dx^1)^2]$$
$$+DN^{-2}(x^2)^{-5\sqrt{2}}(dx^2)^2-(4D)^{-1}N^{-2/3}(x^2)^{-3\sqrt{2}}(dx^3)^2. \qquad (10.1)$$

All the other solutions of the first family are of Petrov type I (general).

b) The Raval-Vaidya [22] and Gödel solutions [5]

Wright [6] and Wainwright [8] have shown that these solutions are of Petrov type D.

c) The Lanczos [23] solution

Wright [6] has shown that it is of Petrov type I.

11. Geometry of the space-time

a) Geometry of the hypersurfaces $x^2 = \text{const}$

It is easy to verify that the x^2-lines are geodesics. The metrics (5.12) induce, on the hypersurface Σ given by $x^2 = r_0 = \text{const}$, the following metric form:

$$ds^2 = H^{-2}(r_0)\,[(dx^0+r_0dx^1)^2-G^{-1}W(r_0)\,(dx^1)^2]+G\varrho^{-1}(r_0)H^3(r_0)\,(dx^3)^2 \qquad (11.1)$$

This form represents a flat Riemann space. Therefore we can embed it into the Minkowski space and investigate its geometry there.

Every hypersurface S in the Minkowski space defines (at least in its neighbourhood) a coordinate system in the following way: we choose 3 coordinates in the hypersurface quite arbitrarily, and through each point of the hypersurface we draw an orthogonal geodesic (i.e. the straight line). On each straight line we choose a parameter λ such that $\lambda = \lambda_0$ on S. Then the condition $\lambda = \text{const} \neq \lambda_0$ defines another hypersurface of the same topology, and the coordinate net on it is transported automatically from $\lambda = \lambda_0$.

Let us make such a construction for our Σ. Let $y^2 \equiv \lambda$, and inside each hypersurface $y^2 = \text{const}$ let use the coordinates (y^0, y^1, y^3) which coincide with (x^0, x^1, x^3) on Σ. Let $y^2(\Sigma) = r_0$, and consequently the metric induced on Σ is (11.1). In such a coordinate system the metric of the Minkowski space assumes the form:

$$ds^2 = [\alpha(y^2)dy^0 + \beta(y^2)dy^1]^2 - [\gamma(y^2)dy^1]^2 - (dy^2)^2 - [\varepsilon(y^2)dy^3]^2, \tag{11.2}$$

where the corresponding Riemann tensor is equal to 0. The equations $R_{ijkl} = 0$ have four sets of solutions:

$$ds^2 = (\alpha dy^0 + \beta dy^1)^2 - (\gamma dy^1)^2 - (dy^2)^2 - (\varepsilon dy^3)^2, \tag{11.3}$$

where $\alpha, \beta, \gamma, \varepsilon = \text{const}$,

$$ds^2 = (\alpha dy^0 + \beta dy^1)^2 - (\gamma dy^1)^2 - (dy^2)^2 - C^2(y^2)^2(dy^3)^2, \tag{11.4}$$

where $\alpha, \beta, \gamma, C = \text{const}$,

$$\begin{aligned} ds^2 = {} & \varepsilon C^2[(y^2)^2 - B^2/4C^4]\,(dy^0 + K dy^1)^2 + (B/E)dy^0 dy^1 \\ & + (BK/E - \varepsilon C^2/E^2)\,(dy^1)^2 - (dy^2)^2 - (\varepsilon dy^3)^2, \end{aligned} \tag{11.5}$$

where $B, C, E, K, \varepsilon = \text{const}$, $\varepsilon = \pm 1$.

$$ds^2 = (\alpha dy^0 + \beta dy^1)^2 - C^2(y^2)^2(dy^1)^2 - (dy^2)^2 - (\varepsilon dy^3)^2, \tag{11.6}$$

where $\alpha, \beta, C, \varepsilon = \text{const}$.
In each case the metric may be put in the Cartesian form $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ by a simple transformation of coordinates. From those transformations we can recognize the geometrical meaning of the coordinates (y^0, y^1, y^2, y^3).

In (11.3) (y^1, y^2, y^3) are orthogonal cartesian coordinates. The presence of the term $dy^0 dy^1$ means that the time is measured by a clock moving in the y^1-direction. Here the surfaces $y^0, y^2 = \text{const}$ are euclidean planes, and the y^1-lines are straight.

In (11.4) (y^1, y^2, y^3) are cylindrical coordinates: y^1 is measured along the generator of the cylinder, y^2 along the radius, and y^3 is the azimuthal coordinate. The surfaces $y^0, y^2 = \text{const}$ are cylinders, but the y^1-lines are again straight.

In (11.5) two subcases $\varepsilon = +1$ and $\varepsilon = -1$ must be treated separately. If $\varepsilon = +1$, then the Minkowski space is parametrized by a collection of observers moving in the y^2-direction with constant, but different accelerations. Their world-lines are labelled with the value of acceleration y^2 (compare Bondi's lecture in [24]). Simultaneously the observer has a constant velocity in the y^1-direction and uses a clock moving in the y^1-direction with another constant velocity. Here the (y^1, y^2, y^3)-lines are straight.

Now let $\varepsilon = -1$. Then (y^1, y^2, y^3) are again the cylindrical coordinates but now $Cy^{1'}$ is the azimuthal angle, y^2 is the radial coordinate, and y^3 is measured along the generator.

Finally, in (11.6) (y^1, y^2, y^3) are cylindrical coordinates with the same meaning as above. Hovewer, this time the observer does not rotate although he still uses a revolving clock for time-measurements.

We see from (1.35) that u^α is tangent to the hypersurfaces $x^2 = \text{const}$ and every particle of the fluid moves still inside one hypersurface $x^2 = \text{const}$. Therefore the trajectory of every particle can be described in one of the metrics (11.3)–(11.6) instead of (5.12).

The non-zero projection of u^α onto x^1-line means that, in the hypersurface $y^0 = \text{const}$, the particles of the fluid follow the y^1-lines. On the other hand the acceleration vector $\dot{u}^\alpha = u^\alpha_{;\varrho} u^\varrho$ is tangent to the x^2-lines, and the vorticity vector (1.39) is tangent to the x^3-lines. In (11.3), (11.4) and the first subcase of (11.5) the y^1-lines are straight and consequently the accelaration, if present, can have only the y^0 and y^1 components. In the second subcase of (11.5) and in (11.6) the motion along y^1-lines is a typical rotational motion with the angular velocity parallel to the y^3-line and acceleration parallel to the y^2-line. This is in perfect agreement with our space-time. So both these metrics may be the models of our manifold. We decide that (11.5) with $\varepsilon = -1$ is a better model, since in the presence of rotating mater it is not possible to define a nonrotating observer. Consequently, the invariance with respect to the additional Killing vector δ^μ_1 means axial symmetry.

For comparison the reader is asked to see a nice discussion of Gautreau and Hoffman [15] concerning a similar problem.

b) Structure of the space-time

Thinking in a non-relativistic, three-dimensional way we can say that the space consists of co-axial cylinders. Matter rotates in such a way that every particle moves still along the azimuthal circle of a fixed cylinder. The velocities of all the particles moving on the same cylinder are equal but they vary from one cylinder to the other. The density of matter, the pressure, and all the other scalars are constant on a fixed cylinder but they also may vary in the radial direction.

c) Geometrical meaning of the assumptions (1.42)

The assumption $\partial_u g_{\alpha\beta} = 0$ meant stationarity what was obvious from the beginning. The assumption $\partial_w g_{\alpha\beta} = 0$ meant that the space-time is homogeneous in the direction of the vorticity vector, i.e. the axis of cylinders. It is interesting that these two assumptions imply, as it appeared from the field equations, that then the spacetime is necessarily axially symmetric. This property *was not* contained in our assumptions.

12. *Physical properties of the solutions*

a) Invariants of the velocity field and the red-shift

We denote θ — the expansion, σ — the scalar of shear, and $\dot{u}^\alpha$ — the vector of acceleration (for definitions see e.g. [25]). As we already noticed in Sec. 1e [1] we have:

$$\theta = \sigma = 0. \tag{12.1}$$

Moreover we have:

$$\dot{u}^\alpha = W\varrho H,_2 \delta_2^\alpha \tag{12.2}$$

for the first family, and:

$$\dot{u}^\alpha = 0 \tag{12.3}$$

for the second and third family.

In consequence of (12.1) the formula for the redshift $d\lambda/\lambda$ (as measured by an observer travelling with a fixed particle of the fluid and receiving light signals from a nearby particle), given in [25], simplifies to:

$$d\lambda/\lambda = -\dot{u}_\alpha \delta_\perp x^\alpha = -H^{-1}H,_2\delta_\perp x^2, \tag{12.4}$$

where:

$$\delta_\perp x^\alpha \overset{\text{def}}{=} (\delta_\beta^\alpha - u^\alpha u_\beta)\delta x^\beta, \tag{12.5}$$

δx^β being an infinitesimal vector pointing from the observer to the particle sending the light signals.

From (12.4) we see that in the second and third families there is no redshift, whereas in the first family it is strongly anisotropic: equal to 0 when $\delta_\perp x^\alpha \perp \dot{u}^\alpha$ and maximal when $\delta_\perp x^\alpha || \dot{u}^\alpha$. This contradicts observations. More realistic models require at least non-zero shear.

b) Some remarks concerning the equation of state

In our first family ϱ and p are functions of one variable x^2, and therefore the equation of state is given in a parametric form:

$$\varrho = \varrho(x^2), \quad p = p(x^2) \tag{12.6}$$

The functional dependence of ϱ and p on x^2 is in general so complicated that it is not possible to obtain an explicit equation $\varrho = \varrho(p)$. We only emphasize that the equation of state appeared to be a consequence of the Einstein field equations. Usually one expects that the equation of state can be postulated independently of the field equations, hence our result may seem surprising. However, this is always so if there exists such a set of coordinates in which the metric tensor depends only on one variable. If we look into the tables of chapter 5 from [1] we see that the most general metrics contain six independent arbitrary constants. All of them appear in the formulas for ϱ and p. The equation $\varrho = \varrho(p)$ with six arbitrary parameters represents, in fact, a large class of equations of state.

13. An interesting special case

a) Solutions with cut-off hypergeometric series

The ordinary or confluent hypergeometric series can degenerate to elementary functions when their parameters assume some special values. The most obvious cases are when the first or second parameter of the ordinary hypergeometric function and the first parameter of the confluent hypergeometric function are negative integer. Then the series becomes finite, i. e. degenerates to a polynomial.

It is seen at once from Table VI that this is impossible in type VI solutions. It is also impossible in type III. However, in the types I, II and V the hypergeometric series may be cut off at some values of parameters. Plebański[2] even suggested that then it would be possible to obtain the equation of state in the form of the van der Waals isotherms. This question has not yet been investigated.

b) A special solution of type IV.

If we take the type IV solution with $M = 0$ or $N = 0$ then p and ϱ become power functions of x^2 and obey the polytrope-type equation:

$$p\varrho^{-\gamma} = \text{const}, \tag{13.1}$$

where:

$$\gamma = 6\,\frac{a-1}{5a-6\pm(a^2-a+1)^{1/2}}, \tag{13.2}$$

upper sign corresponding to $N = 0$ and lower to $M = 0$.

The condition $a > 1$ implies that $\gamma < 0$, $1 < \gamma < 4/3$ or $\gamma > 3/2$. It is seen that realistic values of γ may be chosen just by signature requirements ($a > 1$ is such a one).

14. Concluding remarks

Our solutions appeared to be unrealistic both as models of stars (because the portion of matter is necessarily infinite) and models of the Universe (because of the redshift-anisotropy). They may be of some importance only for relativistic hydrodynamics as a mathematical theory. If one intends to obtain a model of star, then the assumption $\partial_w g_{\alpha\beta} = 0$ should be abandoned, but $\partial_u g_{\alpha\beta} = 0$ might be retained. Conversely, to obtain a realistic model of the Universe the assumption $\partial_u g_{\alpha\beta} = 0$ must be abandoned, otherwise the shear and expansion scalars will vanish and the redshift will be still anisotropic.

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[2] Private communication.

However, the coordinates (1.32) have an essential defect. Most of the authors postulate the metric form for a stationary and axially symmetric space-time as follows:

$$ds^2 = e^{2\alpha}(dx^0 + \Omega d\varphi)^2 - e^{2\beta} d\varphi^2 - h_{AB} dx^A dx^B,$$

where A, $B = 1, 2$, and Ω is called "the angular velocity of dragging inertial frames". The limit $\Omega \to 0$ corresponds to the static spacetime in which matter does not rotate. In the coordinates (1.32) we have $\Omega = x^2$, and we cannot pass to the limit $\Omega \to 0$ without changing the coordinate system.

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