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BIANCHI TYPE I COSMOLOGICAL MODELS WITH PURE MAGNETIC FIELD

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The note presents the possible solutions for electromagnetic fields in a source-free region when the space-time admits an Abelian group of motions with three space-like generators (Bianchi Type I group). The entire set of solutions can be recovered by a simple transformation of an already known group of solutions.

1. Introduction

In the literature the line-element of the form:

$$ds^2 = D(x)\,dt^2 - A(x)\,dx^2 - B(x)\,dy^2 - C(x)\,dz^2 \tag{1}$$

has been discussed in detail (Raychaudhuri [1]) for source-free non-null electromagnetic fields.

Also, various solutions of the source-free electromagnetic field have been obtained (Datta [2], Rosen [3, 4], Jacobs [5]) for a similar diagonalized line-element, whose metric tensor components are only functions of time. It should be mentioned that recently a good number of authors have studied such Bianchi type I solutions for other various physical situations.

While Raychaudhuri [6], Heckmann–Schucking [7], Thorne [8] Doroshkevich [9], Jacobs [10] have studied Bianchi Type I solutions for perfect fluid and incoherent dust distributions, without any electromagnetic fields, others like Robinson [11], Zel'dovich [12], Shikin [13], Doroshkevich [9], Thorne [8], Jacobs [5] have studied the same type of solutions for perfect fluid and incoherent dust distributions in presence of the source-free magnetic field.

The purpose of the present note is to point out that the entire set of Bianchi Type I solutions for source-free electromagnetic fields (already found by Datta [2]) may be recovered by a simple transformation of the line-element given by (1). However, the study of Datta also seems to be incomplete. Then, it is also shown that the already known solutions of various other authors can be recovered from these solutions.

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2. *Equations, process of transformation, and solution*

As it has been pointed out by Rainich [14], Misner and Wheeler [15], a source-free Riemannian space can be the seat of a non-null electromagnetic field, only if:

$$R = 0, \tag{2}$$

$$R^\mu_\nu R^\nu_\lambda = \delta^\mu_\lambda(\tfrac{1}{4} R_{\alpha\beta}R^{\alpha\beta}), \tag{3}$$

$$R^4_4 > 0, \tag{4}$$

$$\alpha_{\beta,\gamma} - \alpha_{\gamma,\beta} = 0, \tag{5}$$

where

$$\alpha_\delta \equiv (-g)^{1/2}\varepsilon_{\delta\lambda\mu\nu}R^{\lambda\gamma;\mu}R^\nu_\gamma/(R_{\alpha\beta}R^{\alpha\beta}). \tag{6}$$

Equations (2)–(6) are given in usual notations [1, 14, 15].

However, Raychaudhuri [1] has shown that the line-element (1) satisfies the conditions (2)–(9) in the following possible cases:

(A) $$R^4_4 = R^1_1 = -R^2_2 = -R^3_3,$$

(B) $$R^4_4 = -R^1_1 = R^2_2 = -R^3_3,$$

(C) $$R^4_4 = -R^1_1 = -R^2_2 = R^3_3,$$

together with the relation (4). Here x, y, z, t are, respectively, denoted as 1, 2, 3, 4 directions.

Now, if one performs a transformation $x \to it$, $t \to ix$, in line-element (1), it will apparently change g_{11} to $-g_{44}$ and g_{44} to $-g_{11}$ leaving other components unchanged, and the argument of the function changes x to it. The imaginary i may be absorbed in the arbitrary constants of integration that appear, making the new $g_{\mu\nu-s}$ real functions of t. Then the transformed line-element can be written as:

$$ds^2 = A(t)dt^2 - D(t)dx^2 - B(t)dy^2 - C(t)dz^2. \tag{7}$$

If the Ricci tensors for the line-element (7) are denoted by a bar overhead, in order to differentiate them from the Ricci tensors of the line-element (1), then $\overline{R}_{\mu\nu} - S$ contain only the derivatives with respect to t, while $R_{\mu\nu} - S$ contain only the derivatives with respect to x. Now, if the space determined by the line-element (7) is the seat of source-free non-null electromagnetic fields, $\overline{R}_{\mu\nu} - S$ must satisfy the conditions (2)–(6) which are equivalent to

(D) $$\overline{R}^4_4 = \overline{R}^1_1 = -\overline{R}^2_2 = -\overline{R}^3_3,$$

(E) $$\overline{R}^4_4 = -\overline{R}^1_1 = \overline{R}^2_2 = -\overline{R}^3_3,$$

(F) $$\overline{R}^4_4 = -\overline{R}^1_1 = -\overline{R}^2_2 = \overline{R}^3_3,$$

together with

$$\overline{R}^4_4 > 0. \tag{4'}$$

However, the change of variable from t to it would change the signature of $\overline{R}_{\mu\nu}-S$ as the $\overline{R}_{\mu\nu}-S$ would involve the products of two Christoffel symbols or the derivatives of these symbols with respect to t. Then one finds

$$\overline{R}{}^4_4 \equiv -R^1_1,$$

$$\overline{R}{}^1_1 \equiv -R^4_4,$$

$$\overline{R}{}^2_2 \equiv -R^2_2,$$

$$\overline{R}{}^3_3 \equiv -R^3_3,$$

except that on one side the t-derivatives are involved, and on the other, the x-derivatives.

Considering the above mentioned similarity, one can say that the case (D) is equivalent to the case (A), case (E) goes over to case (C), and case (F) to case (B). Also the condition (4′) is automatically satisfied if the condition (4) is satisfied for cases (A), (B), and (C). However, it should be noted here that the cases (B), (C) of line element (1) were identical except for the change of the labels of coordinates y and z, respectively. But in our case, (D), (E), and (F) are all identical except for the interchange of x, y, and z coordinates, respectively.

From the above analysis one can conclude that $A(t)$, $B(t)$, $C(t)$, and $D(t)$ of Eq.(7) can be directly obtained from the solutions of $A(x)$, $B(x)$, $C(x)$, and $D(x)$ of Eq. (1) simply if x is replaced by t.

Since all the three cases (A), (B), and (C) for the line-element (1) have been studied exhaustively by Raychaudhuri [1], we here present only the solutions. We also show that the solutions, obtained from case (B) or (C) by process of change of variable, can be suitably transformed to identical solutions in case (D).

Case (D)

As in Raychaudhuri's paper, if one writes

$$\alpha = \ln (c/B)$$

and

$$\beta = \ln (Bc),$$

then the following solutions can be written for existing electromagnetic fields.

However, since $R_{\mu\nu}$ can determine the electromagnetic field tensor except for a duality rotation, here the electromagnetic field tensor may be considered as a purely magnetic or purely electric one, or a combination of the two, obtained by duality rotation. For simplicity sake, we give here expressions for purely magnetic field.

SOLUTION 1

$$\dot{\alpha} = 0, \qquad \dot{\beta} \neq 0.$$

$$ds^2 = \frac{dt^2}{(a/t^2+l/t)} - (a/t^2+l/t)dx^2 - t^2(dy^2+dz^2),$$

where a and l are constants.

$$F_{23} = \pm(-a)^{1/2}.$$

Here a must be negative and the solution is regular everywhere except for $t = 0$, $t = \pm\infty$, and $t < |a/l|$.

Rosen's [4] special solution for a pure magnetic field along x_1-direction,

$$ds^2 = \frac{b_1^2 dt^2}{(1 \pm \cos t)^4} - (\sin t)^2 (dx_1)^2 - \frac{(dx_2)^2 + (dx_3)^2}{(1 \pm \cos t)^2},$$

where b_1 is a constant, may be transformed to the form,

$$ds^2 = \frac{dt^2}{\left(\frac{2b_1}{t} - \frac{b_1^2}{t^2}\right)} - \left(\frac{2b_1}{t} - \frac{b_1^2}{t^2}\right) dx_1^2 - t^2(dx_2^2 + dx_3^2).$$

Then obviously Rosen's solution becomes a special case of the above mentioned solution.

SOLUTION 2

$$\dot{\alpha} \neq 0, \dot{\beta} \neq 0 \qquad \text{and} \qquad \mu \neq 0$$

$$ds^2 = (C_1 t^\mu + C_2 t^{-\mu})^2 dt^2 - (C_1 t^\mu + C_2 t^{-\mu})^{-2} (dx)^2$$
$$- (C_1 t^\mu + C_2 t^{-\mu})^2 [t^\lambda dy^2 + t^{(2-\lambda)} dz^2],$$

where C_1, C_2 are constants, $\mu^2 = \lambda(2-\lambda)/4$ and λ is another constant.

$$F_{23} = \pm 2\mu (C_1 C_2)^{1/2}.$$

Here $\mu^2 C_1 C_2 > 0$ and the solution is regular everywhere except at $t = 0$ and $t = \pm\infty$. Obviously if μ is real, C_1 and C_2 must both have the same sign, whereas if μ is imaginary C_1 and C_2 are of the opposite sign.

Rosen's [3] general solution of a magnetic field along x_1-direction,

$$ds^2 = \frac{b_1^2 (\tan t/2)^{2(b_2+b_3)}}{\sin^4 t} (dt)^2 - (\sin t)^2 (dx_1)^2$$
$$- \frac{(\tan t/2)^{2b_2}}{(\sin t)^2} (dx_2)^2 - \frac{(\tan t/2)^{2b_3}}{(\sin t)^2} (dx_3)^2,$$

with b_1 and b_2 $(= 1/b_3)$ being constants, may be transformed to the above form and the constants in the two expressions are related by:

$$\mu = \frac{1}{(b_2 + b_3)}, \quad C_1 = \frac{1}{2\sqrt{K_1}}, \quad C_2 = \frac{\sqrt{K_1}}{2},$$

and

$$K_1 = \left(\frac{b_2 + b_3}{b_1}\right)^{2/(b_2+b_3)}.$$

SOLUTION 3

$$\dot{\alpha} \neq 0, \; \dot{\beta} \neq 0, \; \mu = 0, \quad \text{i.e. } \lambda = 0 \text{ or } 2.$$

$$ds^2 = \left[\ln\left(\frac{t+}{a}\right)\right]^2 (dt)^2 - \left[\ln\left(\frac{t+l}{a}\right)\right]^{-2} (dx)^2 - \left[\ln\left(\frac{t+l}{a}\right)\right]^2 [t^2 dy^2 + dz^2],$$

$$F_{23} = \pm \frac{t}{(t+l)}\left[\frac{l}{t}\cdot \ln\left(\frac{t+l}{a}\right) - 1\right]^{1/2}.$$

Here the metric is regular everywhere except at $t = \pm\infty$ and $t/a \leqslant (-l/a)$, and further to satisfy the condition (4′),

$$\left[l/t \cdot \ln\left(\frac{t+l}{a}\right)\right] > 1.$$

Case E / Case F

By process of change of variable from Raychaudhuri's solution, in this case there exists only one solution for positive value of $\bar{R}_4^4$ for all values of time.

SOLUTION 1

$$ds^2 = t^{2\lambda}(C_1 t^{\mu} + C_2 t^{-\mu})^2 [dt^2 - dx^2] - (C_1 t^{\mu} + C_2 t^{-\mu})^{-2} dy^2 - t^2 (C_1 t^{\mu} + C_2 t^{-\mu})^2 dz^2,$$

where $\mu^2 = \lambda \neq 0$.

$$F_{31} = \pm 2\mu (C_1 C_2)^{1/2}.$$

Here $\mu^2 C_1 C_2 > 0$ and the solution is regular everywhere except at $t = 0$ and $t = \pm\infty$.

By suitable transformation of the variables t, y, and z this metric becomes identical with the metric of Solution 2, Case D, if the labels of x and y coordinates are interchanged. If the constants of Solution 2, Case D are denoted by a bar overhead, then the inter--relation between the constants in the two cases is:

$$C_1 = \frac{2\bar{C}_1 K_1^{(\bar{\mu}+1)}}{(2-\bar{\lambda})}, \quad C_2 = \frac{2\bar{C}_2 K_1^{(-\bar{\mu}+1)}}{(2-\bar{\lambda})},$$

$$\mu = \frac{2\bar{\mu}}{(2-\bar{\lambda})}, \quad \lambda = \frac{\bar{\lambda}}{(2-\bar{\lambda})} \quad \text{and} \quad K_1 = \left(1 - \frac{\bar{\lambda}}{2}\right)^{\frac{2}{(2-\bar{\lambda})}}.$$

After detailed analysis, it has been found that Jacob's [5] solution is a special case of this solution if the labels of y and z coordinates are interchanged. In his solution, $(C_1)_{\text{our}} = (C_2)_{\text{our}} = \frac{1}{2}$, $\bar{c}_1$, $\bar{c}_2$ of his solution are $\bar{c}_1 = 1/\bar{c}_2 = \mu_{\text{our}}$ and $(\beta)_{\text{Jacobs}} = \mu^2_{\text{our}}$.

SOLUTION

When

$$\mu^2 = \lambda = 0$$

The author mentions this case because the solution given in Raychaudhuri's paper is erroneous, though the ultimate conclusion, "R_4^4 turns out to be negative except when

$C_1 = 0$ in which case the space is flat", still holds. The correct solution for the line-element (1) is

$$as^2 = (C_1 \ln x + C_2)^2(dt^2 - dx^2) - (C_1 \ln x + C_2)^{-2}dy^2 - x^2(C_1 \ln x + C_2)^2 dz^2.$$

For the line-element (7) as well, the above conclusion is found to be correct.

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REFERENCES

[1] A. K. Raychaudhuri, *Ann. Phys.* **11**, 501 (1960).
[2] B. K. Datta, *Nuovo Cimento* **36**, 109 (1965).
[3] G. Rosen, *J. Math. Phys.* **3**, 313 (1962).
[4] G. Rosen, *Phys. Rev.* **136B**, 297 (1964).
[5] K. C. Jacobs, *Astrophys. J.* **155**, 379 (1969).
[6] A. K. Raychaudhuri, *Proc. Phys. Soc.* **72**, 263 (1958).
[7] O. Heckmann, E. Schucking, *Introduction to Current Research*, Ed. L. Witten, John Wiley and Sons Inc. 1962.
[8] K. S. Thorne, *Astrophys. J.* **148**, 51 (1967).
[9] A. G. Doroshkevich, *Astrophysics* **1**, 138 (1965).
[10] K. C. Jacobs, *Astrophys. J.* **153**, 661 (1968).
[11] B. R. Robinson, *Proc. Nat. Acad. Sci. USA* **47**, 1852 (1961).
[12] Ya. B. Zel'dovich, *Sov. Astron. AJ.* **8**, 700 (1965).
[13] I. S. Shikin, *Sov. Phys.-Dokl. (USA)* **11**, 944 (1967).
[14] G. Y. Rainich, *Trans. Am. Math. Soc.* **27**, 106 (1925).
[15] C. W. Misner, J. A. Wheeler, *Ann. Phys.* **2**, 525 (1957).