A RECURSIVE METHOD FOR THE REPRESENTATION MATRICES OF SU(n)

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A simple procedure is described for expressing the generators of SU(n) as differential operators in the parameters of the group. It is then shown how recursion relations for the matrix elements of the "boost operator" leading from SU($n-1$) to SU(n) can be obtained by treating the representation functions as basis functions and operating on them by the differential operators. The matrix elements of the "boost" appear in the present theory as solutions of ordinary differential equations. The operators for SU(3) and SU(4) are constructed, and some recursion relations and special matrix elements are derived by way of illustration.

1. *Introduction*

The unitary unimodular groups SU(n) occupy a position of paramount importance in the present-day physics. These groups have been used extensively to classify multi--electron states of atoms [1] and states of nuclei. For $n = 2, 3, 4, 6, 12$ they have been used as possible symmetries of elementary particles [2]. The experimental evidence in support of the SU(3) symmetry is now overwhelming, and some of the results obtained by using this symmetry are likely to be of permanent value. This is not quite true of the higher groups, though the SU(3) transformations form a subgroup of the larger symmetry. Nevertheless, persistent efforts are being made to gain a deeper insight into the fundamental processes by using the higher groups, and it has become imperative to study the structural properties of these groups in greater detail. The mathematical problems that arise in this connection can be divided into the following broad categories: (a) Construction of the state vectors, (b) Determination of the matrices of the generators, (c) Determination of the representation matrices for finite transformations, (d) Classification and construction of tensor operators, (e) Structure of the Clebsch-Gordan series, (f) Evaluation of the Clebsch-Gordan coefficients (CGC). Some of these problems appear to have been solved

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for general values of n. In the present paper we investigate Problem (c) and propose a solution which, in principle, works for any n. A satisfactory solution of the problem is of considerable theoretical interest and can be used for calculating physically important quantities like the CGC occurring in the study of reactions with two incoming and two outgoing particles. As is well-known, the CGC can be determined by integrating the product of three representation functions [3] if an analytic expression for the latter can be found.

The first step in the derivation of the representation matrices of SU(n) is to introduce a convenient set of parameters for the group. This was done a long time ago by Murnaghan [4] by writing the general group element as a product of simpler factors each containing only two parameters. By successive similarity transformations the Murnaghan matrix for SU(n) can be brought into a form with a "boost operator" in the middle and two SU($n-1$) matrices on the two sides [5]. The problem is, thus, reduced to the calculation of the matrix element (m.e.) of the "boost" and of the subgroup SU($n-1$). In the following sections we describe a recursive method for the determination of the m.e. of the boost leading from SU($n-1$) to SU(n). The method is based on the theorem that the m.e. of an irreducible representation (IR) behave like basis functions when operated on by the generators expressed as differential operators in the parameters of the group. From considerations of linear independence it is evident that the application of a generator to a m.e. of an IR written as a sum of products of simpler factors will give a set of recursion relations connecting different m.e. of the boost. Starting from a simple special case for which the m.e. can be determined by other means one can then construct the general m.e. by repeated use of the recursion relations. This constitutes a powerful method of constructing the representation matrices of SU(n), at least, for lower values of n. Further, it often proves possible to dispense with the iterative procedure and determine the m.e. more elegantly by solving an ordinary differential equation. The details of the method are worked out here for SU(3) and SU(4). The treatment of these special cases gives us an idea of the results to be expected for general values of n.

2. *Differential operators for the generators*

Let H be a group of n parameters α_i ($i = 1, 2, \dots n$) and let G_i be its generators. In order to express the generators as differential operators we multiply an element $H(\alpha)$ of the group by $I+\sum \varepsilon_i G_i$. The multiplication changes the element $H(\alpha)$ into $H(\alpha+\Delta\alpha)$ lying in its neighbourhood in the group manifold. Equating the product of $H(\alpha)$ and $I+\sum \varepsilon_i G_i$ to $H(\alpha+\Delta\alpha)$ and solving a system of linear equations we obtain the values of $\Delta\alpha_i$ in terms of ε_i. From the basic theorems on the representation theory it can be easily shown that the differential operator for G_i is just the sum $\sum \frac{\Delta\alpha_j}{\varepsilon_i}\frac{\partial}{\partial\alpha_j}$ with all other ε's put equal to zero. However, since the multiplication of $H(\alpha)$ by $I+\sum \varepsilon_i G_i$ can be carried out either from the right or from the left, the procedure gives two kinds of operators for the same generator. Applied to a m.e. $D^{(\varrho)*}_{\nu\nu}(\alpha)$ of an irreducible representation (ϱ) of H an operator of the first kind generates a mixture $\sum (G_i)_{\lambda\nu} D^{(\varrho)*}_{\nu'\lambda}(\alpha)$ and an operator of the second kind generates a mixture $\sum (G_i)_{\nu'\lambda} D^{(\varrho)*}_{\lambda\nu}(\alpha)$.

Let us now see what forms the operators take in the case of SU(3) and SU(4) when the above procedure is adopted for constructing them. The groups can be parametrized by writing their general element in the form

$$\mathscr{T}\mathrm{SU}(3) = e^{-i\beta Y}e^{-i\alpha_3 T_3}e^{-i\alpha_2 T_2}e^{-i\gamma T_3}e^{-i\nu N}e^{i\gamma' T_3}e^{i\alpha'_2 T_2}e^{i\alpha'_3 T_3}, \tag{1}$$

$$\mathscr{S}\mathrm{SU}(4) = e^{-i\eta Z}e^{-i\beta Y}e^{-i\alpha_3 T_3}e^{-i\alpha_2 T_2}e^{-i\gamma T_3}e^{-i\nu N}e^{-i\mu M}e^{i\gamma' T_3}$$

$$\times e^{i\alpha'_2 T_2}e^{i\alpha'_3 T_3}e^{-i\bar{\nu} N}e^{-i\bar{\gamma} T_3}e^{-i\bar{\alpha}_2 T_2}e^{-i\bar{\alpha}_3 T_3}e^{-i\bar{\beta} Y}, \tag{2}$$

where [6],

$$T = \frac{1}{2}\begin{pmatrix}\sigma & \\ & 0\end{pmatrix}, \quad N = (A_1^3 + A_3^1), \quad M = (A_3^4 + A_4^3), \quad Y = \frac{2}{\sqrt{3}}H_2,$$

$$Z = \frac{\sqrt{6}}{2}H_3, \quad (A_i^j)_{mn} = \delta_{im}\delta_{jn} - \frac{1}{n}\delta_{ji}\delta_{mn},$$

$$H_r = [2r(r+1)]^{-1/2}(A_1^1 + \dots A_r^r - rA_{r+1}^{r+1}). \tag{3}$$

The traceless matrices A_i^j $(i \neq j)$ and H_r $(r = 1, \dots, n-1)$ are the n^2-1 generators of SU(n). They satisfy the commutation relations

$$[A_i^j, A_k^l] = \delta_k^j A_i^l - \delta_i^l A_k^j. \tag{4}$$

With the above notation the differential operators for the generators of SU(3) and SU(4) can be written as

$$I_3 = -i\frac{\partial}{\partial\alpha_3}, \quad Y = -i\frac{\partial}{\partial\beta}, \quad Z = -i\frac{\partial}{\partial\eta},$$

$$I_\pm(\alpha_3, \alpha_2, \gamma) = i\exp(\pm i\alpha_3)\left(\cot\alpha_2\frac{\partial}{\partial\alpha_3} \mp i\frac{\partial}{\partial\alpha_2} - \csc\alpha_2\frac{\partial}{\partial\gamma}\right), \tag{5}$$

$$L_+ = \frac{1}{2}\exp\left\{i\left(\beta - \frac{1}{2}\alpha_3 + \frac{1}{2}\gamma\right)\right\}\left[\csc\frac{\alpha_2}{2}\cot\nu\left(\frac{\partial}{\partial\gamma} - \frac{\partial}{\partial\alpha_3}\right)\right.$$

$$-2i\cos\frac{\alpha_2}{2}\cot\nu\frac{\partial}{\partial\alpha_2} + \sin\frac{\alpha_2}{2}\tan\nu\left(\frac{\partial}{\partial\gamma} - \frac{3}{2}\frac{\partial}{\partial\beta}\right)$$

$$\left.-i\sin\frac{\alpha_2}{2}\frac{\partial}{\partial\nu} + 4\sin\frac{\alpha_2}{2}\csc 2\nu\frac{\partial}{\partial\gamma'}\right] + i\exp\left\{i\left(\beta - \frac{1}{2}\alpha_3 - \frac{1}{2}\gamma\right)\right\}$$

$$\times\cos\frac{\alpha_2}{2}\csc\nu I_-(-\gamma', -\alpha'_2, -\alpha'_3), \tag{6}$$

$$iA_2^4 = \exp\left\{i\left(-\frac{1}{2}\alpha_3 - \frac{1}{2}\gamma + \frac{1}{3}\beta + \eta\right)\right\}\cos\frac{\alpha_2}{2}\left[-\csc\mu\bar{L}_+ + i\cot\mu\cot\nu\right.$$

$$\times I_-(-\gamma', -\alpha_2', -\alpha_3')\Big] + \exp\left\{i\left(-\frac{1}{2}\alpha_3 + \frac{1}{2}\gamma + \frac{1}{3}\beta + \eta\right)\right\}$$

$$\times\left[\csc\mu\cos\nu\sin\frac{\alpha_2}{2}\bar{K}_+ + \sin\nu\sin\frac{\alpha_2}{2}\left\{-\frac{2}{3}\tan\mu\frac{\partial}{\partial\eta} - \frac{1}{2}i\frac{\partial}{\partial\mu}\right.\right.$$

$$\left.+\frac{3}{2}\csc 2\mu\left(\frac{\partial}{\partial\bar{\beta}} - 2\frac{\partial}{\partial\bar{\gamma}} - 2\frac{\partial}{\partial\alpha_3'} + \frac{2}{3}\frac{\partial}{\partial\gamma'}\right) - \frac{1}{4}\tan\mu\left(\frac{\partial}{\partial\beta} - 2\frac{\partial}{\partial\gamma}\right)\right\}$$

$$+\cot\mu\csc\nu\cos\frac{\alpha_2}{2}\left(-i\frac{\partial}{\partial\alpha_2} - \csc\alpha_2\frac{\partial}{\partial\alpha_3} + \cot\alpha_2\frac{\partial}{\partial\gamma}\right)$$

$$\left.+\frac{1}{2}\cot\mu\cos\nu\sin\frac{\alpha_2}{2}\left(-i\frac{\partial}{\partial\nu} + 4\csc 2\nu\frac{\partial}{\partial\gamma} + 2\cot\nu\frac{\partial}{\partial\gamma'}\right)\right], \tag{7}$$

$$A_3^4 = \exp\left\{i\left(\eta - \frac{2}{3}\beta\right)\right\}\left[\csc\mu\sin\nu\bar{K}_+ + \frac{1}{2}i\cos\nu\frac{\partial}{\partial\mu} - \frac{1}{2}i\cot\mu\right.$$

$$\times\sin\nu\frac{\partial}{\partial\nu} - \frac{3}{2}\csc 2\mu\cos\nu\left(\frac{\partial}{\partial\bar{\beta}} - 2\frac{\partial}{\partial\bar{\gamma}} - 2\frac{\partial}{\partial\alpha_3'} + \frac{2}{3}\frac{\partial}{\partial\gamma'}\right)$$

$$+\frac{2}{3}\tan\mu\cos\nu\frac{\partial}{\partial\eta} + \frac{3}{2}\left(\frac{1}{2}\cot\mu\sec\nu + \frac{1}{6}\tan\mu\cos\nu\right)\frac{\partial}{\partial\beta}$$

$$\left.-\frac{1}{2}(\tan\mu\cos\nu + \cot\mu\sec\nu)\frac{\partial}{\partial\gamma} - \cot\mu\tan\nu\sin\nu\frac{\partial}{\partial\gamma'}\right]. \tag{8}$$

$\bar{K}_+$, $\bar{L}_+$ are raising operators of the second kind operating on the SU(3) subgroup on the r.h.s. of $\exp(-i\mu M)$ in Eq. (2). By the substitutions, $\alpha_3 \to -\alpha_3$, $\alpha_2 \to \pi+\alpha_2$, in L_+, A_2^4 and the substitutions, $\alpha_3' \to -\alpha_3'$, $\alpha_2' \to \pi+\alpha_2'$, in $\bar{K}_+$ one obtains the operators, K_+, A_1^4 and $\bar{L}_+$, respectively. The remaining operators obey the symmetries

$$K_- = -(K_+)^*, \quad L_- = -(L_+)^*, \quad A_4^2 = -A_2^{4*}, \quad A_4^3 = -A_3^{4*}.$$

The expressions (5)–(8) have been obtained by solving the system of linear equations for $\Delta\alpha$. To test the correctness of the expressions we have evaluated some of the commutators, in particular, the commutator $[L_+, A_3^4]$ which must equal A_2^4. Since

$$\bar{I}_- = I_-(-\gamma'-\alpha_2'-\alpha_3'), \quad \bar{I}_3 = i\frac{\partial}{\partial\gamma'}, \quad [\bar{I}_-, \bar{K}_+] = \bar{L}_+, \quad [\bar{I}_3, \bar{K}_+] = \tfrac{1}{2}\bar{K}_+,$$

the commutator of L_+ with the first term of (8) is seen to have the value

$$[L_+, \exp\cdot i(\eta-\tfrac{2}{3}\beta)\cdot\sin\nu\csc\mu\bar{K}_+] = -i\exp\cdot i(\eta+\tfrac{1}{3}\beta-\tfrac{1}{2}\alpha_3+\tfrac{1}{2}\gamma)$$

$$\times\sin\frac{\alpha_2}{2}\cos\nu\csc\mu\bar{K}_+ + i\exp\cdot i\left(\eta + \frac{1}{3}\beta - \frac{1}{2}\alpha_3 - \frac{1}{2}\gamma\right)\cdot\cos\frac{\alpha_2}{2}\csc\mu\bar{L}_+.$$

These are precisely the terms involving $\bar{K}_+$ and $\bar{L}_+$ in (7). The remaining terms of the commutator have been calculated in a straightforward manner and an exact agreement with the expression (7) has been found. It is pointed out here that the appearance of $\bar{I}_-$, $\bar{K}_+$, $\bar{L}_+$ in (6), (7), (8) is a direct consequence of the structure of the system of linear equations for $\Delta\alpha$. The explicit forms of these operators have been determined, but were not required either for deriving the expressions (6), (7), (8), or for setting up the recurrence relations (25) of Sec 4. Analogous simplifications are expected to occur in the case of unitary groups of higher dimensions.

3. *Recurrence relations for SU(3)*

The basis functions of an IR of SU(3) are characterized by three numbers j, μ, δ which are all integral or all half-integral. j, μ are the quantum numbers of the isotopic spin and its z-component and δ is connected with the hyperchange Y by the relation $Y = 2\delta - \frac{2}{3}(p-q)$. By Eq. (1) the m. e. of an IR of SU(3) connecting the states $|j\mu\delta\rangle$ and $|j'\mu'\delta'\rangle$ can be written as

$$\mathfrak{T} \equiv \langle j, \mu, \delta|\mathscr{T}|j', \mu', \delta'\rangle = \sum_{mj} e^{-i\beta\gamma}\mathscr{D}^{j}_{\mu m}(-\alpha_3, -\alpha_2, -\gamma) i^{\bar{m}'-\bar{m}}$$

$$\times\langle \bar{j}, \bar{m}, \bar{\delta}|K|j, m, \delta\rangle D^{\bar{j}}_{\bar{m}\bar{m}'}(-2v)\langle \bar{j}, \bar{m}', \bar{\delta}|K|j', m', \delta'\rangle \mathscr{D}^{j'}_{m'\mu'}(\gamma', \alpha_2', \alpha_3'), \qquad (9)$$

where $\delta - m = \delta' - m'$, $K = \begin{pmatrix} 1 & \\ & i\sigma_2 \end{pmatrix}$, and $\mathscr{D}^{j}_{m'm}(\alpha, \beta, \gamma) = e^{-im'\alpha} D^{j}_{m'm}(\beta)\, e^{-im\gamma}$ are elements of rotation matrices. The operators of Sec. 2, operating on (9), change the quantum numbers of the final state, j, μ, δ, without affecting those of the initial state. To study the result of such operations we give μ the special value $j-1$. $K_+\mathfrak{T}$ is then found to contain products of D-functions of the type

$$\mathscr{D}^{j+1/2}_{j-1/2\,m+1/2}(-\alpha_3, -\alpha_2, -\gamma)\mathscr{D}^{j'}_{m'\mu'}(\gamma', \alpha_2', \alpha_3') \qquad (10)$$

summed over m. On the other hand, since $\mathfrak{T}$ behaves like a basis function, K_+ operating on it must also give

$$K_+\langle j, j-1, \delta|\mathscr{T}|j', \mu', \delta'\rangle = \langle j+\tfrac{1}{2}, j-\tfrac{1}{2}, \delta+\tfrac{1}{2}|K_+|j, j-1, \delta\rangle$$

$$\langle j+\tfrac{1}{2}, j-\tfrac{1}{2}, \delta+\tfrac{1}{2}|\mathscr{T}|j', \mu', \delta'\rangle + \langle j-\tfrac{1}{2}, j-\tfrac{1}{2}, \delta+\tfrac{1}{2}|K_+|j, j-1, \delta\rangle$$

$$\langle j-\tfrac{1}{2}, j-\tfrac{1}{2}, \delta+\tfrac{1}{2}|\mathscr{T}|j', \mu', \delta'\rangle.$$

Equating the expressions for $K_+\mathfrak{T}$ obtained by the two procedures gives the recurrence relations

$$i(j+m+1)^{1/2}\left[2j\cot v - (3\delta - p + q - m)\tan v - \frac{\partial}{\partial v} - 4m'\csc 2v\right](j, m, \delta : m')^*$$

$$-2i[(j-m)(j'+m'+1)(j'-m')]^{1/2}\csc v(j, m+1, \delta : m'+1)^*$$

$$= 2\left[\frac{2j+1}{2j+2}(j+\delta+q+2)(j+\delta+1)(p-j-\delta)\right]^{1/2}\left(j+\frac{1}{2}, m+\frac{1}{2}, \delta+\frac{1}{2} : m'\right)^*, \qquad (11)$$

$$i(j-m)^{1/2}\left[(2j+2)\cot v+(3\delta-p+q-m)\tan v+\frac{\partial}{\partial v}+4m'\csc 2v\right](j, m, \delta : m')^*$$

$$-2i[(j+m+1)(j'+m'+1)(j'-m')]^{1/2}\csc v(j, m+1, \delta : m'+1)^*$$

$$=2\left[\frac{2j+1}{2j}(j-\delta+p+1)(j-\delta)(q-j+\delta+1)\right]^{1/2}\left(j-\frac{1}{2},\ m+\frac{1}{2},\ \delta+\frac{1}{2}:m'\right)^*. \tag{12}$$

Similarly, L_- operating on $\mathfrak{T}$ gives

$$i(j-m+1)^{1/2}\left[2j\cot v+(3\delta-p+q-m)\tan v-\frac{\partial}{\partial v}+4m'\csc 2v\right](j, m, \delta : m')^*$$

$$-2i[(j+m)(j'-m'+1)(j'+m')]^{1/2}\csc v(j, m-1, \delta : m'-1)^*$$

$$=2\left[\frac{2j+1}{2j+2}(p+j-\delta+2)(j-\delta+1)(q-j+\delta)\right]^{1/2}\left(j+\frac{1}{2},\ m-\frac{1}{2},\ \delta-\frac{1}{2}:m'\right)^*, \tag{13}$$

$$i(j+m)^{1/2}\left[(2j+2)\cot v-(3\delta-p+q-m)\tan v+\frac{\partial}{\partial v}-4m'\csc 2v\right](j, m, \delta : m')^*$$

$$+2i[(j-m+1)(j'-m'+1)(j'+m')]^{1/2}\csc v(j, m-1, \delta : m'-1)^*$$

$$=-2\left[\frac{2j+1}{2j}(q+j+\delta+1)(j+\delta)(p-j-\delta+1)\right]^{1/2}\left(j-\frac{1}{2},\ m-\frac{1}{2},\ \delta-\frac{1}{2}:m'\right)^*. \tag{14}$$

Other recurrence relations may be obtained by applying K_-, L_+ and the operators of the first kind to $\mathfrak{T}$.

By combining the relations (11)–(14) convenient expressions for special m. e. of $\exp(-ivN)$, such as those given in Sec. 3 of I [7], may be obtained. As an example we consider the m. e. with the state $|000\rangle$ on the r. h. s. For $j' = 0$ the middle terms in all the four relations vanish. Changing j, m, δ to $j+\frac{1}{2}$, $m+\frac{1}{2}$, $\delta+\frac{1}{2}$ in (14) and combining it with (11) we then obtain the equation

$$\left[\frac{d^2}{dv^2}+2\cot(2v)\frac{d}{dv}-\csc^2 v(\mu'-\mu)^2-\sec^2 v(\mu'+\mu)^2+4I(I+1)\right]$$

$$\times\{\sin v(j, m, \delta : 0)^*\}=0, \tag{15}$$

with

$$I=\tfrac{1}{2}(p+q+1),\quad \mu'=\tfrac{1}{2}(q-p)+\delta+j+\tfrac{1}{2},\quad \mu=\tfrac{1}{2}(q-p)+\delta-j-\tfrac{1}{2}.$$

The solution

$$(j, m, \delta : 0)^*=\csc v D^{1/2(p+q+1)}_{1/2(q-p)+\delta+j+1/2,\ 1/2(q-p)+\delta-j-1/2}(-2v) \tag{16}$$

with the appropriate normalization factor and primes on j, m, δ, agrees with function (26) of I. In relation (11) now put $m=-j-1$, and use the symmetry

$$(j, m, \delta : j', m', \delta')=(j', m', \delta' : j, m, \delta), \tag{17}$$

obtaining

$$\left(j', m'-1, \delta' : j+\frac{1}{2},\ -j-\frac{1}{2},\ j+\frac{1}{2}\right)^* = i\left[\frac{(2j+2)(j'+m')(j'-m'+1)}{(q+2j+2)(2j+1)(p-2j)}\right]^{1/2}$$
$$\times \csc \nu (j', m', \delta' : j, -j, j)^*.$$

Repeated use of this relation with $(j', m', \delta' : 0, 0, 0)$ on the r. h. s. then gives expression (24) of I for $(j', m', \delta' : j, -j, j)$ with the correct sign and numerical factor.

Next, let us consider the m. e. with an *arbitrary* state on the right and the state $\langle \frac{1}{2}(p+q), m, \frac{1}{2}(p-q)| \equiv \langle H|$ on the left. The technique of expansion in a series of basis functions employed in I does not give a convenient expression for this m. e., but, as we shall see presently, the use of the recurrence relations does.

For $j = \frac{1}{2}(p+q)$, $\delta = \frac{1}{2}(p-q)$, the terms on the r. h. s. of (11) and (13) vanish. If m, m' are both increased by unity in (13) and the equations are combined, then for $(\sin \nu)^{-2j-1}$ $(H:m')$ a differential equation is obtained which is of the same form as (15) but with

$$l = \tfrac{1}{2}(\delta'-m'-1), \quad \mu' = -\tfrac{1}{2}(\delta'+m'+2j'+1), \quad \mu = -\tfrac{1}{2}(\delta'+m'-2j'-1).$$

The solution is the D-function

$$D^{1/2(\delta'-m'-1)}_{-1/2(\delta'+m'+2j'+1),\ -1/2(\delta'+m'-2j'-1)}(-2\nu). \tag{18}$$

But, since $l+\mu' = -j'-m'-1 < 0$, $l-\mu = -j'+\delta'-1 < 0$ and l can have either sign, this must be an analytic continuation of the usual D-function, defined by the equation

$$D^j_{m'm}(2\alpha) = \left[\frac{(j+m')!(j-m)!}{(j+m)!(j-m')!}\right]^{1/2} \frac{1}{(m'-m)!} (\cos \alpha)^{m'+m} (\sin \alpha)^{m'-m}$$
$$\times F(-j+m', j+m'+1, m'-m+1; \sin^2 \alpha). \tag{19}$$

Although a compact expression for $(H:m')$ has been found, it seems strange that the expression should depend on j', m', δ' only and not on m which has been kept arbitrary in the above derivation. To remove this conceptual difficulty we go back to the fundamentals of the SU(3) representations and recall that $|j', m', \delta'\rangle$ is a linear combination of coupled states of pairs of angular momenta j_1', j_2' with the same values of j', m', δ' $(= j_1'-j_2')$. The following relations, therefore, hold

$$\delta' = j_1'-j_2', \quad -m' = j_1'+j_2'-r, \quad r \geqslant 0, \quad \delta-m = \delta'-m' = 2j_1'-r.$$

If $m = -j$, then $\delta'-m' = p = 2j_1'-r, j_1' = \frac{1}{2}p$, and, if $m = j$, then $\delta'-m' = -q$ $= -2j_2'+r$. This implies that

$$j'+m' = 0,\ j'+\delta' = p, \text{ for } m = -j,$$
$$j'-m' = 0,\ j'-\delta' = q, \text{ for } m = j.$$

If $m = -j+1$, then $\delta'-m' = p-1$, and, either (a) $j_1' = \frac{1}{2}(p-1)$, $r = 0$, $j'+m' = 0$, $j'+\delta' = p-1$, or (b) $j_1' = \frac{1}{2}p$, $r = 1$, $j'+m' = 0$, $j'+\delta' = p-1$, or (c) $j_1' = \frac{1}{2}p$, $r = 1$,

$$j'+m' = 1,\ j'+\delta' = p.$$

In all these cases j', m', δ' are connected by two relations, and it can be safely assumed that m does occur, though in an involved manner, in the expression for $(H\!:\!m')$.

It is instructive to test the correctness of the expression for $(H\!:\!m')$ in some special cases for which the calculations can be carried out easily. By Eqs (18) and (19),

$$(H : m') \equiv (\tfrac{1}{2}(p+q), m, \tfrac{1}{2}(p-q) : j', m', \delta')$$

$$= \text{const.} (\cos v)^{-\delta'-m'} (\sin^2 v)^{j-j'} F(-j'-\delta', -j'-m', -2j'; \sin^2 v). \tag{20}$$

For $j' = m'$, or $j' = \delta'$, this reduces to an expression which follows from (22) or (23) of I when the primed and the unprimed quantities are interchanged. A more interesting example is furnished by case (c) above which is not covered by the formulae listed in Sec. 3 of I. The general m. e. of exp $(-ivN)$ can be written as

$$(j, m, \delta : j', m', \delta') = \sum_{\bar{j}} (-)^{\delta-\delta'} (2\bar{j}+1)(2j+1)^{1/2}(2j'+1)^{1/2}$$

$$\times \begin{Bmatrix} \bar{j}, \frac{1}{2}(q+2\delta), \frac{1}{2}(p-\delta-m) \\ j, \frac{1}{2}(p-\delta+m), \frac{1}{2}q \end{Bmatrix} D^{\bar{j}}_{\bar{m}\bar{m}'}(-2v) \begin{Bmatrix} \bar{j}, \frac{1}{2}(q+2\delta'), \frac{1}{2}(p-\delta'-m') \\ j', \frac{1}{2}(p-\delta+m), \frac{1}{2}q \end{Bmatrix}, \tag{21}$$

where, $2\bar{m} = -p+q+3\delta+m$, $2\bar{m}' = -p+q+3\delta'+m'$, and the curly brackets denote $6-j$ symbols of SU(2). For $m = -j+1$, $j = \frac{1}{2}(p+q)$, $\delta = \frac{1}{2}(p-q)$, $\delta'-m' = p-1$, $j'+m' = 1$, $j'+\delta' = p$, the expression simplifies and takes the form (20).

4. *Recurrence relations for SU(4)*

The basis functions of an irreducible representation (λ, μ, ν) of SU(4) carry six labels, p, q, j, m, Y and Z. Of these only one belongs to SU(4) proper and the remaining five serve as the representation and the state labels of the SU(3) subgroup. By Eq. (2) the general m. e. of an IR can be written as a triple sum

$$\mathfrak{S} \equiv \langle p, q, j, m, Y, Z | \mathscr{S} | p', q', j', m', Y', Z' \rangle = \sum_{\bar{j}, \bar{m}, \bar{Y}} e^{-i\eta Z} e^{-i\beta Y}$$

$$\times \mathscr{D}^{j}_{m\bar{m}}(-\alpha_3, -\alpha_2, -\gamma) \langle p, q, j, \bar{m}, Y | e^{-ivN} | p, q, \bar{j}, \bar{m}', \bar{Y} \rangle$$

$$\times \langle p, q, \bar{j}, \bar{m}', \bar{Y}, Z | e^{-i\mu M} | p', q', \bar{j}, \bar{m}', \bar{Y}', Z' \rangle$$

$$\times \langle p', q', \bar{j}, \bar{m}', \bar{Y}' | SU(3) | p', q', j', m', Y' \rangle. \tag{22}$$

The reduction in the number of summations over intermediate states is due to the existence of relations like

$$\tfrac{1}{2} Y - \bar{m} = \tfrac{1}{2}\bar{Y} - \bar{m}', \; \tfrac{2}{3} Z + \bar{Y} = \tfrac{2}{3} Z' + \bar{Y}'. \tag{23}$$

To derive recurrence relations for the m. e. of $e^{-i\mu M}$ we put $j = m = \delta = p/2$ in (22) and proceed as in the case of SU(3). The application of the operator A_1^4 to $\mathfrak{S}$ then yields the relation [8]

$$
\begin{aligned}
&A_1 \csc\mu[(\bar{j}+\overline{m}')(\bar{j}-\tfrac{1}{2}, \overline{m}'-\tfrac{1}{2}, \bar{\delta}-\tfrac{1}{2}Z : \bar{\delta}'-\tfrac{1}{2}Z')\\
&\quad -(\bar{\delta}-\overline{m}')(\bar{j}-\tfrac{1}{2}, \overline{m}'+\tfrac{1}{2}, \bar{\delta}-\tfrac{1}{2}Z : \bar{\delta}'-\tfrac{1}{2}Z')]\\
&-A_2 \csc\mu[(\bar{j}-\overline{m}'+1)(\bar{j}+\tfrac{1}{2}, \overline{m}'-\tfrac{1}{2}, \bar{\delta}-\tfrac{1}{2}Z : \bar{\delta}'-\tfrac{1}{2}Z')\\
&+(\bar{\delta}-\overline{m}')(\bar{j}+\tfrac{1}{2}, \overline{m}'+\tfrac{1}{2}, \bar{\delta}-\tfrac{1}{2}Z : \bar{\delta}'-\tfrac{1}{2}Z')]+[(p-\bar{j}-\bar{\delta}+1)(p+\bar{j}-\bar{\delta}+2)]^{1/2}\\
&\times\left[\frac{3}{2}\csc^2\mu\left(\overline{Y}'-\frac{2}{3}\overline{m}'\right)-\frac{1}{2}\frac{\partial}{\partial\mu}-\frac{1}{12}\tan\mu(8Z+p-2q-6\overline{m})+\frac{1}{2}\cot\mu(p+q+\bar{\delta}+\overline{m}')\right]\\
&\quad\times(2\bar{j}+1)(\bar{j}, \overline{m}', \overline{Y}, Z : \overline{Y}', Z') = 6i(2\bar{j}+1)(p+1)(p+q+2)\\
&\quad\times(Z+1, p+1, q\|X\|Z, p, q)(p+1, q, \bar{j}, \overline{m}', \overline{Y}, Z+1 : p', q', \bar{j}, \overline{m}', \overline{Y}', Z'),\\
&A_1 = \tfrac{1}{6}[6(\bar{j}+\bar{\delta})]^{-1/2}[(q+\bar{j}+\bar{\delta})(p+\bar{j}-\bar{\delta}+2)(4p'+2q'-3\overline{Y}'-6\bar{j}+6)\\
&\quad\times(2p'-2q'+3\overline{Y}'+6\bar{j})(2p'+2q'+3\overline{Y}'+6\bar{j}+6)]^{1/2},\\
&A_2 = \tfrac{1}{6}[6(\bar{j}-\bar{\delta}+1)]^{-1/2}[(q-\bar{j}-\bar{\delta})(p-\bar{j}-\bar{\delta}+1)(4p'+2q'-3\overline{Y}'\\
&\quad+6\bar{j}+12)(2p'-2q'-3\overline{Y}'+6\bar{j}+6)(2p'+4q'+3\overline{Y}'-6\bar{j})]^{1/2}, \qquad (24)
\end{aligned}
$$

where the abbreviation $(j, m, Y, Z : Y', Z')$ has been used for $\langle p, q, j, m, Y, Z|\exp(-i\mu M)| |p', q', j, m, Y', Z'\rangle$. This is one of the many relations which can be derived by applying the twelve non-diagonal generators to $\mathfrak{S}$. As we are interested mainly in the methodology, we do not derive the other relations here.

By combining two or more recurrence relations or by the iterative procedure mentioned in the Introduction special m. e. of $\exp(-i\mu M)$ can be obtained. Another useful method is to remove the summations from the expression for $\mathfrak{S}$ by a proper choice of the initial and the final state and then determine the special m. e. as an eigenfunction of a Casimir operator of the group. A good example of a summation-free $\mathfrak{S}$ is obtained by setting $j_{\text{initial}} = j_{\text{final}} = q = p' = 0$ in (22).

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