Vol. B6 (1975) ACTA PHYSICA POLONICA No 3

NON-STATIC CHARGED FLUID DISTRIBUTIONS IN GENERAL RELATIVITY

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(Received December 21, 1973)

Solutions for two different models of non-static charged fluid distributions are given. One of them exhibits cylindrical symmetry, while the other may be considered to be plane symmetric. In both solutions it has been shown that once the contraction starts, the collapse to a singularity becomes accelerated and the entire collapse occurs within a finite proper time.

1. Introduction

It has been recently shown that the shear-free (or isotropic) irrotational expansion or contraction of a charged dust distribution (i. e. with vanishing pressure) is not permissible in general relativity (De [1], Raychaudhuri and De [2]).

In view of the above result it might be of some interest to study shear-free and irrotational motion of models consisting of charged fluid, where the pressure of the matter cannot be neglected. Indeed there are already few such solutions for models exhibiting spherical symmetry (Faulkes [3], Vaidya and Shah [4]) and two special solutions starting with Cylindrical Symmetry (De [5]). While Faulkes from the very outset considered an isotropic form of the line-element, Vaidya and Shah made a few simplifying assumptions, one of which restricts the motion to be shear-free. In the two solutions of De, no definite conclusion concerning the dynamical behaviour of the distributions could be reached.

In the present paper, we present two new classes of solutions, for charged fluid distributions with cylindrical and plane symmetries (cf. Taub [6]), respectively. In both solutions the space is found to be conformally flat and once the contraction sets in, the distributions are found to collapse to a state of infinite density within a finite proper time.

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2. Cylindrical symmetric distributions

Let us consider the non-static cylindrically symmetric line-element in co-moving coordinates as

$$ds^2 = g_{00}(r, t)dt^2 - e^{2\Psi(r,t)}[dr^2 + dz^2] - l(r, t)d\varphi^2. \tag{1}$$

The Einstein–Maxwell equations in usual notation are:

$$R^\alpha_\beta = -8\pi[T^\alpha_\beta - \tfrac{1}{2}\,\delta^\alpha_\beta T], \tag{2}$$

where

$$T^\alpha_\beta = (P+\varrho)v^\alpha v_\beta - \delta^\alpha_\beta P - \frac{1}{4\pi}\,F^{\alpha\nu}F_{\beta\nu} + \frac{\delta^\alpha_\beta}{16\pi}\,(F^{\mu\nu}F_{\mu\nu}),$$

$$F^{\alpha\beta},\, \beta = 4\pi\sigma v^\alpha \tag{3}$$

and

$$*F^{\alpha\beta},\, \beta = 0. \tag{4}$$

From the symmetry consideration only the radial component of electric field is present, and the magnetic field components are assumed to be absent. Since the co-ordinate system is assumed to be co-moving,

$$v^\mu = \frac{\delta^\mu_0}{\sqrt{g_{00}}}. \tag{5}$$

Eq. (2) written out explicitly gives

$$R^1_1 = \frac{1}{g_{00}}\left[\frac{\dot\Psi\dot l}{2l} + \ddot\Psi + 2\dot\Psi^2 - \frac{\dot\Psi \dot g_{00}}{2g_{00}}\right]$$
$$+ e^{-2\Psi}\left[\frac{\Psi' g_{00}'}{2g_{00}} + \frac{\Psi' l'}{2l} - \Psi'' - \frac{1}{2}\frac{g_{00}''}{g_{00}} - \frac{1}{2}\frac{l''}{l} + \frac{1}{4}\left(\frac{g_{00}'}{g_{00}}\right)^2 + \frac{1}{4}\left(\frac{l'}{l}\right)^2\right],$$

$$R^2_2 = \frac{1}{g_{00}}\left[\frac{\dot\Psi\dot l}{2l} + \ddot\Psi + 2\dot\Psi^2 - \frac{\dot\Psi \dot g_{00}}{2g_{00}}\right] - e^{-2\Psi}\left[\frac{\Psi' g_{00}'}{2g_{00}} + \frac{\Psi' l'}{2l} + \Psi''\right],$$

$$R^3_3 = \frac{1}{g_{00}}\left[\frac{1}{2}\frac{\ddot l}{l} - \frac{1}{4}\frac{\dot l \dot g_{00}}{l g_{00}} + \dot\Psi\frac{\dot l}{l} - \frac{1}{4}\left(\frac{\dot l}{l}\right)^2\right] + e^{-2\Psi}\left[-\frac{1}{2}\frac{l''}{l} - \frac{1}{4}\frac{l' g_{00}'}{l g_{00}} + \frac{1}{4}\left(\frac{l'}{l}\right)^2\right],$$

$$R^0_0 = \frac{1}{g_{00}}\left[2\ddot\Psi + 2\dot\Psi^2 + \frac{1}{2}\frac{\ddot l}{l} - \frac{1}{4}\left(\frac{\dot l}{l}\right)^2 - \dot\Psi\frac{\dot g_{00}}{g_{00}} - \frac{1}{4}\frac{\dot l \dot g_{00}}{l g_{00}}\right]$$
$$+ e^{-2\Psi}\left[-\frac{1}{2}\frac{g_{00}''}{g_{00}} - \frac{1}{4}\frac{g_{00}' l'}{g_{00} l} + \frac{1}{4}\left(\frac{g_{00}'}{g_{00}}\right)^2\right],$$

$$R_{01} = \dot\Psi' + \frac{1}{2}\frac{\dot l'}{l} - \frac{1}{4}\frac{\dot l l'}{l^2} - \frac{\dot\Psi l'}{2l} - \frac{\Psi \dot g_{00}'}{2g_{00}} - \frac{1}{4}\frac{\dot l g_{00}'}{l g_{00}}.$$

The above equations must satisfy the following conditions:

$$R_2^2 = R_3^3, \tag{6}$$

$$R_{01} = 0, \tag{7}$$

$$R_2^2 - R_1^1 = -2F^{01}F_{01} = 2F^2(r)e^{-2\Psi}l^{-1}, \tag{8}$$

where $F(r)$ is a function of p only.

$$16\pi P = 2F^{01}F_{01} - (R_0^0 + R_1^1), \tag{9}$$

$$4\pi\varrho = R_1^1 + 4\pi P - F^{01}F_{01}. \tag{10}$$

For the motion to be shear-free, we get $\dot{g}_{ik}/g_{ik} = h(r, t)$, where the dot denotes time-derivative and i, k denote only the space coordinates. $h(r, t)$ is a function of (r, t) only. Then the metric elements of E_q (1) may be suitably written as

$$l = p^2 A(r, t), \tag{11a}$$

$$e^{2\Psi} = e^{2f(r)}A(r, t), \tag{11b}$$

where $f(r)$ depends only on p and $A(r, t)$ on (r, t) only. Then from (7) one can write

$$g_{00} = (\dot{l}/l)^2. \tag{12}$$

From the relations (6) and (11a, b) it comes out that

$$-\frac{\dot{A}'}{\dot{A}} + \frac{1}{2}\frac{A'}{A} = \frac{f'' + f'/r}{f' - 1/r} = C(r), \tag{13}$$

where $C(r)$ depends only on p.

After integration of A-part of Eq. (13), we find

$$A = e^{2D(r)}(R+T)^2, \tag{14}$$

where $D(r) = -\int C(r)\,dr$ and R and T are functions of r and t, respectively. f-part of Eq. (13) cannot be integrated unless $C(r)$ is specified. In the previous paper of De [5], he assumed the form $C(r) = (n+1)/r$, where n is a constant. Let us take $C(r) = 0$.

Then we find

$$A = (R+T)^2$$

and

$$l = p^2(R+T)^2, \tag{15a}$$

$$e^{2\Psi} = p^{2K}(R+T)^2, \tag{15b}$$

K being a constant of integration

$$g_{00} = (R+T)^{-2}, \tag{15c}$$

after suitable transformation of time and space coordinates. However, in order to satisfy Eq. (8) the permissible value of K is found to be zero and

$$F^2(r) = r^2 R'^2. \tag{16}$$

From the relations (9) and (10), the pressure and matter density can be determined and Eq. (3) gives the charge density. The corresponding expressions are

$$4\pi P = -\ddot{T}(R+T) - \tfrac{3}{2}\dot{T}^2, \tag{17a}$$

$$4\pi \varrho = \frac{3}{2}\dot{T}^2 - \frac{R''+R'/r}{(R+T)^3}, \tag{17b}$$

$$4\pi\sigma = \pm \frac{(R''+R'/r)}{(R+T)^3}, \tag{17c}$$

and the line-element

$$ds^2 = (R+T)^{-2}dt^2 - (R+T)^2[dr^2 + dz^2 + r^2 d\varphi^2]. \tag{17d}$$

The functions R and T remain arbitrary subject to the restriction set by physical conditions, $\varrho > 0$, $P \geqslant 0$. Because there may not be any singularity, $(R+T)$ must not vanish anywhere. Then without loss of generality we can accept it to be positive everywhere. Thus in Eq. (17a) $\ddot{T}$ is always negative. Hence we can conclude that for decreasing $(R+T)$ with respect to time, i. e. once the contraction sets in, the collapse of the fluid distribution is accelerated and a singularity is reached at a finite value of t, the coordinate time.

Although the singularity corresponding to $(R+T) = 0$ is associated with an infinite value of g_{00}, nevertheless the following consideration indicates that the proper time $(\int \sqrt{g_{00}}\, dt)$ remains finite if the pressure remains positive.

Choosing an origin of time at the epoch $(R+T) = 0$, we have for $t \to 0$, $(R+T) \to t^\alpha$ where α is a constant and it is greater than zero. Now with $P > 0$. Eq. (17a) indicates that $\ddot{T}$ must tend to infinity with negative value at least as $t^{-\alpha}$ as $t \to 0$. The above two conditions together give $0 < \alpha < 1$ and hence $(\int \sqrt{g_{00}}\, dt)$ converges as $t \to 0$.

If Eq. (17a) differentiated with respect to r, one obtains

$$4\pi P' = -\ddot{T}R' = |\ddot{T}|R'. \tag{18}$$

Now from Eqs (16–18) a number of further conclusions can be drawn as in the following.
(*a*) Anywhere the electric field vanishes, the variation of pressure also vanishes. As on the axis $r = 0$, $R' = 0$ (otherwise singularity appears for ϱ and σ at $r = 0$); the electric field and variation of pressure vanish there.
(*b*) If the charge distribution has the same sign everywhere (this can be obtained by suitably adjusting the arbitrary function R with the above mentioned restriction at $r = 0$), the pressure can only either continuously increase or decrease from the axis $r = 0$.
(*c*) In case when the distribution of charges with the same sign everywhere can be bounded, the pressure will continuously fall from the axis $r = 0$ and becomes zero at the boundary.

3. *Plane symmetric distribution*

Following Taub [6], the symmetry is said to be plane when the space-time admits translations along two perpendicular spatial directions and a rotation in the plane of translation.

Then the line-element may be written as

$$ds^2 = e^{2u}dt^2 - e^{2v}dx^2 - e^{2w}[dy^2 + dz^2], \tag{19}$$

where u, v, w are functions of x and t.

From the condition of shear-free motion, one can write $v = w + f(x)$ and by a suitable transformation of x coordinate the line-element (19) may be reduced to the form

$$ds^2 = e^{2u}dt^2 - e^{2w}[dx^2 + dy^2 + dz^2]. \tag{20}$$

Hence, in this case the space becomes conformally flat for shear-free motion.

Here also, Eqs (6) to (10) hold good, except that Eq. (8) will now take the form

$$R_2^2 - R_1^1 = -2F^{01}F_{01} = 2F^2(x)e^{-4w}. \tag{21}$$

However, it is obvious that the relation (6) is automatically satisfied for the metric (20). We omit here the explicit expressions for the Ricci tensors.

From the condition (7) one obtains

$$e^u = f(t)\dot{w}, \tag{22}$$

where $f(t)$ is an arbitrary function of t only. Condition (21) yields

$$2F^2(x) = e^{2w}\left[w'' - w'^2 + \frac{\dot{w}''}{\dot{w}} - \frac{2\dot{w}'w'}{\dot{w}}\right]. \tag{23}$$

Let us now consider a trial solution, $e^w = (X+T)^n$ (24), where X and T are arbitrary functions of x and t, respectively, and n is a constant. However, the relation (23) is found to be consistent only when $n = 1$.

Thus the metric may be explicitly written after a suitable transformation of t as

$$e^{2w} = (X+T)^2, \quad e^{2u} = (X+T)^{-2} \quad \text{and} \quad F^2(x) = X'^2. \tag{24}$$

Using Eqs (9), (10) and (3), we get

$$4\pi P = -\ddot{T}(X+T) - \tfrac{3}{2}\dot{T}^2, \tag{25a}$$

$$4\pi\varrho = \frac{3}{2}\dot{T}^2 - \frac{X''}{(X+T)^3}, \tag{25b}$$

and

$$4\pi\sigma = \pm\frac{X''}{(X+T)^3}. \tag{25c}$$

Obviously $\ddot{T}$ will always remain negative in this case also, and following the arguments in the previous part one can show that the integral ($\int \sqrt{g_{00}}\, dt$) converges. Then the conclusion remains identical: from any finite dimension of the distribution to ultimate collapse, the lapse of proper time is finite.
Differentiating Eq. (25a) with respect to x, one can write

$$4\pi P' = -\ddot{T}X' = |\ddot{T}|X'. \tag{26}$$

Oviously we can conclude that anywhere the electric field vanishes, the variation of pressure also vanishes.

The authors' thanks are due to Professor A. K. Raychaudhuri for helpful suggestions.

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