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NEW SUMMATION TECHNIQUE IN PERTURBATION THEORY*

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A new summation technique in perturbation theory is developed. By means of rearrangement of the ordinary perturbation expansion we can regroup terms with respect to order of correlation. The various correlations partially factorize from the rest which is totally factorizable. This factorization makes the series summable to infinite order in the coupling constant.

1. *Introduction*

The reasons for studying higher orders in perturbation theory are many. Summed up results will be important not only for strong interactions, but also in weak coupling theories like QED and scalar electrodynamics, to get an idea of asymptotic properties or the bound state structure. Also for a strong Yang-Mills theory in the asymptotic free region non-perturbative results would be required for an analytic continuation down to available energies.

A well known but questionable approach goes via the eikonal model. Due to the neglect of recoil effects, however, the loss of information in these amplitudes as compared to the original model is not known. In passing we notice here that the recoil is defined relative to the straight line path approximation (the c-number part of the current operator) which gives the eikonal model. Therefore, instead of trying to find a justification of this model in some narrow region, we should rather derive the recoil corresponding to the full model. To minimize the calculations we here study the generalized infinite ladder (with all criss-crosses) in a scalar: φ^2: ϕ-model and then extend to scalar electrodynamics. All essential results from our earlier papers [1–3] are summarized.

We make a simple rearrangement of the ordinary perturbation expansion and regroup terms with respect to the order of correlation. The various correlation terms partially factorize from the rest which is totally factorizable. This permits one to difine forminvariant correlation functionals irrespective to which individual quanta are involved and we can then

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easily sum to all orders in the coupling constant. Each correlation term is proportional to a power of the coupling constant, which is twice the order of correlation. This justifies the eikonal approximation in a weak coupling theory since the eikonal result, except for kinematical correlations which are everywhere present, is uncorrelated.

2. *The correlation rearrangement*

The masses of the φ- and ϕ-quanta are denoted m and μ, respectively. As usual we define (Fig. 1)

$$t = q_t^2, \qquad q_t = p_a - p_{a'},$$
$$s = q_s^2, \qquad q_s = p_a + p_b, \tag{2.1}$$

and the corresponding $(n+1)$-th order amplitude is given by

$$-iM_{n+1}(s,t) = \frac{(-ig)^2}{(n+1)!} \int \prod_{l=1}^{n+1} \left[\frac{d^4 k_l}{(2\pi)^4} \Delta_F(k_l) \right]$$
$$\times I \cdot (2\pi)^4 \cdot \delta^4(p_a - p_{a'} - \sum_{l=1}^{n+1} k_l), \tag{2.2}$$

with $\Delta_F(k_l) = i(k^2 - \mu^2 + i\varepsilon)^{-1}$.

We then integrate over arbitrary one of the $(n+1)$ k-spaces, say the r-th, and by means of the δ-function we can write

$$\Delta_F(k_r) = \int d^4 x \Delta_F(x) e^{iq_t \cdot x - i \sum_{l \neq r}^{n+1} k_l \cdot x}. \tag{2.3}$$

After this choice of r we can say which parts of the through-going momenta are in- and out-going and obtain

$$I_{\text{sym}} = \sum_{r=1}^{n+1} I_{r\,\text{sym}} = \sum_{r=1}^{n+1} \sum_{\text{perm}} I_r, \tag{2.4}$$

where

$$I_{r\,\text{sym}} = \sum_{\substack{\text{perm} \\ ''s''}} \prod_{\substack{i=a,a' \\ b,b'}} \prod_{l=\alpha(i)}^{\beta(i)} A_{j_l}^{(i)}. \tag{2.5}$$

$$A_{j_l}^{(i)} = \frac{ig}{2\varepsilon_i p_i \cdot K_{j_l} + K_{j_l}^2 + p_i^2 - m_i^2 + i\varepsilon}, \tag{2.6}$$

and

$$K_{j_l} = \sum_{t=j_{\alpha(i)}}^{\beta(i)} k_t. \tag{2.7}$$

The numbers α, β and ε are given by

i	a	a'	b	b'
α	1	$r+1$	1	$s+1$
β	$r-1$	$n+1$	$s-1$	$n+1$
ε	-1	$+1$	$+1$	-1

(2.8)

The "s"-summation in (2.5) is a double sum over s_1 and s_2 such that $s_1+s_2 = s-1$, where $s-1$ is the number of quanta absorbed by the b-prong. Out of these s_1 do not first cross the r-th line whereas s_2 do cross the r-th line (Fig. 1).

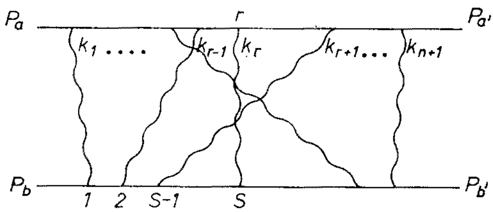

Fig. 1. A generalized infinite ladder

We then apply what one could call a recoil correlation rearrangement [1–3], which after summation over perturbations leads to the simple form

$$\sum_{\substack{\text{perm}\\(i)}} \prod_{l=\alpha(i)}^{\beta(i)} A_{jl}^{(i)} = \left\{ \prod_{l=\alpha(i)}^{\beta(i)} f_l^i + \sum_{\substack{s<t\\ \alpha(i)}}^{\beta(i)} f_s^i f_t^i \frac{-x_{st}^i}{1+x_{st}^i} \prod_{\substack{l\neq s,t\\ \alpha(i)}}^{\beta(i)} f_l^i + \begin{matrix}\text{higher}\\ \text{correlations}\end{matrix} \right\}, \tag{2.9}$$

with

$$f_l^i = \frac{ig}{2\varepsilon_i p_i \cdot k_l + k_l^2 + p_i^2 - m_i^2 + i\varepsilon}, \tag{2.10}$$

$$x_{st}^i = \frac{2\sum\limits_{\substack{s<t\\ \alpha(i)}}^{\beta(i)} k_s \cdot k_t - (p_i^2 - m_i^2)}{2\varepsilon_i p_i(k_s+k_t) + k_s^2 + k_t^2 + 2(p_i^2 - m_i^2)}. \tag{2.11}$$

Inserting (2.9) and (2.4) and defining the pair correlation "current"

$$\chi_{st}^i = f_s^i f_t^i \frac{-x_{st}^i}{1+x_{st}^i}, \tag{2.12}$$

after some straight forward combinatorics we get

$$I_{\text{sym}}^{n+1} = \sum_{r=1}^{n+1} I_{r\,\text{sym}} = \left\{ \prod_{l=1}^{n} J_l^A J_l^B + \sum_{s<t}^{n} (\chi_{st}^A J_s^B J_t^B + J_s^A J_t^A \chi_{st}^B + \chi_{st}^A \chi_{st}^B) \right.$$

$$\left. \times \prod_{l\neq s,t}^{n} J_l^A J_l^B + \begin{matrix}\text{higher}\\ \text{correlations}\end{matrix} \right\}, \tag{2.13}$$

where the total currents are defined by

$$J^Q = \sum_{i\in Q} f^i, \quad \chi^Q = \sum_{i\in Q} \chi^i, \tag{2.14}$$

$$Q = A \equiv \{aa'\} \text{ or } B \equiv \{bb'\}.$$

In order to exploit the factorization properties we then define the functionals

$$U = i\int \frac{d^4k}{(2\pi)^4}\, \Delta_F(k) e^{-ikx} J^A(k) J^B(k), \tag{2.15}$$

$$P = i^2 \int \frac{d^4k_1 d^4k_2}{(2\pi)^8}\, \Delta_F(k_1)\Delta_F(k_2) e^{-i(k_1+k_2)x} \times \{\chi^A_{12} J^B_1 J^B_2 + J^A_1 J^A_2 \chi^B_{12} + \chi^A_{12}\chi^B_{12}\}, \tag{2.16}$$

which after insertion gives

$$-iM_{n+1} = g^2 \int d^4x e^{iqx} \Delta_F(x) \frac{1}{(n+1)!}\left\{(iU)^n + \binom{n}{2} i^2 P (iU)^{n-2} + \dots\right\}$$

$$= g^2 \int d^4x e^{iqx} \Delta_F(x) \left\{1 + \frac{i^2 P}{2!} \frac{\delta^2}{\delta(iU)^2} + \dots\right\} \frac{(iU)^n}{(n+1)!}. \tag{2.17}$$

Because of the partial factorization, as exhibited in (2.17), we can now easily sum to an infinite order in n and obtain

$$M(s,t) = ig^2 \int d^4x \Delta_F(x) e^{iqx} \left\{1 + \frac{i^2 P}{2!} \frac{\delta^2}{\delta(iU)^2} + \dots\right\} \frac{e^{iU}-1}{iU}. \tag{2.18}$$

The importance of higher order correlations will be discussed after the next example, and we are here content with the first correction term to the eikonal model.

3. *Scalar electrodynamics*

The same procedure goes through for this model with certain modifications. The propagators (2.6) are now replaced by

$$A^{(i)}_{\mu_l} = ie\, \frac{2(p_i + \varepsilon_i(k_1 + \dots + k_{l-1})_{\mu_l}}{2\varepsilon_i p_i K_l + K_l^2 + i\varepsilon} \tag{3.1}$$

and the quantity I becomes $I_r \cdot V_r$ where V_r is the product of the two vertices of the r-th quantum

$$V_r = -e^2 (P_a + P_{a'}) \cdot (P_b + P_{b'}), \tag{3.2}$$

$$P_i = p_i + \varepsilon_i \sum_{l=\alpha(i)}^{\beta(i)} k_l, \quad i = a, a', b, b'. \tag{3.3}$$

In this theory we also have sea-gull terms, which after summation over permutations, leave contributions to the pair correlation effect and higher order correlations. In fact, as we will see, these sea-gulls are presicely needed to make the pair correlation tensor conserved. Proceeding exactly as in the previous case we obtain

$$I_{r\,\text{sym}} V_r = \sum_{\substack{\text{prem}\\ \text{``s''}}} \prod_{\substack{i=a,a'\\ b,b'}} \{\prod_{l=\alpha(i)}^{\beta(i)} f^i_{\mu_l} + \sum_{\substack{s<t\\ \alpha(i)}}^{\beta(i)} \chi^i_{\mu_s\mu_t} \prod_{\substack{l\neq s,t\\ \alpha(i)}}^{\beta(i)} f^i_{\mu_l} + \ldots\} V_r(P), \tag{3.4}$$

where the current elements are defined by

$$f^i_\mu = ie\,\frac{2P_{i\mu}+\varepsilon_i k_\mu}{2\varepsilon_i p_i k + k^2 + i\varepsilon}, \tag{3.5}$$

$$\chi^i_{\mu_s\mu_t} = \left\{ f^i_{\mu_s} ie\,\frac{2\varepsilon_i k_s \mu_t}{y^i_{st}} + f^i_{\mu_t} ie\,\frac{2\varepsilon_i k_t \mu_s}{y^i_{st}} - f^i_{\mu_s} f^i_{\mu_t} x^i_{st} - (ie)^2\,\frac{2g_{\mu_s\mu_t}}{y^i_{st}} \right\} \frac{1}{1+x^i_{st}}, \tag{3.6}$$

and

$$y^i_{st} = 2\varepsilon_i p_i (k_s + k_t) + k_s^2 + k_t^2. \tag{3.7}$$

Thus if there were no recoil effects in the vertex function V_r (3.2) we would again obtain the amplitude (2.18) with the currents in (2.10–12) replaced by (3.5–6). As is easily checked the currents (3.5) sum up to a conserved current J^Q_μ

$$J^Q_\mu = \sum_{i\in Q} f^i_\mu, \quad \chi^Q_{\mu\nu} = \sum_{i\in Q} \chi^i_{\mu r} (Q = A, B), \tag{3.8}$$

whereas in the pair currents (3.6) the numerator recoil in the two first terms, the denominator recoil in the third term and the sea-gulls, which are proportional to $g_{\mu\nu}$, sum up to a conserved tensor current for each separate prong.

The vertex function V_r, however, includes recoil effects and we will now demonstrate that these can be exactly treated by means of a generalization of Low's theorem [4–5]. Although the original version of this theorem applies for soft emitted quanta we here demonstrate an extension which is correct even for hard virtual (exchanged) quanta.

Formally we can expand V_r according to

$$V_r(P_i) = \left\{ 1 + \sum_{s=1}^{\infty} \frac{1}{s!} (\hat{K}^i_{\beta(i)})^s \right\} V_0(p_i), \tag{3.9}$$

where

$$\hat{K}^i_{\beta(i)} = \varepsilon_i K_{\beta(i)} \left\{ \frac{\partial}{\partial p_i} \right\}_{\text{on } V_0}. \tag{3.10}$$

This provides a simple factorizable form

$$V_r(P_i) = e^{\hat{K}^i_{\beta(i)}} V_0(p_i) = \prod_{l=\alpha(i)}^{\beta(i)} e^{\hat{k}^i_l} V_0(p_i), \tag{3.11}$$

with

$$\hat{k}_l^i = \varepsilon_i k_l \left\{ \frac{\partial}{\partial p_i} \right\}_{\text{on } V_0}. \tag{3.12}$$

We have here introduced the notation $V_0 = \lim_{P_i \to p_i} V_r$.

However, because of the explicit form of V_r, as given by (3.2), only the first order derivatives, with respect to each separate external momentum, are non-zero. We can therefore exactly include these recoil effects by the replacements

$$f_\mu^i \to \hat{f}_\mu^i \equiv f_\mu^i e^{\hat{k}^i} = f_\mu^i + f_\mu^i \hat{k}^i, \tag{3.13}$$

$$\chi_{\mu_s \mu_t}^i \to \hat{\chi}_{\mu_s \mu_t}^i \equiv \chi_{\mu_s \mu_t}^i e^{(\hat{k}_s{}^i + \hat{k}_t{}^i)} = \chi_{\mu_s \mu_t}(1 + \hat{k}_s^i + \hat{k}_t^i), \tag{3.14}$$

$$V_r \to V_0. \tag{3.15}$$

However, now the currents $\hat{f}_\mu^i$ do not add up to a conserved total current and therefore a leakage must take place somewhere, like in Low's case. The difference is that in our case we know explicitly the form V_0 of the unknown blob (Fig. 2). Therefore (3.13–3.14)

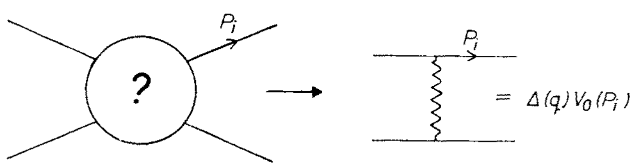

Fig. 2. A known core function

are now also valid in the ultraviolet part of spectrum. Adding such a leakage current l and making use of the fact that it should not include a pole part, covariance and charge conservation give it's exact form

$$l_\mu^i = -ie \frac{\partial}{\partial p_i^\mu}. \tag{3.16}$$

In Low's case the leakage took place via emission from the unknown blob (Fig. 3) since that was not included from the beginning. In our case the only things left out are

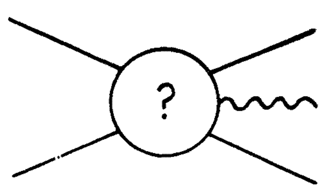

Fig. 3. An unknown core function

the sea-gull formations with the r-th quantum involved. These had to be dropped in order to remove the δ-function in a simple way by integration over the k_r-space. However, before making a graphical interpretation we go a few steps further. Instead of (3.13) the "full" current is now given by

$$\hat{J}^i_\mu = J^i_\mu + D^i_\mu, \qquad D^i_\mu = J^i_\mu \hat{k}^i - ie\, \frac{\partial}{\partial p^i_\mu}\,. \tag{3.17}$$

After some detailed analysis one finds that no such leakage currents should be added to the pair current [2]. With the total currents defined like in (3.8) we further find

$$\prod_{l=1}^{n} \hat{J}^A_{\mu_l} \hat{J}^{B\mu_l} = \prod_{l=1}^{n} J^A_{\mu_l} J^{B\mu_l} + \sum_{s=1}^{n} (D^A_{\mu_l} J^{B\mu_l} + J^A_{\mu_l} D^{B\mu_l} + D^A_{\mu_l} D^{B\mu_l}) \cdot \prod_{l \neq s}^{n} J^A_{\mu_l} J^{B\mu_l}$$

$$+ \sum_{s<t}^{n} (J^A_{\mu_s} D^A_{\mu_t} D^{B\mu_s} J^{B\mu_t} + D^A_{\mu_s} J^A_{\mu_t} J^{B\mu_s} D^{B\mu_t}) \prod_{l \neq s,t}^{n} J^A_{\mu_l} J^{B\mu_l}, \tag{3.18}$$

and like in the previous case we define functionals according to

$$U = i \int \frac{d^4k}{(2\pi)^4}\, \Delta_F(k) e^{-ikx} J^A_\mu J^{B\mu}, \tag{3.19}$$

$$\hat{S} \equiv i \int \frac{d^4k}{(2\pi)^4}\, \Delta_F(k) e^{-ikx} \{D^A_\mu J^{B\mu} + J^A_\mu D^{B\mu} + D^A_\mu D^{B\mu}\}, \tag{3.20}$$

$$\hat{P} = i^2 \int \frac{d^4k_1 d^4k_2}{(2\pi)^8}\, \Delta_F(k_1) \Delta_F(k_2) e^{-i(k_1+k_2)}$$

$$\times \{J^A_{\mu_1} D^A_{\mu_2} D^{B\mu_1} J^{B\mu_2} + D^A_{\mu_1} J^A_{\mu_2} J^{B\mu_1} D^{B\mu_2} + \hat{\chi}^A_{\mu_1\mu_2} \hat{J}^{B\mu_1} \hat{J}^{B\mu_2} + \hat{\chi}^B_{\mu_1\mu_2} \hat{J}^{A\mu_1} \hat{J}^{A\mu_2} + \hat{\chi}^A_{\mu_1\mu_2} \hat{\chi}^{B\mu_1\mu_2}\}. \tag{3.21}$$

Insertion into an expression like (2.2), plus the sea-gull contributions, of course, then gives an amplitude of the form

$$-iM_{n+1}(s,t) = \frac{1}{(n+1)!} \int d^4x e^{iqx} \Delta_F(x)$$

$$\times \left\{1 + \frac{i\hat{S}}{1!}\, \frac{\delta}{\delta(iU)} + \frac{i^2\hat{P}}{2!}\, \frac{\delta^2}{\delta(iU)^2} + \ldots\right\} (iU)^n V_0, \tag{3.22}$$

which sums up to

$$M(s,t) = i \int d^4x e^{iqx} \Delta_F(x) \left\{1 + \frac{i\hat{S}}{1!}\, \frac{\delta}{\delta(iU)} + \frac{i^2\hat{P}}{2!}\, \frac{\delta^2}{\delta(iU)^2} + \ldots\right\} \frac{e^{iU}-1}{iU}\, V_0 \tag{3.23}$$

The graphical interpretation of the leakage current parts in the three terms in (3.20) is given in Fig. 4 a, b and c. The similar but pair correlated two first terms in (3.21) correspond to graphs of type given in Fig. 5. The sea-gull parts in the last three terms of (3.21) are examplified in Fig. 6 a, b and c.

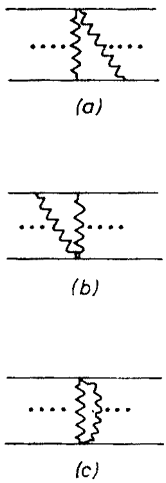

Fig. 4

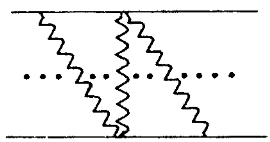

Fig. 5

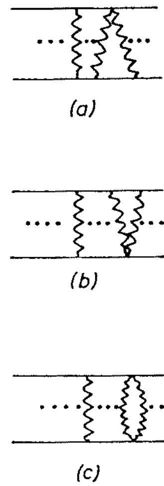

Fig. 6

Fig. 4. Seagulls with the r-th and one other photon
Fig. 5. Seagulls with the r-th and two other photons
Fig. 6. Seagull formations

4. Discussion

The first term in (3.23) is the usual eiknonal model, which is here obtained without any assumption of large external momenta or similar approximation. The technical clue is the rearrangement of the full ladder expansion with respect to order (c) of correlation. The correlation terms in (3.23) are proportional to α^{c+1} which is easily seen in (3.19–3.21). Therefore, in a weak coupling theory like this, the correlation expansion decreases with increasing order of correlation in agreement with our working hypothesis.

By a detailed analysis we find that in scalar electrodynamics and QED the eikonal model includes exactly just one (arbitrary) exchanged photon. The correlation terms $\hat{S}$ and $\hat{P}$ in (3.23) includes exactly two and three exchanged photons, respectively, and the remaining photons are treated in the eikonal approximation. In passing we notice that in the scalar: φ^2: ϕ-model the summed up result (2.18 includes no $\hat{S}$-term and therefore the eikonal model includes exactly two exchanged quanta and the rest within the eikonal approximation.

Neglecting the triple and higher correlations in this expansion gives an error of order α^4, and we can, therefore, after adding the radiative corrections to the corresponding order, reproduce exactly the ordinary perturbation result to the sixth order. As was demonstrated in an earlier paper [2], vacuum polarization graphs to this order do not spoil this factorization scheme, but merely modify $g_{\mu\nu}$ in the photon propagators. In a subsequent work it will be demonstrated how other virtual corrections can also be treated in this recoil--correlation expansion approach.

After these technical questions are solved, we can then derive electromagnetic form-factors and recoil corrections to the bound states of the eikonal amplitude [6].

It will be also interesting to see in a strong coupling theory whether the correlation expansion decreases with the increasing correlation order *c*. This question will be probably important also for an eventual application of the earlier suggested multiperipheral model [3] where this correlation-rearrangement is exploited.

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