ON BARGMANN TRANSFORMATION OF RELATIVISTIC GREEN'S FUNCTIONS

By K. Lukierska

Institute of Physics, Wrocław Technical University*

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The paper contains an investigation of Bargmann transformations of free relativistic Green's functions.

We prove the equivalence of two types of triplets on the space of entire analytic functions.

We present explicit analytic expression for the Bargmann transformation of free field propagators in one-dimensional case with any value of the mass parameter. In four-dimensional Minkowski space we derive Bargmann transformations for massless particles. In the case of massive particles it has been shown that the Bargmann transformations can be obtained in the form of power series representing entire functions. An analytic representation of the unitarity and causality conditions is given and the invariance of the growth and type with respect to Poincaré transformations is shown.

1. Introduction

The use of the conventional Hilbert space is not sufficient for the description of Quantum Field Theory. Recently several generalizations of conventional Hilbert space methods have been developed (cf. e.g. [1]). One of them is the method of dual Hilbert spaces of functional power series [2–4].

In Sec. 2 of this paper we prove the equivalence of the triplets in the spaces of entire functions introduced by Bargmann [5–6] and Rzewuski [2]. Bargmann transformation transforms distributions into entire functions. Since in many cases it is more convenient to deal with entire functions than with distributions, we attempt in this paper to transform some of the distributions appearing in relativistic Quantum Field Theory into the Bargmann space.

In Sec. 3 we obtain all Green's functions in one-dimensional case and in Minkowski space for massless particles. According to the prediction of the present theory, because all Green's functions, with the exception of the two-dimensional case with mass equal to zero, are Schwartz distributions, they become entire analytic functions after the Bargmann

* Address: Instytut Fizyki, Politechnika Wrocławska, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland.

transformation. In our examples they have growth index 2 and type 1/2. The use of the power series expansions of analytic functions has this positive feature that it allows to perform the estimations based on the functional power series method. This fact is used in Sec. 3 of this paper to estimate the analytic Bargmann transformations of the free particle propagators, when mass is not equal to zero and straightforward calculation is very complicated.

An application of the analytic representation for the investigation of Quantum Field Theory seems to be interesting and has been undertaken already for Euclidean field theory (cf. e.g. [7], [8]).

Sec. 4 of this paper contains analytic formulation of the axioms of Quantum Field Theory for the scalar field.

The unitarity condition gives a restriction for the generating functional for the S-matrix: $|S[g; \alpha, \beta]| \leqslant e^{\frac{1}{2}\|\alpha\|^2+\frac{1}{2}\|\beta\|^2}$. The local causality condition takes an integral form. The lack of invariance of the Bargmann transformation with respect to the Poincaré transformations does not give a Lorentz invariant theory, but in that paper we pay only attention to the analytical properties which remain invariant in the sense that the growth index and type are the same for all Poincaré frames.

2. *Relation between certain Hilbert spaces of power series and entire analytic functions*

We introduce here a method of obtaining triplets in the space of entire functionals as described in [2]. We define, in the linear space of entire functions, Banach spaces $\mathscr{T}_\lambda$ and $\mathscr{T}_{\lambda^{-1}}$ having the norms:

$$|||f|||_\lambda := \sup_z |f(z)| \exp\{-\tfrac{1}{2}\lambda^{-2}|z|^2\} \quad \text{for} \quad 0<\lambda<1, \tag{2.1}$$

$$|||f|||_\lambda := \sup_z |f(z)| \exp\{-\tfrac{1}{2}\lambda^{2}|z|^2\} \quad \text{for} \quad 0<\lambda<1. \tag{2.2}$$

We define further, in the same space of entire functions, Hilbert spaces $\mathscr{H}_\lambda$, $\mathscr{H}_{\lambda^{-1}}$ with the help of scalar products:

$$\langle f, g\rangle_\lambda := \int e^{-|z|^2} f(\lambda z) g(\lambda \bar{z}) d\left(\frac{z}{\sqrt{\pi}}\right) \quad \text{for} \quad 0<\lambda<1 \tag{2.3}$$

and

$$\langle f, g\rangle_{\lambda^{-1}} := \int e^{-|z|^2} f(\lambda^{-1} z) g(\lambda^{-1} \bar{z}) d\left(\frac{z}{\sqrt{\pi}}\right) \quad \text{for} \quad 0<\lambda<1, \tag{2.4}$$

where

$$\int d\left(\frac{z}{\sqrt{\pi}}\right) := \int \frac{dx}{\sqrt{2\pi}} \int \frac{dy}{\sqrt{2\pi}}, \qquad z = \frac{1}{\sqrt{2}}(x+iy),\ x, y \in R. \tag{2.5}$$

Because of the relations:

$$\mathscr{H}_\lambda \supset \mathscr{T}_{\lambda'} \supset \mathscr{H}_{\lambda'}, \qquad \mathscr{H}_{\lambda'^{-1}} \supset \mathscr{T}_{\lambda^{-1}} \supset \mathscr{H}_{\lambda^{-1}} \quad \text{when} \quad \lambda < \lambda', \tag{2.6}$$

we can alternatively define

$$\mathcal{T} := \bigcup_{0<\lambda<1} \mathcal{T}_{\lambda^{-1}} = \bigcup_{0<\lambda<1} \mathcal{H}_{\lambda^{-1}}, \tag{2.7}$$

$$\mathcal{T}' := \bigcap_{0<\lambda<1} \mathcal{T}_{\lambda} = \bigcap_{0<\lambda<1} \mathcal{H}_{\lambda}. \tag{2.8}$$

Taking into account that the scalar product in the central Hilbert space $\mathcal{H}$ is defined as

$$\langle f, g \rangle := \int e^{-|z|^2} f(z) g(z) d\left(\frac{z}{\sqrt{\pi}}\right), \tag{2.9}$$

the following relation is evident

$$\mathcal{T}' \supset \mathcal{H} \supset \mathcal{T}. \tag{2.10}$$

As was shown in [2] $\mathcal{T}' \subset \mathcal{T}^*$, where $\mathcal{T}^*$ is the space of linear continuous functionals on $\mathcal{T}$. The idea of Bargmann transformation in the functional power series formulation appears in the natural way. Let us introduce the entire functions

$$f(z) = \sum_n (n!)^{-1/2} (f_n, z^n), \tag{2.11}$$

$$g(z) = \sum_n (n!)^{-1/2} (g_n, z^n), \tag{2.12}$$

belonging to the Hilbert space determined by the scalar product:

$$\langle f, \dot{g} \rangle = \sum (f_n, g_n). \tag{2.13}$$

It can be shown [2] that the scalar product (2.13) is equivalent to (2.9) and to

$$\langle f, g \rangle = \int \varphi(x) \bar{\psi}(x) d\left(\frac{x}{\sqrt{2\pi}}\right), \tag{2.14}$$

where

$$\varphi(x) = \sum_n f_n H_n(x) = \int A(z, x) f(\bar{z}) d\mu(z), \tag{2.15}$$

$$d\mu(z) = e^{-|z|^2} \frac{dx dy}{2\pi}, \qquad z = \frac{1}{\sqrt{2}}(x+iy). \tag{2.16}$$

$H_n(x)$ are the Hermite functions. The $A(z, x)$ is the kernel of the Bargmann unitary operator [5]. It has the form

$$A(z, x) = e^{-\frac{1}{2}z^2 - \frac{1}{4}x^2 + xz}. \tag{2.17}$$

The inverse transformation to (2.15) is:

$$f(z) = \int A(z, x) \varphi(x) \frac{dx}{\sqrt{2\pi}}. \tag{2.18}$$

One finds

$$\int d\mu(z)A(z,x)A(\bar{z},y) = \delta(x-y), \tag{2.19}$$

$$\int \frac{dx}{\sqrt{2\pi}} A(z,x)A(\bar{z}',x) = e^{z\bar{z}'}, \tag{2.20}$$

and $e^{z\bar{z}'}$ plays a role of the reproducing kernel:

$$\int d\mu(z)e^{z\bar{z}'}f(\bar{z}') = f(z). \tag{2.21}$$

In papers [5], [6] Bargmann proves that the space of test functions S and Schwartz distributions S', transformed by means of the Bargmann transformation to the space of analytic functions are defined respectively as follows

$$\mathscr{F} = \bigcap_{k=-\infty}^{\infty} \mathscr{F}^k = \bigcap_{k=-\infty}^{\infty} H^k, \tag{2.22}$$

$$\mathscr{F}' = \bigcup_{k=-\infty}^{\infty} \mathscr{F}^k = \bigcup_{k=-\infty}^{\infty} H^k, \tag{2.23}$$

where $\mathscr{F}^k$ is the Banach space of entire functions defined by the norm:

$$|||f|||_k := \sup_z \Theta_k^{-1}(z)|f(z)|, \tag{2.24}$$

$$\Theta_k^{\alpha}(z) = e^{-\alpha\frac{|z|^2}{2}}(1+|z|^2)^{\frac{k}{2}}. \tag{2.25}$$

The norm in H^k is defined as

$$\|f\|_k^2 := \int |f(z)|^2 d^k\mu(z), \tag{2.26}$$

$$d\mu^k(z) = \Theta_{2k}^2(z)dz, \tag{2.27}$$

$$dz = dxdy, \quad z = x+iy. \tag{2.28}$$

Equality of the spaces $\mathscr{T}$ and $\mathscr{F}$ is shown in the following theorem:

Theorem: $\mathscr{T} = \mathscr{F}$

The proof consists of two parts: first we show that $f \in \mathscr{T} \Rightarrow f \in \mathscr{F}$ (part a) and further $f \in \mathscr{F} \Rightarrow f \in \mathscr{T}$ (part b):

a) from (2.2) we have

$$f \in \mathscr{T} \Leftrightarrow \bigvee_{0<\lambda<1} |||f|||_{\lambda^{-1}} := \sup_z |f(z)| \exp\{-\tfrac{1}{2}|z|^2\} < \infty, \tag{2.29}$$

where

$$z = \frac{1}{\sqrt{2}}(x+iy), \quad \lambda : 0 < \lambda < 1, \tag{2.30}$$

hence we get the estimation

$$|f(z)| \leqslant |||f|||_{\lambda^{-1}} \exp\{\tfrac{1}{2}\lambda^2|z|^2\}. \tag{2.31}$$

Because of (2.26)

$$f\in\mathscr{F} \Leftrightarrow \bigwedge_{k=0}^{\infty} \|f\|_k^2 := \int |f(z)|^2 e^{-|z|^2}(1+|z|^2)^k dz. \tag{2.32}$$

From (2.31) and (2.32) we obtain

$$\bigwedge_{k=-\infty}^{\infty} \|f\|_k^2 \leqslant \int e^{-|z|^2(1-\lambda^2)}(1+|z|^2)^k dz = \frac{4\pi e^{1-\lambda^2}}{(1-\lambda^2)^{k+1}}\,\Gamma(k+1, 1-\lambda^2) < \infty, \tag{2.33}$$

where $\Gamma(\alpha, x)$ is gamma function and λ is fixed.

b) $f\in\mathscr{F} \Rightarrow f$ is entire analytic function and from (2.24) it follows:

$$\bigwedge_{k=-\infty}^{\infty} |||f|||_k := \sup |f(z)|e^{-\frac{|z|^2}{2}}(1+|z|^2)^{\frac{k}{2}} < \infty, \tag{2.34}$$

what leads to the relation:

$$f\in\mathscr{F} \Leftrightarrow |f(z)| \leqslant e^{(1/2-\varepsilon)|z|^2}, \quad \varepsilon > 0. \tag{2.35}$$

Substituting (2.35) into (2.2) one can obtain that the condition:

$$|||f|||_{\lambda^{-1}} := \sup_z |f(z)|e^{-\frac{\lambda^2}{2}|z|^2} < \infty \tag{2.36}$$

is fulfilled if $\lambda = \sqrt{1-2\varepsilon}$.

We may note that the norm (2.26) cannot be generalized to the case when z is an element of an infinite dimensional space, in contradiction to norms (2.3), (2.4) and (2.9) (cf. e.g. [2]).

3. *Free particle propagators*

The free particle propagators occur as integral kernels. The Bargmann transformation of any kernel $K(z, \bar{z}')$ is given by the following formula:

$$\hat{K}(z, \bar{z}') = \iint \frac{dxdy}{2\pi} A(z, x)K(x, y)A(\bar{z}', y). \tag{3.1}$$

In one-dimensional case after the Bargmann transformation the inhomogeneous Klein-Gordon equation is equivalent to the following one:

$$\left[\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} + z^2 - 1 - m^2\right]\hat{\Delta}^{(?)}(z, \bar{z}') = e^{z\bar{z}'} \tag{3.2}$$

and its homogeneous solutions are:

$$\hat{\Delta}(z, \bar{z}') = \frac{2}{m} e^{z\bar{z}' + 1/2(z-\bar{z}')^2 - 2m^2} \sin [2m(z-\bar{z}')], \tag{3.3}$$

$$\hat{\Delta}^{(1)}(z, \bar{z}') = 2m e^{z\bar{z}' + 1/2(z-\bar{z}')^2 - 2m^2} \cos [2m(z-\bar{z}')]. \tag{3.4}$$

For $\hat{\bar{\Delta}}(z, \bar{z}')$ which is the solution of the inhomogeneous equation we have the formula:

$$\hat{\bar{\Delta}}(z, \bar{z}') = \frac{1}{2im} e^{zz' + 1/2(z-\bar{z}')^2 - 2m^2} \left\{ e^{-2im(z-\bar{z}')} Erf\left[\sqrt{2}\left(im - \frac{z-z'}{2}\right)\right] + e^{2im(z-\bar{z}')} Erf\left[\sqrt{2}\left(im + \frac{z-\bar{z}'}{2}\right)\right]\right\}, \tag{3.5}$$

where $Erf(z)$ is the probability integral.

We may obtain further the particular solution of (3.2) by considering the suitable linear combinations of those kernels.

In two-dimensional case the Green's functions for massless particles are not Schwartz distributions, and they are well defined only if one introduces regularization (cf. e.g. [10]) leading to the appearance of indefinite metric. The regularized expressions, however, after performing the Bargmann transformation do not satisfy the transformed Klein-Gordon equation.

In three-dimensional case, after the integration over angles, we were not able to perform the last integration over the radial variable. In four-dimensional case, for $m = 0$ using the analytic extension we obtain the following kernels:

$$\hat{\Delta}(z, \bar{z}') = \frac{1}{\pi^{3/2}} e^{z_0\bar{z}'_0 + z\bar{z}' + 1/4(z_0-\bar{z}'_0)^2 + 1/4(z-\bar{z}')^2} \frac{\mathrm{sh}\,[\frac{1}{2}(z_0-\bar{z}'_0)\,|z-\bar{z}'|]}{|z-\bar{z}'|}, \tag{3.6}$$

where

$$|z-\bar{z}'| = \sqrt{(z_1-\bar{z}'_1)^2+(z_2-\bar{z}'_2)^2+(z_3-\bar{z}'_3)^2},$$

$$\hat{\Delta}^{(1)}(z, \bar{z}') = \frac{e^{z_0\bar{z}'_0 + z\bar{z}' + 1/4(z_0-\bar{z}'_0)^2 + 1/4(z-\bar{z}')^2}}{2\pi^{3/2}|z-\bar{z}'|} \left\{ e^{1/2(z_0-\bar{z}'_0)|z-\bar{z}'|} Erf\left[\frac{i}{2}|z-\bar{z}'| - \frac{i}{2}(z_0-\bar{z}'_0)\right] + e^{-1/2(z_0-\bar{z}'_0)|z-\bar{z}'|} Erf\left[\frac{i}{2}|z-\bar{z}'| + \frac{i}{2}(z_0-\bar{z}'_0)\right]\right\}, \tag{3.7}$$

$$\hat{\bar{\Delta}}(z, \bar{z}') = \frac{e^{z_0\bar{z}'_0 + z\bar{z}' + 1/4(z_0-\bar{z}'_0)^2 + 1/4(z-\bar{z}')^2}}{4\pi^{3/2}|z-\bar{z}'|}$$

$$\times \{ e^{-\frac{i}{2}(z_0-\bar{z}'_0)|z-\bar{z}'|} Erf[\tfrac{1}{2}|z-\bar{z}'| - \tfrac{1}{2}(z_0-\bar{z}'_0)] + e^{1/2(z_0-\bar{z}'_0)|z-\bar{z}'|} Erf[\tfrac{1}{2}|z-\bar{z}'| + \tfrac{1}{2}(z_0-\bar{z}'_0)]\}. \tag{3.8}$$

The index of growth is equal to 2 and type is equal to 1/2. The equation corresponding to (3.2) in four-dimensional case has the form:

$$\left[\frac{\partial^2}{\partial z_1^2}+\frac{\partial^2}{\partial z_2^2}+\frac{\partial^2}{\partial z_3^2}-\frac{\partial^2}{\partial z_0^2}-2\left(z_1\frac{\partial}{\partial z_1}+z_2\frac{\partial}{\partial z_2}+z_3\frac{\partial}{\partial z_3}-z_0\frac{\partial}{\partial z_0}\right)\right.$$

$$\left.+z_1^2+z_2^2+z_3^2-z_0^2-2+m^2\right]\Delta^{(?)}(z,\bar{z}') = e^{z_0\bar{z}'_0+z\bar{z}'} \tag{3.9}$$

where $\hat{\Delta}^{(?)}(z,\bar{z}')$ denotes any inhomogeneous solution (similary as in one-dimensional case (see Eq. (3.2)).

For $m \neq 0$ in all dimensions greater than one the calculations of the Bargmann transformation of Green's functions are very complicated, however we can express them as the product $g(z_0, \bar{z}'_0, z, \bar{z}')f(z_0-\bar{z}_0, z-\bar{z}')$ where $g(z_0, \bar{z}'_0, z, \bar{z}')$ is an analytic function and $f(z_0-z_0, z-\bar{z}')$ is given by the integral representation. For instance in four dimensional case

$$\hat{\Delta}(z,\bar{z}') = 4\,\frac{e^{z_0\bar{z}'_0+z\bar{z}'-2m^2}}{|z-\bar{z}'|}\int\limits_0^\infty \frac{ds}{(2\pi)^2}\,\frac{s}{\sqrt{s^2+m^2}}\sin\left[|z-\bar{z}'|s\right]\sin\left[(z_0-\bar{z}'_0)\sqrt{s^2+m^2}\right], \tag{3.10}$$

where $|z-\bar{z}'| = \sqrt{(z_1-\bar{z}'_1)^2+(z_2-\bar{z}'_2)^2+(z_3-\bar{z}'_3)^2}$. In every case the analyticity of $f(z_0-\bar{z}'_0, z-\bar{z}')$ we can test using the method of the functional power series [2].

This method permits to test the analyticity of the functions of many variables when we can estimate their expansions coefficients without the idea of the associative radii [11] which is more complicated.

The method is following: let us expand $f(\xi)$ in the double power series:

$$f(\xi) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\sqrt{n!m!}}\,a_{n,m}\xi_0\xi_1, \qquad \xi = (\xi_0, \xi_1). \tag{3.11}$$

In our cases $\xi_0 = z_0-\bar{z}'$, $\xi_1 = \sqrt{\sum_{i=1}^{N}(\bar{z}_i-\bar{z}'_i)^2}$; here $N = 1, 2, 3$ for two, three and four dimensions respectively.

This power series is equivalent to the functional series:

$$f(\xi) = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{i_1=0}^{1}\cdots\sum_{i_n=0}^{1} f_{i_1\ldots i_n}\xi_{i_1}\cdots\xi_{i_n}$$

$$= \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\sum_{m=0}^{n}\binom{n}{m}f_{n-m}\xi_0^{n-m}\xi_1^m, \tag{3.12}$$

where

$$f_{n-m,m} = f_{\underbrace{0\,\ldots\,0}_{n-m}\;\underbrace{1\,\ldots\,1}_{m}}. \tag{3.13}$$

Comparing coefficients of equal powers of ξ_0 and ξ_1, in (3.11) and (3.12) we have:

$$\frac{1}{\sqrt{(n-m)!m!}}\, a_{n-m,m} = \frac{1}{\sqrt{n!}}\binom{n}{m} f_{n-m,m}, \tag{3.14}$$

hence

$$f_{n-m,m} = \sqrt{\frac{(n-m)!m!}{n!}}\, a_{n-m,m}. \tag{3.15}$$

But we can also write (3.11) in the form:

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (f_n \xi^n), \quad \text{where} \quad \xi^2 = \xi_0^2 + \xi_1^2. \tag{3.16}$$

Introducing the complex numbers γ_1 and γ_2, satisfying $|\gamma_1|^2 + |\gamma_2|^2 = 1$, we can rewrite (3.16).

$$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left\{ \sum_{m}^{n} \binom{n}{m} f_{n-m,m} \gamma_0^{n-m} \gamma_1^m \right\} \xi^n, \tag{3.17}$$

where

$$f_n = \sum_{m=0}^{n} \binom{n}{m} f_{n-m,m} \gamma_0^{n-m} \gamma_1^m, \tag{3.18}$$

substituting (3.15) to (3.18) we have

$$f_n = \sum_{m=0}^{n} \gamma_0^{n-m} \gamma_1^m \binom{n}{m} \sqrt{\frac{(n-m)!m!}{n!}}\, a_{n-m,m} = \sum_{m=0}^{n} \sqrt{\frac{n!}{(n-m)!m!}}\, a_{n-m,m} \gamma_0^{n-m} \gamma_1^m. \tag{3.19}$$

From Cauchy-Hadamard's theorem the convergence radius of the single power series (3.19) is

$$R = \limsup_{n\to\infty} \left(\sum_{m=0}^{n} \sqrt{\frac{1}{(n-m)!m!}}\, a_{n-m,m} \gamma_0^{n-m} \gamma_1^m \right)^{\frac{1}{n}}, \tag{3.20}$$

It can be showed that for every $f(z_0-\bar{z}'_0, z-\bar{z}')$ connected with the given Green's function $R = \infty$. From this fact the analyticity of Bargmann transformations of relativistic Green's functions results.

4. Formulation of the axioms of Quantum Field Theory in the Bargmann space

A. Unitarity condition

Unitarity condition (cf. e.g. [13]) in the Bargmann spaces takes the form:

$$\Omega[g] * \Omega[g] = 1, \tag{4.1}$$

where $\Omega[g]$ denotes the generating functional for the vacuum expectation value of the chronological product of current operators transformed to z-space.

$$\omega(z_1, \ldots, z_n) = \frac{\partial}{\partial g(z_1)} \cdots \frac{\partial}{\partial g(z_n)} \Omega[g]\Big|_{g=0}, \tag{4.2}$$

$$\Omega[g] = \sum \frac{1}{n!} \int d\mu(z_1) \ldots \int d\mu(z_n)\omega(z_1, \ldots, z_n)g(z_1) \ldots g(z_n), \tag{4.3}$$

here $g(z)$ is the test function.

$$* \equiv \exp \frac{\delta}{\delta g} \hat{\Delta}^+ \frac{\delta}{\delta g}, \tag{4.4}$$

$$d\mu(z_i) = e^{-|z_i|^2} dx_i dy_i, \quad z_i = \frac{1}{\sqrt{2}}(x_i + iy_i), \quad i = 1, 2, \ldots n, \tag{4.5}$$

$$\hat{\Delta}^+(z, \bar{z}') = \iint \frac{d^4x}{(2\pi)^2} \frac{d^4y}{(2\pi)^2} A(z, x)\Delta^+(x-y)A(\bar{z}', y). \tag{4.6}$$

Unitarity condition imposes the restriction on growth and type of the functional Ω [2]. Using the connection of S and Ω [12]

$$s[g; \alpha, \beta] = e^{\alpha\beta}\Omega[g + g_0[\alpha, \beta]], \tag{4.7}$$

where

$$g[\alpha, \beta; z'] = \int d\mu(z) \{\alpha(z)f_+(z, z') + \beta(z)f_-(z, z')\}, \tag{4.8}$$

f_+ and f_- are the Bargmann transformations of the orthonormal solutions of Klein-Gordon equation:

$$f_\pm(z, z') = \frac{1}{(2\pi)^{3/2}} \int \frac{d^4x}{(2\pi)^2} \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} A(z', x)A(z, \vec{k}) \frac{e^{\pm i\vec{k}\vec{x} \mp i\omega(\vec{k})x_0}}{\sqrt{2\omega(\vec{k})}}, \tag{4.9}$$

$$\int d\mu(z)\alpha(z)\beta(z) < \infty, \quad \bar{z} = (z_1, z_2, z_3). \tag{4.10}$$

We have the restriction

$$|S[0;\alpha,\beta]| \leqslant \|S\| e^{1/2\|\alpha\|^2+1/2\|\beta\|^2}. \tag{4.11}$$

$\|S\|$ is the norm of the bounded operator. Due to the unitarity condition we may put $\|S\| = 1$ and because of (4.7) we have further:

$$|\Omega[g+g_0[\alpha,\beta]]| \leqslant e^{1/2\|\alpha-\beta\|^2}. \tag{4.12}$$

B. Causality condition

In general the causality condition in Minkowski space R^4 has the form (cf. e.g. [13])

$$\frac{\delta}{\delta q(y)} J[x] + (x \leftrightarrow y) = i\sigma[q;x,y] \quad \text{for} \quad x \lesssim y, \tag{4.13}$$

where $J[x]$ is the generating functional for the retarded functions

$$J(x;y_1,\dots,y_n) = \frac{\delta}{\delta q(y_1)} \cdots \frac{\delta}{\delta q(y_n)} J[x]\bigg|_{q=0}, \tag{4.14}$$

$$J[x] = \sum_{n=1}^{\infty} \frac{1}{n!} \int dy_1 \dots \int dy_n J(x;y_1,\dots,y_n) q(y_1) \dots q(y_n), \tag{4.15}$$

here $q(y)$ is the test function.

The connection between $J[x]$ and $\Omega[q]$ is following:

$$J[x] = -i\bar{\Omega}[q] * \frac{\delta}{\delta q(x)} \Omega[q] \tag{4.16}$$

where $* \equiv \exp \dfrac{\delta}{\delta q} \Delta^+ \dfrac{\delta}{\delta q}$ and $\sigma[q;x,y]$ is the quasilocal term.

It is evident that (4.13) is equivalent to

$$\Theta(x_0-y_0) \frac{\delta}{\delta q(y)} J[x] + (x \leftrightarrow y) = i\sigma[q;x,y], \tag{4.17}$$

if we take into account that (4.17) is Lorentz invariant.

After the Bargmann transformation the equation (4.17) turns into the following expression:

$$\frac{\delta J'[z]}{\delta g(z')} + \frac{\delta J'[z']}{\delta g(z)} + \int d\mu(\xi_0) \int d\mu(\xi'_0) e^{z_0\xi^*_0 + z'_0\xi'^*_0}$$

$$\times Erf\left(\frac{z_0 - z'_0 + \xi^*_0 - \xi'^*_0}{\sqrt{2}}\right) \left[\frac{\delta J[\xi'_0, z']}{\delta g(\xi_0, z)} - \frac{\delta J[\xi_0, z]}{\delta g(\xi'_0, z')}\right] = \frac{i}{\sqrt{2}\,\pi} \sigma[g;z,z'], \tag{4.18}$$

where

$$g(z) = \int \frac{d^4 y}{(2\pi)^2} A(z, y) q(y) dy, \tag{4.19}$$

$$J'[z] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d\mu(z_1) \dots \int d\mu(z_n) J[z; z_1, \dots, z_n] g(z_1) \dots g(z_n), \tag{4.20}$$

$$J'(z; z_1, \dots, z_n) = \int \frac{d^4 x}{(2\pi)^2} A(z, x) \int \frac{dy_1}{(2\pi)^2} \dots \int \frac{dy_n}{(2\pi)^2} A_n(z_1, \dots, z_n; y_1, \dots, y_n) \times J[x; y_1, \dots, y_n], \tag{4.21}$$

$$A_n(z_1, \dots, z_n; y_1, \dots, y_n) = A(z_1, y_1) \dots A(z_n, y_n). \tag{4.22}$$

We note that the Lorentz invariance of (4.17) corresponds in z-space to the validity of the condition (4.18) transformed to any Poincaré frame by means of the kernels (4.24)–(4.27) from the section C. This problem will be treated in detail in subsequent work.

C. Invariant analytic properties of scalar field in Bargmann space

The general Lorentz transformation can be always written as the product of three Lorentz transformations (cf. e.g. [13])

$$\alpha = \varrho_1 \sigma \varrho_2, \tag{4.23}$$

where σ is the transformation which does not change the variables x_1 and x_2, whereas ϱ_1 and ϱ_2 are rotations in space which do not change the variable x_0. In z-space, the kernel corresponding to rotation R is:

$$\hat{O}^R(z, \bar{z}') = \int \frac{dx}{(2\pi)^2} A(z, x) A(\bar{z}', R\vec{x}, x_0). \tag{4.24}$$

Performing the calculation one gets

$$\hat{O}^R(z, \bar{z}') = e^{z_i R \bar{z}'_i + z_0 \bar{z}'_0}. \tag{4.25}$$

The kernel corresponding to Lorentz transformation (rotation about the angle β in x_0 and x_1 plane) can be written as follows

$$\hat{O}(z, \bar{z}') = e^{1/2(z^2 + z'^2)} \frac{1}{\cosh \beta} e^{\frac{1}{\cosh \beta}(z_0 \bar{z}'_0 + z_3 \bar{z}'_3) + \operatorname{tgh} \beta (z_0 z_3 - \bar{z}'_0 \bar{z}'_3)} \tag{4.26}$$

and the operation of the translation a_μ is given by

$$T^a(z, \bar{z}') = e^{-1/8 a^2_\mu + 1/2(\bar{z}'_\mu - z_\mu) + z_\mu \bar{z}'_\mu}. \tag{4.27}$$

We see, therefore, that two major properties of free fields in Bargmann space — its growth and type — remain invariant under the Poincaré transformation.

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