THE SPIN-1 ELECTROMAGNETIC VERTEX FUNCTION

By A. J. Phares*

International Centre for Theoretical Physics, Trieste**

(Received September 27, 1974)

The e^3 contribution to the electromagnetic vertex function of spin-1 particles is calculated in the framework of source theory. A magnetic moment of arbitrary strength κ is included in the primitive interaction. In particular, the quadrupole form factor is shown to be finite with no need of adding any contact term if and only if $\kappa = 1$. A general discussion of the contact terms is included in the present work and the explicit expressions of the form factors are obtained along with their asymptotic behaviour, when $\kappa = 1$. The dynamically induced quadrupole moment, in that case, is calculated to be $\alpha/18\pi$, and the associated spectral form is superconvergent.

1. General presentation of the e^3 contribution to the vertex function

The causal process considered is described in Fig. 1. An extended photon source emits a virtual photon of momentum Q and spectral mass M, which decays into a pair of spin-1 particles of momenta p_2 and p'_2. These particles then interact by exchanging a spacelike photon of momentum $p_1-p_2 = p'_1-p'_2, p_1$ and p'_1 being the momenta of the scattered particles which are absorbed by the sources K_1 and $K_{1'}$.

In order to calculate the vacuum persistence amplitude for this causal process, one can follow Schwinger's method [1] used in spin-0 and spin-$\frac{1}{2}$ cases. Basically, one must evaluate the two-particle effective emission source equivalent to the extended photon source. This is done by using the same technique as presented by the author elsewhere [2]. This method is not as easy to apply as it is in the spin-0 and -$\frac{1}{2}$ cases. There exists a short way of writing down the final result using the so-called "causal rules" [3]. Finally, the e^3 contribution to the vacuum amplitude is ($d\omega_p = d\vec{p}/(2\pi)^3\, 2p^0$):

$$A_v = -\tfrac{1}{2}\, e^2 \int d\omega_{p_1} d\omega_{p_2} d\omega_{p'_1} d\omega_{p'_2} (2\pi)^4 \delta(p_1+p'_1-p_2-p'_2)$$

$$\times K_1^b(-p_1)\bar{g}_{b\mu}(p_1) eq J^{\mu\lambda a} \bar{g}_{\lambda c}(p'_1) K_1^c(-p'_1) A_a(Q)/(p_1-p_2)^2. \quad (1)$$

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* On leave of absence from the National Council for Scientific Research, Beirut, Lebanon.

** Address: International Centre for Theoretical Physics, Miramare-Grignano, P. O. Box 586, Trieste, Italy.

The third-rank tensor $J^{\mu\lambda a}$ involves contributions coming from the internal part of the diagram of Fig. 1,

$$J^{\mu\lambda a} = T^{\mu\nu\alpha}(p_1, -p_2, p_1-p_2)g_{\alpha\beta}T^{\lambda\sigma\beta}(p_1', -p_2', p_1'-p_2')\bar{g}_{\nu\kappa}(p_2)\bar{g}_{\sigma\delta}(p_2')T^{\kappa\delta a}(p_2, p_2', Q). \tag{2}$$

Here the T tensor is the value associated with each vertex of the diagram under consideration and depends on the incoming and outgoing momenta,

$$T^{\mu\nu\alpha}(p_1, -p_2, p_1-p_2) = g^{\mu\nu}(p_1+p_2)^{\alpha}-(1+\kappa)\,(g^{\mu\alpha}p_1^{\nu}+g^{\nu\alpha}p_2^{\mu}), \tag{3}$$

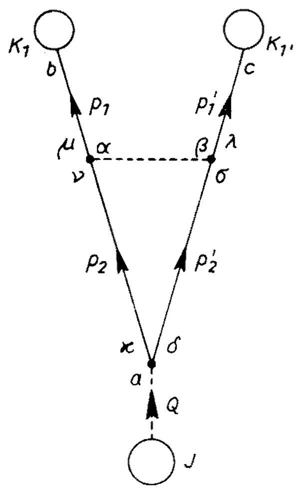

Fig. 1. The e^3 contribution to the spin-1 electromagnetic vertex function

and $\bar{g}_{\mu\nu}(p)$ is associated with each spin-1 particle line of mass m,

$$\bar{g}^{\mu\nu}(p) = g^{\mu\nu}+p^{\mu}p^{\nu}/m^2. \tag{4}$$

Finally, A_μ is the electromagnetic field, e the electric charge and q the 2×2 charge matrix.

Introducing unity expressed as

$$1 = \int d\omega_Q dM^2(2\pi)^3\delta(p_2+p_2'-Q). \tag{5}$$

Eq. (1) becomes

$$A_v = -e^2\int d\omega_{p_1}d\omega_{p'_1}dM^2 d\omega_Q(2\pi)^4\delta(p_1+p_1'-Q)$$
$$\times K_1^b(-p_1)\bar{g}_{b\mu}(p_1)\,\tfrac{1}{2}\,eqI^{\mu\lambda a}\bar{g}_{\lambda c}(p_1')K_1^c(-p_1')A_a(Q), \tag{6}$$

with

$$I^{\mu\lambda a} = \int d\omega_{p_2}d\omega_{p'_2}(2\pi)^3\delta(p_2+p_2'-Q)J^{\mu\lambda a}/(p_1-p_2)^2. \tag{7}$$

First of all, gauge invariance is secured as

$$Q_a I^{\mu\lambda a} = 0. \tag{8}$$

Furthermore, crossing symmetry implies that

$$I^{\mu\lambda a}(p_1, p_1') = -I^{\lambda\mu a}(p_1', p_1), \tag{9}$$

and this condition is manifestly satisfied.

2. *Form factors*

According to the above general presentation, $I^{\mu\lambda a}$ is a third-rank tensor involving two independent vectors, say Q and $p_1 - p_1'$ for convenience, restricted to satisfy the constraints (8) and (9). The most general structure of such a tensor is (dropping the index 1):

$$I^{\mu\lambda a}(p, p') = f(g^{\mu a}Q^\lambda - g^{\lambda a}Q^\mu) + (p-p')^a \begin{Bmatrix} a g^{\mu\lambda} + bQ^\mu Q^\lambda + c(p-p')^\mu (p-p')^\lambda \\ + d[Q^\mu(p-p')^\lambda - (p-p')^\mu Q^\lambda] \end{Bmatrix}$$
$$+ g[Q^\mu(p-p')^\lambda + Q^\lambda(p-p')^\mu]Q^a + M^2 g[g^{\mu a}(p-p')^\lambda + g^{\lambda a}(p-p')^\mu], \tag{10}$$

where a, b, c, d, f, g are scalars depending on the spectral mass M. These scalars can be determined in terms of the following linearly independent dimensionless amplitudes:

$$\begin{aligned} B_1 &= (p-p')_a I_\lambda^{\lambda a}/(p-p')^2, \\ B_2 &= Q_\mu Q_\lambda (p-p')_a I^{\mu\lambda a}/Q^2(p-p')^2, \\ B_3 &= (p-p')_\mu (p-p')_\lambda (p-p')_a I^{\mu\lambda a}/(p-p')^4, \\ B_4 &= (p-p')_\mu I^{\mu a} a/(p-p')^2, \\ B_5 &= Q_\mu (p-p')_\lambda (p-p')_a I^{\mu\lambda a}/Q^2(p-p')^2, \\ B_6 &= Q_\mu I^{\mu\lambda}\lambda/Q^2. \end{aligned} \tag{11}$$

In the expression (6) of A_v, $I^{\mu\lambda a}$ is sandwiched between the transverse projection operators $\bar{g}_{b\mu}(p_1)$ and $\bar{g}_{\lambda c}(p_1')$, so that the terms in $I^{\mu\lambda a}$ containing either p_1^μ or $p_1'^\lambda$ do not contribute to the overall expression of A_v. Then, in the Lorentz gauge, the contributing terms of $I^{\mu\lambda a}$ are

$$I^{\mu\lambda a} \to a[(p-p')^a g^{\mu\lambda} - (1+\kappa)(g^{\mu a}Q^\lambda - g^{\lambda a}Q^\mu)]$$
$$+ [(1+\kappa)a + f + M^2 g](g^{\mu a}Q^\lambda - g^{\lambda a}Q^\mu) + (b-c+2d)(p-p')^a Q^\mu Q^\lambda. \tag{12}$$

In order to obtain the proper physical significance of the terms in the above expression, one must take into account the primitive interaction normalization condition. To do so, let us write the vacuum amplitude for the process considered in Fig. 2, where a photon source emits a virtual photon which decays into a pair of spin-1 particles absorbed by the sources K_1 and $K_{1'}$:

$$i \int (dx) j^a(x) A_a(x) = -i \int d\omega_p d\omega_{p'} K_1^b(-p) \bar{g}_{b\mu}(p_1) \tfrac{1}{2}\, eq I_e^{\mu\lambda a} \bar{g}_{\lambda c}(p') K_{1'}^c(-p') A_a(Q). \quad (13)$$

Here j^a is the electric current vector in source free regions ($G^{a\nu}$ and φ^ν are the spin-1 tensor and vector fields)

$$j^a = G^{a\nu} ieq \varphi_\nu - \partial_\nu(\varphi^a ieq \varphi^\nu), \quad (14)$$

and

$$I_e^{\mu\lambda a} = g^{\mu\lambda}(p-p')^a - (1+\kappa)(g^{\mu a}Q^\lambda - g^{\lambda a}Q^\mu). \quad (15)$$

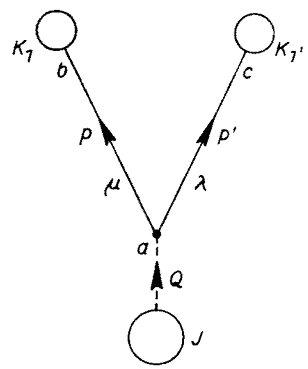

Fig. 2. The primitive interaction

From the primitive interaction normalization condition, Eqs (13)–(15), the first two terms of Eq. (12) must be associated with the electric charge and magnetic form factors, respectively, while the third term is associated with the dynamically induced electric quadrupole moment. The corresponding weight functions will be called

$$f_e(M^2) = a = \tfrac{1}{2}(B_1 - B_2 - B_3), \quad (16)$$

$$f_m(M^2) = (1+\kappa)a + f + M^2 g = \tfrac{1}{2}(1+\kappa)(B_1 - B_2 - B_3) + \tfrac{1}{2}(-B_3 + B_4 + B_5 - B_6), \quad (17)$$

$$f_q(M^2) = m^2(b - c + 2d)$$

$$= \frac{1}{2}\frac{m^2}{M^2}(B_1 - 3B_2 - B_3) + \frac{1}{2}\frac{m^2}{(p-p')^2}(B_1 - B_2 - 5B_3 + 2B_4 + 6B_5 - 2B_6). \quad (18)$$

Finally, the vacuum amplitude (6) is equivalent to

$$A_\nu = ie^2 \int d\omega_{p_1} d\omega_{p'_1} dM^2 i d\omega_Q (2\pi)^4 \delta(p_1 + p'_1 - Q) K_1^b(-p_1) \bar{g}_{b\mu}(p_1) \tfrac{1}{2}\, eq$$

$$\times \left[f_e(M^2) I_e^{\mu\lambda a} + f_m(M^2) I_m^{\mu\lambda a} + \frac{1}{m^2} f_q(M^2) I_q^{\mu\lambda a} \right] \bar{g}_{\lambda c}(p'_1) K_1^c(-p'_1) A_a(Q), \quad (19)$$

where the tensors I_e, I_m and I_q are immediately identified by comparison with (12). Clearly, Eq. (19) is made up of three parts which are separately gauge invariant, indeed, under the causal conditions being considered, since

$$Q_a I^{\mu\lambda a}_{e,m,q} = 0, \quad Q(p_1 - p'_1) = 0. \tag{20}$$

The space-time extrapolation of the result (19) consists in generalizing the causal calculation to any arrangement of the sources. This can be done by considering separately each of the three parts involved in (19).

First, we shall consider the electric charge term:

$$A_v^{(e)} = ie^2 \int d\omega_{p_1} d\omega_{p'_1} dM^2 i d\omega_Q (2\pi)^4 \delta(p_1 + p'_1 - Q)$$
$$\times K_1^b(-p_1) \bar{g}_{b\mu}(p_1) \tfrac{1}{2} eq I_e^{\mu\lambda a} f_e(M^2) \bar{g}_{\lambda c}(p'_1) K_1^c(-p'_1) A_a(Q).$$

It can also be written as

$$A_v^{(e)} = ie^2 \int d\omega_{p_1} d\omega_{p'_1} dM^2 i d\omega_Q \exp[iQ(x-x')] \exp[-i(p_1+p'_1)x] (dx)(dx')$$
$$\times K_1^b(p_1) \bar{g}_{b\mu}(p_1) \tfrac{1}{2} eq I_e^{\mu\lambda a} f_e(M^2) \bar{g}_{\lambda c}(p'_1) K_1^c(-p'_1) A_a(x'). \tag{21}$$

Let us note that the electric current vector evaluated in a region that is prior to the action of the detection sources is given by

$$j^a(x) = -\int d\omega_{p_1} d\omega_{p'_1} K_1^b(-p_1) \bar{g}_{b\mu}(p_1) \tfrac{1}{2} eq I_e^{\mu\lambda a}$$
$$\times \bar{g}_{\lambda c}(p'_1) K_1^c(-p'_1) \exp[-i(p_1+p'_1)x], \tag{22}$$

so that (21) yields

$$A_v^{(e)} = -ie^2 \int dM^2 f_e(M^2) \int (dx)(dx') j^a(x) i d\omega_Q \exp[iQ(x-x')] A_a(x'). \tag{23}$$

Now we are in a position to perform the space-time extrapolation of the causal calculation by making the following replacement [1]:

$$i \int d\omega_Q \exp[iQ(x-x')] \to \int \frac{(dk)}{(2\pi)^4} \exp[ik(x-x')] \left[\frac{1}{k^2+M^2-i\varepsilon} + \text{c.t.}\right], \tag{24}$$

where there no longer is reference to the causal arrangement. The contact term (c.t.) is a polynomial in the momentum variable k^2. These additional terms must be chosen so as to meet the charge normalization requirement.

The action term that combines the primitive interaction with the interaction induced modification that have just been evaluated can be presented as

$$\int (dx)(dx') j^a(x) F_e(x+x') A_a(x'), \tag{25}$$

where the form factor F_e expressed in momentum space is

$$F_e(k^2) = 1 - e^2 \int_{4m^2}^{\infty} dM^2 f_e(M^2) \left[\frac{1}{k^2+M^2-i\varepsilon} + \text{c.t.}\right]. \tag{26}$$

Charge normalization requires that $F_e(k^2 = 0) = 1$, so that

$$\frac{1}{k^2+M^2-i\varepsilon} + \text{c.t.} = \frac{1}{k^2+M^2-i\varepsilon} - \frac{1}{M^2} + \frac{k^2}{M^4} + \ldots \tag{27a}$$

$$= (-k^2/M^2)^n \frac{1}{k^2+M^2-i\varepsilon}, \tag{27b}$$

where n represents the number of subtracted terms in (27a). *A priori*, the only restriction that one has on n is $n \geqslant 1$. In the next section it will become apparent that the smallest value of n which makes the spectral form in (26) finite is $n = 3$.

Before discussing the other form factors, one should point out that $F_e(k^2)$ is a particular expression of the charge form factor. As a matter of fact, the reduction (12) holds only in the Lorentz gauge. In App. A, the space-time extrapolation is performed in a gauge-invariant way showing explicitly the tensorial structure of the general expression of the form factor, which reduces to $F_e(k^2)g_{\mu\nu}$ in the Lorentz gauge.

Following exactly the same discussion as for the charge form factor, the magnetic form factor can be written as follows:

$$F_m(k^2) = -e^2 \int_{4m^2}^{\infty} dM^2 \frac{f_m(M^2)}{k^2+M^2-i\varepsilon}\left(-\frac{k^2}{M^2}\right)^n, \tag{28}$$

where we have taken into account the magnetic moment normalization condition $F_m(k^2 = 0) = 0$.

As to the dynamically induced quadrupole moment coupling, there are no physical restrictions at all on the number of contact terms and the way to choose them. It happens that the spectral form is finite without adding any contact term, when $\kappa = 1$. In that case,

$$F_q(k^2) = e^2 \int_{4m^2}^{\infty} dM^2 f_q(M^2) \frac{1}{k^2+M^2-i\varepsilon}. \tag{29}$$

3. *Weight functions*

Although the calculations are long, there are no basic difficulties in determining the weight functions $f_e(M^2)$, $f_m(M^2)$ and $f_q(M^2)$. We proceed by steps and construct new amplitudes, C_i, analogous to the B_i's, except that in Eqs (11) the I-tensor is replaced by the tensor $J_{\mu\lambda a}$, whose expression is given by Eq. (2). Then, the infrared problem arises naturally when performing the $d\omega_{p_2}\, d\omega_{p'_2}$ integrations, since the zero mass of the photon is responsible for the unlimited range of the Coulomb potential. This suggests that the difficulty is only superficial and will disappear when additional soft photon processes are considered. The general way of bypassing the problem is by imagining that the photon has a very small but finite mass μ. According to the definition of the C_i's, these amplitudes are related to the B_i's in exactly the same manner as $J_{\mu\lambda a}$ is related to $I_{\mu\lambda a}$ (see Eq. (7)). Recalling the origin of $(p_1-p_2)^2$ in the structure of the photon propagation function, we are lead to make the following replacement:

$$\frac{1}{(p_1-p_2)^2} \to \frac{1}{(p_1-p_2)^2+\mu^2}\,. \tag{30}$$

On physical grounds and by analogy with the spin-0 and -$\frac{1}{2}$ cases, one expects the magnetic and quadrupole form factors to be μ independent, on one hand, and the charge form factor $\ln\mu$ dependent on the other.

One chooses to perform the $d\omega_{p_2}\, d\omega_{p'_2}$ integrations in the centre-of-mass frame of the incoming and outgoing massive spin-1 particles, i.e. in the rest frame of $Q = p_1+p_1' = p_2+p_2' = (M, \vec{0})$. Let θ be the scattering angle in that frame, then the relationship between B_i and C_i becomes

$$B_i(M^2) = \frac{I_0}{M^2-4m^2}\int_0^1 d\left(\sin^2\frac{\theta}{2}\right)\frac{C_i(M^2,\theta)}{\sin^2\frac{\theta}{2}+\frac{\mu^2}{M^2-4m^2}}\,, \tag{31}$$

with

$$I_0 = \int d\omega_{p_2} d\omega_{p'_2}(2\pi)^3\delta(p_2+p_2'-Q) = \frac{1}{(4\pi)^2}\left(1-\frac{4m^2}{M^2}\right)^{1/2}. \tag{32}$$

It happens that the C_i's have a simple θ dependence, and the only integrals to be evaluated are

$$I_n = \int_0^1 d\left(\sin^2\frac{\theta}{2}\right)\frac{\cos^{n-1}\theta}{\sin^2\frac{\theta}{2}+\mu^2/(M^2-4m^2)}\,, \tag{33}$$

with $n = 1, 2, 3, 4, 5$. These integrals are simply calculated to be

$$I_1 = \ln\frac{M^2-4m^2}{\mu^2}\,,$$

$$I_2 = I_3 = I_1-2, \qquad I_4 = I_5 = I_1-\tfrac{8}{3}\,. \tag{34}$$

In this way the photon mass dependence is isolated in a single term I_1. Finally, we are interested in the linear combinations of the B_i's as appearing in Eqs (16)–(18). The algebra involved in the calculations of the weight functions is based purely on kinematics. In

Appendix B, the basic kinematic expressions used to determine $C_i(M^2, \theta)$ are given. The results are ($x = 4m^2/M^2$):

$$f_e = I_0\left\{-\frac{2x-1}{x-1}(I_1-2)+\frac{1}{3}(1+\kappa)(8x^2-4x-1)\right.$$

$$\left.-\tfrac{1}{3}(4x^2-1)+\tfrac{1}{12}(1+\kappa)^2(-16x^2+2x-3)+\tfrac{1}{2}(1+\kappa)^3x\right\}, \tag{35}$$

$$f_m = I_0\{(2x-1)(x-1)-\tfrac{1}{3}(1+\kappa)(18x^2-6x-1)$$

$$+\tfrac{1}{3}(1+\kappa)^2(18x^2-8x+1)+\tfrac{1}{6}(1+\kappa)^3(-12x^2+6x-1)-\tfrac{1}{2}(1+\kappa)^4x\}, \tag{36}$$

$$f_q = I_0\left\{-\frac{(2x-1)(10x-1)}{12x(x-1)}+(1+\kappa)\frac{16x^2-14x+1}{12x(x-1)}\right.$$

$$\left.-(1+\kappa)^2\frac{28x^2-32x-1}{48x(x-1)}+\frac{1}{6}(1+\kappa)^3\right\}. \tag{37}$$

4. Discussion and conclusion

As expected, $F_e(k^2)$ is logarithmically divergent in the $\mu = 0$ limit for any value of κ. $F_m(k^2)$ and $F_q(k^2)$ are $\ln\mu$ independent as they should be on physical grounds.

As to the number of contact terms needed to make the spectral forms ultraviolet convergent, in general, three are necessary in the expressions of F_e and F_m and only one in the expression of F_q. In the special case where $\kappa = 1$, F_q is finite without adding any contact term. As a matter of fact, the large x behaviour of $f_q(x)$ is ($g = 1+\kappa$):

$$\frac{1}{(4\pi)^2}\frac{1}{12}(2g^3-7g^2+16g-20) = \frac{1}{(4\pi)^2}\frac{1}{12}(g-2)(2g^2-3g+10),$$

which is zero if and only if $g = 2$, i.e. $\kappa = 1$. Furthermore, for the special choice of $\kappa = 1$, $f_q(x) \sim 1/x^2$ for large values of x, and the corresponding spectral form is *superconvergent*.

In gauge theories unifying weak and electromagnetic interactions, the choice of $\kappa = 1$ for the massive W bosons mediating the weak interactions has a dynamical origin [4] and is responsible for the cancellation of some of the divergences [5]. Therefore, there is a strong motivation for choosing $\kappa = 1$ and from now on we shall specialize our previous results to that case:

$$f_e(x) = -I_0\left[\frac{2x-1}{x-1}(I_1-2)+\frac{1}{3}(4x^2+18x+4)\right], \tag{38}$$

$$f_m(x) = -I_0(2x^2+3x-1), \tag{39}$$

$$f_q(x) = I_0/6x(x-1). \tag{40}$$

Making the change of variable

$$v = \left(1 - \frac{1}{x}\right)^{1/2}, \tag{41}$$

and fixing the number of contact terms to three ($n = 3$) in the expressions of $F_e(k^2)$ and $F_m(k^2)$, Eqs (26), (28) and (29) become

$$F_e(k^2) = 1 - \frac{\alpha}{2\pi}\left(\frac{k^2}{4m^2}\right)^3 \left\{ \int_0^1 dv \, \frac{(1+v^2)(1-v^2)^2 \ln \frac{4m^2}{\mu^2} - \frac{v^2}{1-v^2}}{1+(k^2/4m^2)(1-v^2) - i\varepsilon} \right.$$

$$\left. - \frac{2}{3} \int_0^1 dv \, \frac{v^6 + 10v^4 - 16v^2 + 3}{1+(k^2/4m^2)(1-v^2) - i\varepsilon} \right\}, \tag{42}$$

$$F_m(k^2) = -\frac{\alpha}{2\pi}\left(\frac{k^2}{4m^2}\right)^3 \int_0^1 dv \, \frac{v^2(4-v^2-v^4)}{1+(k^2/4m^2)(1-v^2) - i\varepsilon}, \tag{43}$$

$$F_q(k^2) = \frac{\alpha}{12\pi} \int_0^1 dv \, \frac{1-v^2}{1+(k^2/4m^2)(1-v^2) - i\varepsilon}. \tag{44}$$

Let us define

$$J_r = \int_0^1 dv \, \frac{v^{2r}}{1+(k^2/4m^2)(1-v^2) - i\varepsilon}, \qquad r = 0, 1, 2, \dots . \tag{45}$$

which satisfies the recursion relation

$$J_{r+1} = \left(1 + \frac{4m^2}{k^2}\right) J_r - \frac{4m^2}{k^2} \frac{1}{2r+1}. \tag{46}$$

J_0 is easily evaluated to be

$$J_0 = \frac{2m^2}{k^2}\left(1 + \frac{4m^2}{k^2}\right)^{-1/2} \ln \frac{\left(1 + \frac{4m^2}{k^2}\right)^{1/2} + 1}{\left(1 + \frac{4m^2}{k^2}\right)^{1/2} - 1}. \tag{47}$$

Now using the recursion relation (46), one obtains

$$J_r = \left(1 + \frac{4m^2}{k^2}\right)^r J_0 - \frac{4m^2}{k^2} \sum_{l=0}^{r-1} \frac{\left(1 + \frac{4m^2}{k^2}\right)^l}{2(r-l)-1}. \tag{48}$$

According to Eqs (45)–(48), one can obtain the following explicit expressions for the form factors (except for part of $F_e(k^2)$):

$$F_e(k^2) = 1 - \frac{\alpha}{2\pi}\left(\frac{k^2}{4m^2}\right)^3 \left\{ \int_0^1 dv \, \frac{(1+v^2)(1-v^2)^2 \ln \dfrac{4m^2}{\mu^2} \dfrac{v^2}{1-v^2}}{1+(k^2/4m^2)(1-v^2)-i\varepsilon} \right.$$

$$\left. -\tfrac{2}{3}(J_3+10J_2-16J_1+3J_0) \right\}, \tag{49}$$

$$F_m(k^2) = 1 - \frac{\alpha}{2\pi}\left(\frac{k^2}{4m^2}\right)^3 (4J_1 - J_2 - J_3), \tag{50}$$

$$F_q(k^2) = \frac{\alpha}{12\pi}(1-J_0). \tag{51}$$

So far, we have been assuming, implicitly, that $-k^2 < 4m^2$. For $-k^2 > 4m^2$, the phase of the logarithm involved in $J_0(k^2)$ (and $J_r(k^2)$ in general) is appropriately chosen to give the imaginary part of the corresponding integral.

Using the fact that the behaviours of $J_0(k^2)$ in the neighbourhood of the origin and at infinity are given by

$$J_0(k^2) \underset{k^2 \to 0}{\sim} 1 - \frac{2}{3}\left(\frac{k^2}{4m^2}\right),$$

$$J_0(k^2) \underset{k^2 \to \infty}{\sim} \frac{2m^2}{k^2}\left(1 - \frac{2m^2}{k^2}\right) \ln \frac{k^2}{m^2}, \tag{52}$$

one deduces that

$$F_q \underset{k^2 \to 0}{\sim} \frac{\alpha}{18\pi}\left(1 - \frac{1}{10}\frac{k^2}{m^2}\right),$$

$$F_q \underset{k^2 \to \infty}{\sim} \frac{\alpha}{3\pi}\frac{m^2}{k^2}\left[1 - \frac{2m^2}{k^2} \ln \frac{k^2}{m^2}\right].$$

Hence, there is an induced quadrupole moment, $F_q(k^2 = 0) = \alpha/18\pi$, and its dynamical origin is evidenced here in the vanishing of the quadrupole moment form factor, $F_q(k^2)$, for $k^2 \to \infty$. For comparison, one notes that the k^2 fall-off of $F_q(k^2)$ is faster than k^2 fall-off of the electron magnetic moment form factor, which is

$$\frac{\alpha}{\pi}\frac{m^2}{k^2} \ln \frac{k^2}{m^2}.$$

The situation seems different for the charge and magnetic form factors, where

$$F_{\mathrm{m}}(k^2) \underset{k^2\to 0}{\sim} 1+\frac{\alpha}{12\pi}\left(\frac{k^2}{m^2}\right),$$

$$F'_{\mathrm{m}}(k^2) \underset{k^2\to \infty}{\sim} -\frac{\alpha}{18\pi}\left(\frac{k^2}{2m^2}\right)^2 \ln\frac{k^2}{m^2},$$

$$k^2) \underset{k^2\to \infty}{\sim} -\frac{\alpha}{3\pi}\left(\frac{k^2}{4m^2}\right)^2 \ln\frac{k^2}{m^2}.$$

I wish to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, where part of this work was done. I also wish to thank Professor Fonda for reading the manuscript. I am grateful to the National Council for Scientific Research of Lebanon for support.

APPENDIX A

Charge form factor[1]

The expression of $A_{\mathrm{v}}^{(\mathrm{e})}$ as obtained in (23) is gauge invariant under the causal condition being considered, since

$$\partial_a j^a(x) = 0 \quad \text{as} \quad Q(p_1-p_1') = 0. \tag{A1}$$

Being rooted in the kinematics of free particles, this property will not be maintained after the space-time extrapolation is performed. Accordingly, we must rewrite (23) in a way that is without consequence for the causal situation but ensures its gauge invariance in general. Returning to the momentum space, one observes that ($n \geqslant 1$)

$$\frac{1}{M^2}(Q_\mu Q_\nu - Q^2 g_{\mu\nu})\left(-\frac{Q^2}{M^2}\right)^{n-1} A^\nu = A_\mu + Q_\mu \frac{1}{M^2}\left(-\frac{Q^2}{M^2}\right)^{n-1} Q_\nu A^\nu \tag{A2}$$

differs only by a gauge transformation from $A_\mu(Q)$, and can replace it in (23). The substitution is:

$$A_\mu \to \frac{1}{M^{2n}}\partial^\nu(\partial^2)^{n-1}F_{\mu\nu}, \tag{A3}$$

and the resulting space-time extrapolation of (23) is:

$$-ie^2\int_{4m^2}^{\infty}\frac{dM^2}{M^{2n}}f_{\mathrm{c}}(M^2)\int(dx)\,(dx')j_\mu(x)\Delta_+(x-x',M^2)\partial'_\sigma(\partial'^2)^{n-1}F^{\nu\sigma}(x'), \tag{A4}$$

---

[1]Here we follow very closely Schwinger's discussion (Ref. [1]) relative to the spin-0 and $-\frac{1}{2}$ analogues of the present calculation.

with

$$\Delta_+(x-x', M^2)' = \int \frac{(dk)}{(2\pi)^4} \frac{\exp[ik(x-x')]}{k^2+M^2-i\varepsilon}. \tag{A5}$$

The primitive interaction and the above interaction induced modification leads to the action term

$$\int (dx)(dx')j^\mu(x)L_{\nu\mu}(x-x')A^\nu(x'), \tag{A6}$$

with

$$L_{\mu\nu}(x-x') = g_{\mu\nu}\delta(x-x')$$

$$-(\partial_\mu\partial_\nu - g_{\mu\nu}\partial^2)(\partial^2)^{n-1}e^2 \int \frac{dM^2}{M^{2n}} f_e(M^2)\Delta_+(x-x', M^2), \tag{A7}$$

and the four-dimensional momentum space equivalent expression is

$$L_{\mu\nu}(k) = g_{\mu\nu} + (k_\mu k_\nu - g_{\mu\nu}k^2)(-k^2)^{n-1} \times e^2 \int \frac{dM^2}{M^{2n}} f_e(M^2) \frac{1}{k^2+M^2-i\varepsilon}. \tag{A8}$$

One immediately deduces that, in the Lorentz gauge, the above tensor reduces to

$$L_{\mu\nu}(k) = g_{\mu\nu}F_e(k^2), \tag{A9}$$

where we recognize the appearance of the scalar charge form factor $F_e(k^2)$ as obtained in (26)–(27).

APPENDIX B

Kinematics of the e^3 contribution to the spin-1 electromagnetic vertex function

$x = M^2/4m^2$
θ = scattering angle in the centre-of-mass frame

Notations

$\bar{g}_{\mu\nu}(p_2) \equiv \bar{g}_{\mu\nu}, \quad \bar{g}_{\mu\nu}(p_2') \equiv \bar{g}'_{\mu\nu}.$

Let p and q be any 4-vectors; we define

$p_\mu q_\nu \bar{g}^{\mu\nu} \equiv (pq\bar{g}), \quad p_\mu q_\nu \bar{g}'^{\mu\nu} \equiv (pq\bar{g}'),$

$\bar{g}_{\mu\nu}\bar{g}'^{\mu\nu} \equiv (\bar{g}\cdot\bar{g}'), \quad p_\mu \bar{g}^{\mu\nu} q_\lambda \bar{g}'_{\lambda\nu} \equiv (p\bar{g})\cdot(q\bar{g}'),$

$p_\mu \bar{g}^{\mu\nu} q^\lambda \bar{g}_{\lambda\nu} \equiv (p\bar{g})\cdot(q\bar{g}), \quad p_\mu \bar{g}'^{\mu\nu} q^\lambda \bar{g}'_{\lambda\nu} \equiv (p\bar{g}')\cdot(q\bar{g}').$

$$p_1p_1' = p_2p_2' = -m^2(2x-1),$$
$$p_1p_2 = m^2[(x-1)\cos\theta - x],$$
$$p_1p_2' = p_1'p_2 = -m^2[(x-1)\cos\theta + x],$$
$$Qp_1 = Qp_1' = Qp_2 = Qp_2' = -2m^2x,$$
$$(p_1-p_2)^2 = 2m^2(x-1)(1-\cos\theta),$$
$$(p_1-p_1')^2 = 4m^2(x-1),$$
$$p_2(p_1-p_1') = -p_2'(p_1-p_1') = 2m^2(x-1)\cos\theta,$$
$$p_1(p_1-p_1') = -p_1'(p_1-p_1') = 2m^2(x-1),$$
$$p_1(p_1-p_2) = p_1'(p_1'-p_2') = m^2(x-1)(1-\cos\theta),$$
$$p_1(p_1+p_2) = p_2(p_1+p_2) = m^2[(x-1)\cos\theta+3x-1],$$
$$(\bar{g}\cdot\bar{g}') = 2+(2x-1)^2,$$
$$(p_1p_2'\bar{g}) = (p_1'p_2\bar{g}') = 2m^2x(x-1)(1-\cos\theta),$$
$$(p_1p_2\bar{g}') = (p_1'p_2'\bar{g}) = 2m^2x(x-1)(1+\cos\theta),$$
$$(p_1p_1\bar{g}) = (p_1'p_1'\bar{g}') = -m^2(x-1)(1-\cos\theta)[(x-1)\cos\theta-(x+1)],$$
$$(p_1p_1\bar{g}') = (p_1'p_1'\bar{g}) = m^2(x-1)(1-\cos\theta)[(x-1)\cos\theta+(x+1)],$$
$$(p_1p_1'\bar{g}) = (p_1p_1'\bar{g}') = m^2(x-1)^2(1-\cos^2\theta),$$

$$[p_1'(p_1+p_2)\bar{g}'] = [p_1(p_1'+p_2')\bar{g}] = m^2(x-1)(1-\cos\theta)[(x-1)\cos\theta+3x-1],$$
$$[p_2(p_1+p_2)\bar{g}'] = [p_2'(p_1'+p_2')\bar{g}] = 2m^2x(x-1)(3+\cos\theta),$$
$$[p_2(p_1-p_1')\bar{g}'] = -[p_2'(p_1-p_1')\bar{g}] = 4m^2x(x-1)\cos\theta,$$
$$[p_1(p_1+p_2)\bar{g}'] = [p_1'(p_1'+p_2')\bar{g}] = m^2(x-1)(1+\cos\theta)[(x-1)\cos\theta+3x-1],$$
$$[p_1(p_1-p_1')\bar{g}] = -[p_1'(p_1-p_1')\bar{g}'] = -2m^2(x-1)(1-\cos\theta)[(x-1)\cos\theta-1],$$
$$[p_1'(p_1-p_1')\bar{g}] = -[p_1(p_1-p_1')\bar{g}'] = -2m^2(x-1)(1+\cos\theta)[(x-1)\cos\theta+1],$$

$$[(p_1+p_2)(p_1-p_1')\bar{g}'] = -[(p_1'+p_2')(p_1-p_1')\bar{g}] = 2m^2(x-1)[(x-1)\cos^2\theta+3x\cos\theta+1],$$
$$[(p_1-p_1')(p_1-p_1')\bar{g}] = [(p_1-p_1')(p_1-p_1')\bar{g}'] = 4m^2(x-1)[(x-1)\cos^2\theta+1],$$

$$(p_1\bar{g})\cdot(p_1\bar{g}) = (p_1p_1\bar{g}),$$
$$(p_1'\bar{g})\cdot(p_1\bar{g}) = (p_1\bar{g}')\cdot(p_1'\bar{g}') = (p_1p_1'\bar{g}),$$
$$(p_1'\bar{g})\cdot(p_1'\bar{g}) = (p_1\bar{g}')\cdot(p_1\bar{g}') = (p_1'p_1'\bar{g}),$$
$$(p_1\bar{g}')\cdot(p_1'\bar{g}) = -m^2(x-1)(1+\cos\theta)[(x-1)(2x+1)\cos\theta+2x^2-x+1],$$

$$(p_1\bar{g})\cdot(p_1'\bar{g}') = m^2(x-1)(1-\cos\theta)\,[(x-1)(2x+1)\cos\theta-2x^2+x-1],$$

$$(p_1\bar{g})\cdot(p_1\bar{g}') = -m^2(x-1)^2(2x+1)(1-\cos^2\theta),$$

$$(p_1\bar{g})\cdot[(p_1-p_1')\bar{g}] = -2m^2(x-1)(1-\cos\theta)\,[(x-1)\cos\theta-1],$$

$$(p_1'\bar{g})\cdot[(p_1-p_1')\bar{g}] = -2m^2(x-1)(1+\cos\theta)\,[(x-1)\cos\theta+1],$$

$$(p_1\bar{g}')\cdot[(p_1-p_1')\bar{g}] = 2m^2(x-1)(1+\cos\theta)\,[(x-1)(2x+1)\cos\theta+1],$$

$$(p_1'\bar{g}')\cdot[(p_1-p_1')\bar{g}] = 2m^2(x-1)(1-\cos\theta)\,[(x-1)(2x+1)\cos\theta-1],$$

$$[(p_1-p_1')\bar{g}]\cdot[(p_1-p_1')\bar{g}'] = 4m^2(x-1)\,[(x-1)(2x+1)\cos^2\theta+1],$$

$$[(p_1-p_1')\bar{g}]\cdot[(p_1-p_1')\bar{g}] = 4m^2(x-1)\,[(x-1)\cos^2\theta+1],$$

$$(p_2\bar{g}')\cdot(p_2'\bar{g}) = -4m^2x(x-1)(2x-1),$$

$$(p_2\bar{g}')\cdot(p_1\bar{g}) = (p_2'\bar{g})\cdot(p_1'\bar{g}') = -2m^2x(x-1)(2x-1)(1-\cos\theta),$$

$$(p_2\bar{g}')\cdot(p_1'\bar{g}) = (p_2'\bar{g})\cdot(p_1\bar{g}') = -2m^2x(x-1)(2x-1)(1+\cos\theta),$$

$$(p_2'\bar{g})\cdot(p_1'\bar{g}) = (p_2\bar{g}')\cdot(p_1\bar{g}') = 2m^2x(x-1)(1+\cos\theta),$$

$$(p_2\bar{g}')\cdot(p_1'\bar{g}') = (p_2'\bar{g})\cdot(p_1\bar{g}) = 2m^2x(x-1)(1-\cos\theta),$$

$$(p_2\bar{g})\cdot(p_1\bar{g}) = (p_2\bar{g}')\cdot(p_1'\bar{g}') = (p_1p_2'\bar{g}),$$

$$(p_2\bar{g}')\cdot[(p_1-p_1')\bar{g}] = -(p_2'\bar{g})\cdot[(p_1-p_1')\bar{g}'] = 4m^2x(x-1)(2x-1)\cos\theta,$$

$$(p_2\bar{g}')\cdot[(p_1-p_1')\bar{g}'] = -(p_2'\bar{g})\cdot[(p_1-p_1')\bar{g}] = 4m^2x(x-1)\cos\theta,$$

$$[(p_1+p_2)\bar{g}']\cdot[(p_1-p_1')\bar{g}] = 2m^2(x-1)\,[(x-1)(2x+1)\cos^2\theta+3x(2x-1)\cos\theta+1],$$

$$(p_2'\bar{g})\cdot[(p_1+p_2)\bar{g}'] = (p_2\bar{g}')\cdot[(p_1'+p_2')\bar{g}] = -2m^2x(x-1)(2x-1)(3+\cos\theta).$$

REFERENCES

[1] J. Schwinger, *Particles, Sources and Fields*, vols 1 and 2, Addison-Wesley, 1970 and 1973.

[2] A. J. Phares, *Phys. Rev. D* (1974).

[3] A. J. Phares, *Harvard thesis*, 1973 (unpublished).

[4] See, for example, Schwinger's model (*Phys. Rev.* **D7**, 908 (1973); **D8**, 960 (1973)) and Weinberg's model of leptons (*Phys. Rev. Lett.* **19**, 1264 (1967)). In these models the choice of the gauge group (U(2) and SU(2)×U(1), respectively) fixes the value $\kappa = 1$.

[5] T. Appelquist, private communication.