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# ON THE DISCONTINUITY FORMULA IN KADYSHEVSKY'S FORMALISM

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The problem of the discontinuities of the Feynman amplitudes is discussed. Kadyshevsky's formalism is used to obtain the Cutkosky formula.

## 1. *Introduction*

The problems connected with the singularities of the amplitudes in the Feynman formalism were discussed by many authors [1–5]. All these discussions were based on the well known Feynman formula

$$\frac{1}{A_1A_2 \dots A_n} = (n-1)! \iint \frac{d\alpha_1 \dots d\alpha_n \delta(\sum \alpha_i - 1)}{(\sum \alpha_i A_i)^n}, \tag{1.1}$$

applied to the usual Feynman amplitudes; the expression

$$\frac{1}{A_i} = \frac{1}{q_i^2 - m^2 + i\varepsilon} \tag{1.2}$$

denotes the Feynman propagator in the momentum space. The momenta $q_i$ are combinations of the external momenta $p$ and the independent internal momenta $k$, over which integration is performed

$$f(p) = \int \frac{d^4k_1 \dots d^4k_l}{A_1(p, k) \dots A_n(p, k)}. \tag{1.3}$$

$l$ is the number of independent loops in the graph, as usual. The functions $A(p, k)$ are in the denominator of the integrand and this suggests that the singularities of the integral appear when some $A_i(p, k)$ vanish.

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The considerations presented in [1–5] confirm this suggestion, but it seems to be difficult to perform a detailed and complete analysis of the problem in this way.

For a self-energy graph, for example (Fig. 1), each of the propagators reaches the mass-shell $q^2 = m^2$ when it is integrated over $d^4q_1d^4q_2$, but the amplitude $f(p^2)$ is nonsingular (on

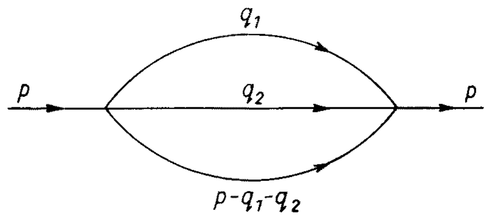


Fig. 1

condition that $p^2 < 9m^2$)). The singularity appears when $p^2 > 9m^2$, namely when all three propagators can be on the mass shell "simultaneously".

The form of the singularities is discussed in [4]. The outline of considerations is given, which leads to the conclusion that if the propagators $A_1 \dots A_m$ are responsible for a given singularity (a cut), then we obtain the discontinuity disc $f(p)$ on this cut by substituting into the expression (1.3) the form

$$\delta_+(q_1^2-m^2) \dots \delta_+(q_m^2-m^2) \tag{1.4}$$

instead of

$$\frac{1}{(q_1^2-m^2+i\varepsilon)(q_2^2-m^2+i\varepsilon) \dots (q_m^2-m^2+i\varepsilon)}. \tag{1.4'}$$

A rigorous proof of this formula, when classical methods are used, is rather complicated.

The method which we want to present here is based on Kadyshevsky's formalism [6]. Its simplicity comes from the fact that the singularities are computed directly from their "source".

## 2. *Kadyshevsky's formalism*

This chapter is based on [6] and included to fix the notation. Kadyshevsky worked out a method of computation for the $S$-matrix elements in perturbation theory, different from the classical Feynman method. The starting point is the well known formula for the $S$-operator

$$S = 1+i\sum_{n=1}^{\infty} T^{(n)},$$

with

$$T^{(n)} = (-i)^{n-1}\int \theta(x_1^0-x_2^0) \dots \theta(x_{n-1}^0-x_n^0)L(x_1) \dots L(x_n)d^4x_1 \dots d^4x_n. \tag{2.1}$$

Making use of the fact that

$$[L(x), L(y)] = 0 \quad \text{for } (x-y)^2 < 0,$$

we can write (2.1) in the form

$$T^{(n)} = (-i)^{n-1} \int \theta(\lambda(x_1-x_2)) \dots \theta(\lambda(x_{n-1}-x_n))L(x_1) \dots L(x_n)d^4x_1 \dots d^4x_n, \tag{2.1'}$$

where

$$\lambda^2 \equiv (\lambda^0)^2-(\lambda)^2 = 1, \quad \lambda^0 > 0.$$

The Heaviside functions $\theta(\lambda x)$ can be represented as

$$\theta(\lambda x) = -\frac{1}{2\pi i}\int\limits_{-\infty}^{+\infty} \frac{e^{-i\tau(\lambda x)}}{\tau+i\varepsilon}\, d\tau, \tag{2.2}$$

and after the Fourier transformation of the field operators we obtain:

$$T^{(n)} = (-i)^{n-1}\left[\frac{1}{(-i)^{n-1}}\int \frac{d\tau_1}{\tau_1+i\varepsilon}\dots\frac{d\tau_{n-1}}{\tau_{n-1}+i\varepsilon}\, e^{-i\tau_1\lambda\cdot(x_1-x_2)}\dots e^{-i\tau_{n-1}\lambda\cdot(x_{n-1}-x_n)}\right.$$
$$\times :\varphi(p_{11})\dots\varphi(p_{1k})::\varphi(p_{21})\dots\varphi(p_{2k}):\dots:\varphi(p_{n1})\dots\varphi(p_{nk}):$$
$$\left.\times\, e^{ix_1\cdot\sum_s p_{1s}}e^{ix_2\cdot\sum_s p_{2s}}\dots e^{ix_n\sum_s p_{ns}}d^4p\,d^4x\right], \tag{2.3}$$

where in the symbols $p_{kl}$ describing the four-momenta, the first index denotes the number of the Lagrange operator and the second one the number of the field operator in a given $L$-operator.

Under the integral we get a product (not a chronological product, but a simple one!) of the normal products and we can apply Wick's theorem to it. A contraction in this case has the form

$$\varphi(k)\varphi(p) \equiv \langle 0|\varphi(k)\varphi(p)|0\rangle = D^+(k)\delta^4(k+p)$$
$$\equiv \delta^4(k+p)\theta(p^0)\delta(p^2-m^2) \equiv \delta^4(k+p)\delta_+(p^2-m^2). \tag{2.4}$$

After this the formula (2.3) disintegrates into a sum of terms according to the various configurations of the contractions. Each term contains a normal product of the field operators (the external lines), some contractions (the propagators) and the factors $1/(\tau_i+i\varepsilon)$.

The integration over the space-time coordinates yields the $\delta^4$-distributions responsible for momentum conservation in the vertices

$$\delta^4(\Sigma\, k_{\text{in}}^{(i)}-\Sigma\, k_{\text{out}}^{(i)}+\lambda\tau_{i-1}-\lambda\tau_i),$$

where $\Sigma k_{\text{in}}^{(i)}$ denotes the whole momentum entering the vertex ($i$) by the external lines and propagators. $\Sigma k_{\text{out}}^{(i)}$ is connected with the lines leaving the vertex ($i$). The "momenta" $\lambda\tau_i$ come from the transformates of the $\theta(\lambda x)$ distributions. The "momentum" $\lambda\tau_1$ leaves the vertex (1) and enters the vertex (2). Next $\lambda\tau_2$ leaves the vertex (2) and goes to (3). At last $\lambda\tau_{n-1}$ binds the vertices ($n-1$) and ($n$), respectively. Thus, to the Feynman graph a line must be added, starting from the vertex (1), going in turn over all other vertices and ending at the last vertex ($n$).

The succession of vertices is determined by the numeration of the Lagrange operators, and more precisely, by the relation between this numeration and the configuration of contractions. This remark will be useful below: one can always deform a graph in such a way, that the added line will not intersect itself in any point. (If necessary, one can imagine the graph as being constructed in three dimensions.) This line we shall call and design as a dashed line. For example, for a Feynman vertex-graph

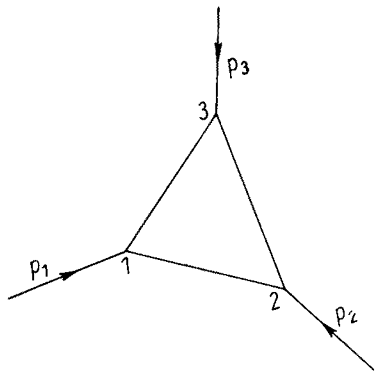


Fig. 2

we have in our formalism

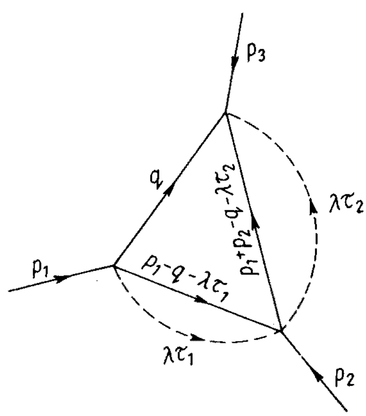


Fig. 3

and an expression obtained after trivial integrations

$$\int d^4p_1 d^4p_2 d^4p_3 \delta^4(p_1+p_2+p_3) : \varphi(p_1)\varphi(p_2)\varphi(p_3) :$$

$$\int d^4q \int \frac{d\tau_1}{\tau_1+i\varepsilon} \int \frac{d\tau_2}{\tau_2+i\varepsilon} \delta_+(q^2-m^2)\delta_+((p_1-q-\lambda\tau_1)^2-m^2)\delta_+((p_1+p_2-q-\lambda\tau_2)^2-m^2), \tag{2.5}$$

where $\delta_+(k^2-m^2)$ denotes the upper mass shell $k_0 > 0$. The limits of integration over $d\tau$ come from kinematics. It is necessary to stress here that to a given Feynman graph (Fig. 2) correspond several Kadyshevsky's graphs (Fig. 3) with different runs of the dashed line. In order to obtain the whole amplitude we must sum up over those graphs.

### *3. The discontinuity formula*

The expression connected with certain Kadyshevsky's graph contains a product of terms $(\tau_i+i\varepsilon)^{-1}$ representing the segments of the dashed line and a product of distributions $\delta_+(p^2-m^2)$ connected with propagators. The singularities[1] of the amplitude result from the denominators $(\tau_i+i\varepsilon)$. So the amplitude becomes singular if $0 \in [a_i, b_i]$. On the contrary, if zero does not belong to any interval $[a_i, b_i]$, then the amplitude is real, and the possibility of making this statement comes from the formalism used. To see this let us collect powers of the imaginary unit in Kadyshevsky's approach. We have $i^{n-1}$ before the integral (2.1′) and $i^{-(n-1)}$ from the $\theta(\lambda x)$ distributions. This two powers cancel each other and the amplitude is real (except the possible residues). In the Feynman approach we have $i^{n-1}$ as before and one power of $(i)$ from every field propagator. So the picture is not so clear in this case.

Let us consider the simplest case, when zero belongs to one interval $[a_i, b_i]$ only. The integral over $d\tau_i$ desintegrates in the principle value (real) and the residue at $\tau_i = 0$ times $(-\pi i)$.

Let us now divide the graph in two parts by a line intersecting the $\tau_i$ dashed line *only* and some propagators if necessary (in the case when the graph is imagined in three dimensions, we have a dividing surface). There is no freedom in this division because the dashed line goes through all vertices and the ends of it are separated (the beginning in the vertex 1 and the end in $n$). This can be presented schematically:

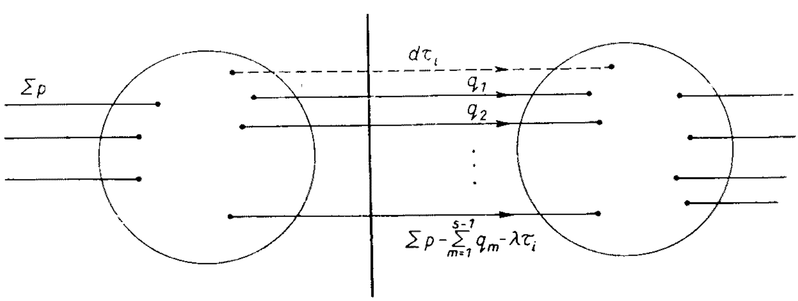


Fig. 4

For simplicity we assume that both parts of the graph regarded as individual graphs are the bound ones. In the case when the dividing line cuts $s$ propagators, they will form

[1] The term "singularity" does not contain the divergences removed by regularization. Here and below we understand it as discontinuities of the amplitude appearing for certain values of the external variables. In agreement with this convention "the amplitude is singular" means "the amplitude has a non-zero imaginary part for real values of external parameters".

$s-1$ loops. So they will have a form:

$$\delta_+(q_1^2-m_1^2)\ \dots\ \delta_+(q_{s-1}^2-m_{s-1}^2)\delta_+((\Sigma\, p-\sum_1^{s-1} q_m-\lambda\tau_i)^2-m_s^2). \qquad (3.1)$$

$\Sigma p$ is the algebraic sum of the external momenta connected with the left-hand side of the graph.

In the circles all remaining propagators and the ends of the dashed line are contained.

An assumption was made that the integral over $d\tau_i$ contains zero on the contour. It is equivalent to the condition that the momentum $\Sigma p$ can be decomposed in $s$ vectors on the suitable mass shells. So it must be $(\Sigma p)^2 \geqslant \sum_{i/1}^{s} m_i^2$ with $\Sigma p_0 > 0$ in the situation in Fig. 4 and with $\Sigma p_0 < 0$ when the direction of the $\tau_i$ line (and of the propagators, of course) is opposite. In the first case we have the upper mass shells and in the second the lower ones. We obtain the discontinuity formula (the residue) by inserting $\tau_i = 0$ in the integrand. The parameter $\tau_i$ exists in the distribution $\delta_+((\Sigma p-\Sigma q-\lambda\tau_i)^2-m_s^2)$ and in some other propagators in the circles (Fig. 4).

It was already noticed that from the circles in Fig. 4 together with the external lines and the halves of propagators treated as external lines, the individual Kadyshevsky diagrams can be made if the parameter $\tau_i$ is put equal zero. In order to make amplitudes from the circles we have to sum up over all possible runs of the dashed line in the circles. It is not the way to obtain all possible runs of the dashed line in the whole diagram of Fig. 4, but we obtain all Kadyshevsky's graphs giving contribution to the discontinuity formula. A simple example makes it clear:

Let us consider a box-graph

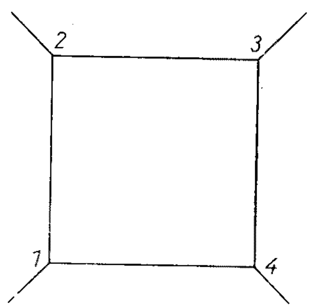


Fig. 5

We assume that only the propagators (1.4) and (2.3) can be on the mass shell simultaneously. So the singularities will exist in graphs:

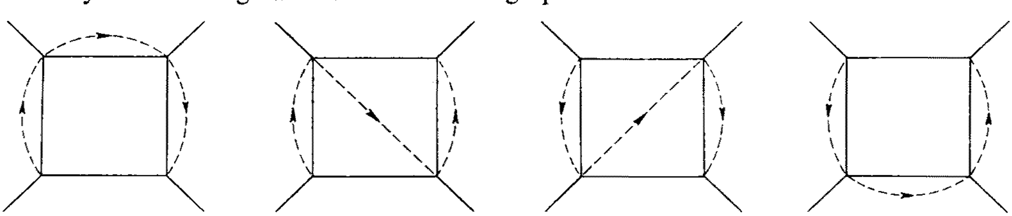

Fig. 6

Other graphs will be regular. For example in the graph

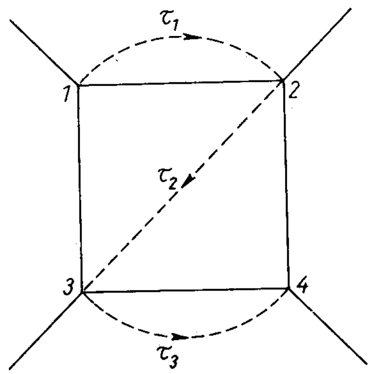


Fig. 7

none of the parameters $\tau_i$ can reach zero because of our kinematical assumptions. (E. g. if $\tau_1$ could be zero, then in the Feynman graph in Fig. 5 the propagators (1.3) and (1.2) could be on the mass shell simultaneously.)

Now the Cutkosky formula is trivial: both circles represent the suitable Feynman amplitudes and the divided propagaters yield the $\delta_+(q^2 - m^2)$ distributions.

We can recapitulate as follows. The discontinuity of the amplitude is connected with the possibility that one (or more) parameter $\tau_i$ can reach zero. In our simplest case (only one $\tau_i$ equals zero) it is equivalent to the possibility that some Feynman propagators reach the mass shell simultaneously. To find them we must divide the graph into two parts cutting the $\tau_i$ dashed line only. The propagators cut this way are what we are looking for. The detailed analysis of the cases when more than one parameter $\tau_i$ can reach zero seems to be difficult. Generally, the limits of integrals over $\tau_i$ are not independent. In particular cases as in Fig. 8

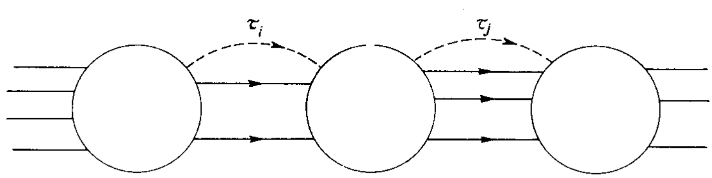


Fig. 8

they are independent, but e. g. in the graph of the structure as in Fig. 9 we have three possibilities:

1. $p^2 > (m_1+m_2)^2$, but $k_2^2 < (m_1+m_3)^2 \rightarrow$ the parameter $\tau_i$ reaches zero, $\tau_j$ does not.

2. $p^2 > (m_1+m_2)^2$, $k_2^2 > (m_1+m_3)^2$, but the propagators (1.2) and (1.3) cannot simultaneously be on the mass shells for the same values of the momentum $q$ as the propagators (1.3) and (2.3). In this case both the variables $\tau_i$ and $\tau_j$ reach zero, but not simultaneously. (The limits $[a_j, b_j]$ depend on $\tau_i$, and when $\tau_i = 0$, then $0 \notin [a_j, b_j]$.)

3. All three propagators can be on the mass shells simultaneously, the variables $\tau_i$, $\tau_j$ reach zero simultaneously (anomalous threshold).

It is not clear whether this approach leads to any significant simplification for more complicated cases, nevertheless it gives some guidance in the problem of possible singulari-

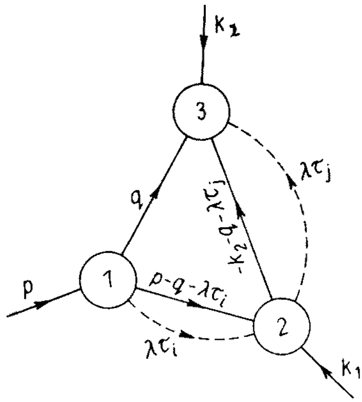


Fig. 9

ties connected with a given Feynman graph. It stresses also the direct dependence between the discontinuities and the possibility for the Feynman propagators to reach the mass shell simultaneously.

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