PARTIAL WAVE ANALYSIS OF THE THREE PARTICLE PROTON DISSOCIATION

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The Illinois partial wave analysis program has been modified to study the dissociation of the proton into proton and two mesons. The formalism used in this modification is described in detail and a geometrical interpretation of all spin rotations and couplings is given.

1. Introduction

In recent years a considerable interest was taken in the Illinois partial wave analysis (PWA) program designed by Ascoli (cf. Ascoli et al. 1970). This program was successfully used for the study of meson dissociation in many reactions, for instance $\pi^-p \to (\pi^-\pi^-\pi^+)p$ between 5 and 25 GeV/c (Ascoli et al. 1970, 1971, 1973a), the same reaction at 25 and 40 GeV/c (Antipov et al. 1973), $K^-p \to (K^-\pi^+\pi^-)p$ at 10 and 16 GeV/c (Deutschmann et al. 1974), $\pi^+p \to (\pi^+\pi^+\pi^-)p$ at 13 GeV/c (Thompson et al. 1974) and at 8, 16 and 23 GeV/c (Otter et al. 1974) and of $\pi^-p \to (\omega\pi^-)p$ at 4, 5, and 7.5 GeV/c (Chaloupka et al. 1974). Let us finally mention the paper by Ascoli et al. (1973b) where by analysing a Deck type model for which the partial waves are known, the reliability of the Illinois program results could be estimated.

The method used in the Illinois PWA program has been described and discussed by Brockway (1970), Bowler (1973) and by Hansen et al. (1974). The last of these papers is the most detailed one and discusses the assumptions made in the program.

While the mesonic diffractive dissociation has been extensively studied, as can be seen from the still incomplete above list of PWA papers, the properties of the nucleon diffractive dissociation are, to a large extent, still a mystery. Although the effective mass and momentum transfer distributions, as well as the decay branching ratios, have already been investigated (cf. Boesebeck et al. 1971), there are still many questions left unanswered. For instance, even the nature of the lowest mass enhancement at 1460 MeV is not yet understood. In this situation, a reliable partial wave analysis of the three particle nucleon dissociation would be very interesting.

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The present paper describes the formalism used in a modification of the Illinois PWA program made to study the proton dissociation into proton and two mesons. This modification was actually implemented and the modified program has been used in preliminary fits to the reaction $\pi^+ p \to \pi^+(p\pi^+\pi^-)$ at 16 and 23 GeV/c.

2. *Partial wave expansion in the three-particle subsystem*

The amplitude for the process

$$a+b \to (1+2+3)+4 \tag{2.1}$$

can be written in the form

$$f_{\lambda_a\lambda_3} = \langle \vec{p}_1\vec{p}_2\vec{p}_3\lambda_3; \vec{p}_4|U|\vec{p}_a\lambda_a, \vec{p}_b\rangle. \tag{2.2}$$

We assume that particles a and 3 are protons (or other spin 1/2 particles) and all other particles are spinless mesons[1]. By λ_a and λ_3 we denote helicities of the protons. Because eventually we sum over λ_a and λ_3, their exact definition is irrelevant.

In the following, an angular momentum expansion of amplitude (2.2) is used. As an example, we write down the expansion, called the 1–3 coupling, where the angular momentum of the (1, 3) subsystem is singled out

$$f_{\lambda_a\lambda_3} = \sum_{JM\nu} \sqrt{\frac{2J+1}{4\pi}}\, D^J_{M\nu}(\varphi, \vartheta, \gamma)^* \sum_{lLj} h^{JLlj}_{M\lambda_a}(s, t, W, s_1, s_2) G^{JLlj}_{\nu\lambda_3}(s_1, s_2, W) \tag{2.3}$$

with

$$G^{JL_2l_2j_2}_{\nu\lambda_3}(1, 3) = \sqrt{\frac{2}{(4\pi)^3}}\sqrt{(2l_2+1)(2L_2+1)} \sum_{\nu_2\lambda_2} d^{1/2}(-\alpha_2)_{\nu_2\lambda_3}$$

$$\times C(l_2, 0; \tfrac{1}{2}, \nu_2|j_2, \nu_2) d^{j_2}(\pi-\chi_2)_{\lambda_2\nu_2} C(L_2, 0; j_2, \lambda_2|J, \lambda_2) d^J(\vartheta_1-\pi)_{\nu\lambda_2}. \tag{2.4}$$

Here J is the total angular momentum of particles (1, 2, 3), M is its projection on the z-axis of an arbitrary reference frame (x, y, z) defined in the cms of particles (1, 2, 3) (and usually taken to be the Gottfried-Jackson frame or the s-channel helicity frame of the reaction $a+b \to (1, 2, 3)+4$), ν is the projection of J on $\vec{p}_3$ in the (1, 2, 3) cms, $l = l_2$ is the relative orbital momentum of particles 1 and 3, $j = j_2$ is the total angular momentum of particles (1, 3), and $L = L_2$ is the orbital angular momentum in the coupling of

[1] This formalism can be also applied to proton-proton collisions with unpolarized target and beam. In this case the index λ_a should be everywhere replaced by the set of indices $(\lambda_a, \lambda_b, \lambda_4)$ and summation over all of them performed in the density matrix formula. In fact, in both cases the summation over λ_a (or over $\lambda_a, \lambda_b, \lambda_4$) occurs only in the formula for the reduced density matrices (4.2). As the Illinois program fits just these density matrix elements rather than amplitudes, no changes in the program are necessary to study the proton-proton collisions instead of the pion-proton collisions. Only some properties of the density matrices may change, for example, more non-zero eigenvalues in the proton-proton case could be expected (cf. Ademollo et al. 1965).

j_2 and particle 2 giving J. Then s_1 and s_2 are squared effective masses of systems (2, 3) and (1, 3), and W is the effective mass of particles (1, 2, 3). Further, angles φ, ϑ, γ describe the orientation of the (1, 2, 3) plane in the (1, 2, 3) cms in the (xyz) frame. Namely, φ, ϑ are polar angles of $\vec{p}_3$ in (xyz) (i. e. ϑ is the angle between z and $\vec{p}_3$, and φ is the angle between the (z, x) plane and the (z, p_3) plane), and γ is the angle between the planes (p_1, p_3) and (z, p_3).

In formula (2.4) summation runs over ν_2 and λ_2 which are the projections of j_2 on the momentum of particle 3 in the (1,3) cms and on the reversed momentum of particle 2 in the same frame. The angle α_2 is between $-\vec{p}_{12}$ and $-\vec{p}_1$ in the particle 3 rest frame

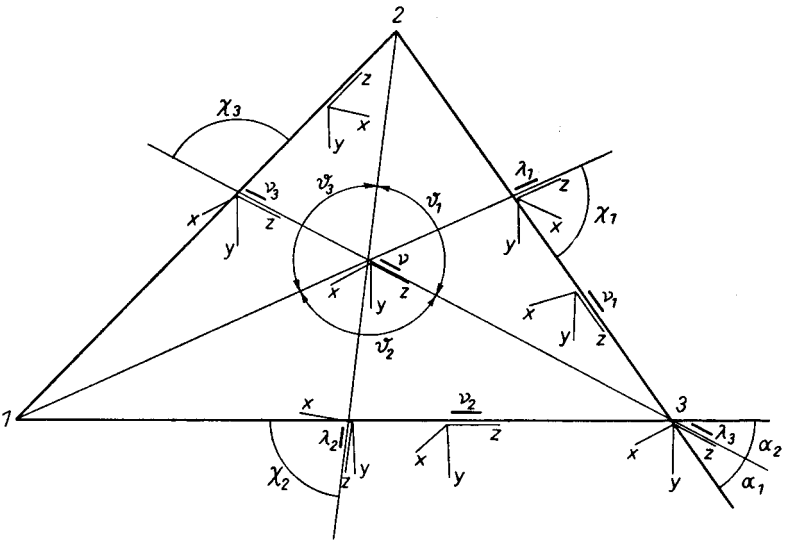

Fig. 1. Kinematics of the three-particle system. The reference frame in the (1, 2, 3) cms (centre of the figure) is the $(x'y'z')$ frame described in the text

and the angle χ_2 is between $-\vec{p}_2$ and $\vec{p}_1$ in the (1, 3) cms. Finally, ϑ_1 is the angle between $\vec{p}_2$ and $\vec{p}_3$ in the (1, 2, 3) cms. These angles and projections (except for φ, ϑ, γ) are shown in the velocity diagram of particles 1, 2 and 3 in Fig. 1 (cf. Wick 1962).

Formulae (2.3) and (2.4) have a simple geometric interpretation. In fact, the D function in Eq. (2.3) is characteristic for angular momentum expansion of the three-particle system (cf. e. g. Berman and Jacob 1965). It says that a rotation through the Euler angles $(\varphi, \vartheta, \gamma)$ is needed to pass from the (xyz) frame to the $(x'y'z')$ frame and that the projection of the total angular momentum J on the z-axis is M and that its projection on the z'-axis is ν. Both these frames are defined in the (1, 2, 3) cms. The (xyz) frame is external to the three particle system and usually chosen as either the Gottfried-Jackson or the s-channel helicity frame of the production reaction. The $(x'y'z')$ frame is attached to the three particle system. In our case it is defined as follows: the z'-axis is along $\vec{p}_3$ and the y'-axis is parallel to the cross product $\vec{p}_1 \times \vec{p}_2$.

The second sum in Eq. (2.3) is the coefficient of the partial wave expansion J, M, ν. This coefficient is developed in turn in terms of the functions G, giving the new coefficients h.

The role of the first expansion was to convert the angles φ, ϑ and γ into discrete variables J, M, ν. Similarly, the second expansion is done to convert the angles ϑ_1, χ_2 and α_2 into L_2, j_2 and l_2.

The geometrical interpretation of G is the following. We begin in the particle 3 (proton) rest system where we know λ_3, i. e. the spin projection of particle 3 on the z'-axis, or on $-\vec{p}_{12} = -\vec{p}_1 - \vec{p}_2$. We want to get eventually ν, i. e. the projection of J on z'. To do this, we rotate first λ_3 through angle $-\alpha_2$ (cf. Fig. 1) obtaining thus ν_2, the projection of the proton spin on $-\vec{p}_1$. This rotation is done around the y-axis which is normal to the (1, 2, 3) plane, and is expressed by the $d^{1/2}$ function in Eq. (2.4). The next step is the Lorentz transformation along p_1 to the (1, 3) cms. This transformation does not change ν_2 and therefore it is not visualized in Eq. (2.4). Then in the (1, 3) frame we perform the $L-S$ coupling of particles 1 and 3 to get their total angular momentum j_2 and relative orbital momentum l_2. The total spin of particles 1 and 3 must be 1/2 (because 1 is spinless and 3 has spin 1/2). The projection of l_2 on the relative momentum is always zero, so the projection of j_2 on the relative momentum must be equal to the projection of the total spin, i. e. ν_2. This fact is expressed by the first Clebsch-Gordan coefficient in Eq. (2.4). The next step, performed in the (1, 3) cms, is done to calculate the projection λ_2 of j_2 on $-\vec{p}_2$. We know now the projection of j_2 on $\vec{p}_3$ and the angle between $\vec{p}_3$ and $-\vec{p}_2$, denoted $\pi-\chi_2$ in Fig. 1. Therefore λ_2 is calculated by performing a rotation through $\pi-\chi_2$, as shown by the second d-function in Eq. (2.4). Then we make a Lorentz transformation along $\vec{p}_2$ into the (1, 2, 3) cms which does not affect λ_2. Once in (1, 2, 3) cms, we perform the second $L-S$ coupling to get the total angular momentum J of (1, 2, 3) and the orbital angular momentum L_2 by coupling j_2 and particle 2. This coupling is done with the second Clebsch-Gordan coefficient in Eq. (2.4). Finally, the last d-function in Eq. (2.4) is used to calculate ν from the known projection λ_2 of J on $-\vec{p}_2$.

The 2–3 coupling is performed in an analogous way. Formula (2.3) still holds (although l, L and j have then a different meaning) and Eq. (2.4) is replaced by

$$G_{\nu\lambda_3}^{JL_1l_1j_1}(2,3) = \sqrt{\frac{2}{(4\pi)^3}}\sqrt{(2l_1+1)(2L_1+1)}\sum_{\nu_1\lambda_1} d^{1/2}(\alpha_1)_{\nu_1\lambda_3}$$

$$\times C(l_1, 0; \tfrac{1}{2}, \nu_1|j_1, \nu_1)d^{j_1}(-\chi_1)_{\lambda_1\nu_1}C(L_1, 0; j_1, \lambda_1|J, \lambda_1)d^J(\pi-\vartheta_2)_{\nu\lambda_1}. \qquad (2.5)$$

This formula can be interpreted similarly as Eq. (2.4), step by step. Note that, similarly as in Eq. (2.4), J, j_1, ν, λ_3, ν_1, λ_1 are half-integers and L_1 and l_1 are integers.

The 1–2 coupling is slightly different from the preceding two, so we discuss it in detail. The formula reads

$$G_{\nu\lambda_3}^{JL_3S_3j_3}(1,2) = \sqrt{\frac{2}{(4\pi)^3}}\sqrt{(2j_3+1)(2L_3+1)}\sum_{\nu_3} d^{j_3}(\pi-\chi_3)_{\nu_3 0}$$

$$\times C(j_3, \nu_3; \tfrac{1}{2}, \lambda_3|S_3, \lambda_3+\nu_3)C(L_3, 0; S_3, \lambda_3+\nu_3|J, \nu). \qquad (2.6)$$

Here we begin by coupling particles 1 and 2 in the (1, 2) cms to get the total angular momentum j_3 of the system (1, 2). The coefficient of this coupling is equal to unity, as the coupled particles are spinless (then there is only one spin amplitude, equal to unity by normalization). The second step is done to obtain ν_3, the projection of j_3 on $\vec{p}_3$. We know already the projection of j_3 on $\vec{p}_2$ which is zero (because the total spin is zero and the projection of the orbital angular momentum is zero, too), so we rotate it through the angle $\pi - \chi_3$, the angle between $\vec{p}_2$ and $\vec{p}_3$. This is the first d-function in Eq. (2.6). The next step is the Lorentz transformation into the (1, 2, 3) cms and then the $L-S$ coupling of (1, 2) with particle 3 to give J. The system (1, 2) has total angular momentum j_3 with projection ν_3 on $\vec{p}_3$ and couples with particle 3 (proton with spin projection λ_3) to give the total spin S_3 and the orbital angular momentum L_3. The coupling is performed with two Clebsch-Gordan coefficients in Eq. (2.6).

Note that the set L_2, l_2, j_2 or L_1, l_1, j_1 has been replaced now by L_3, S_3, j_3 and that Eq. (2.3) still holds after this substitution is made. Now L_3, j_3 and ν_3 are integers and all other discrete variables in Eq. (2.6) are half-integers.

To summarize, Eq. (2.3) represents a general three-particle partial-wave expansion of amplitude (2.2) and the subsequent expansion in a two-particle subsystem. The former expansion is made in terms of the D-functions and the latter in terms of functions in Eqs (2.4–2.6).

3. Basis with positive angular momentum projections

Instead of the ordinary spin basis $|J, M\rangle$ with $M = -J, \ldots, J$ it is more convenient to use here an equivalent basis

$$|JM\eta\rangle = \frac{1}{\sqrt{2}}(|J, M\rangle + \eta\varepsilon(-1)^M|J, -M\rangle) \tag{3.1}$$

where $M > 0$, and $\eta = \pm 1$ is the eigenvalue of the reflection operator in the $x-z$ plane (see Appendix B). Further

$$\varepsilon = P(-1)^{J+1/2} \tag{3.2}$$

and $(-1)^M$ is a shorthand for $\exp(iM\pi)$ as M is here half-integer. The reason for adopting the new basis (3.1) is a simplification of the density matrix ϱ which in the ordinary basis J, M has many interdependent elements. In the $|J, M, \eta\rangle$ basis the ϱ matrix splits into two parts, corresponding to $\eta = \pm 1$

$$\varrho^{J_1J_2}_{M_1M_2\eta} = \varrho^{J_1J_2}_{M_1M_2} + \eta\varepsilon_2(-1)^{-M_2}\varrho^{J_1J_2}_{M_1-M_2}, \tag{3.3}$$

with $M_1, M_2 > 0$ and $\varepsilon_2 = P_2\ (-1)^{J_2}$.

Note that in the case of spinless meson dissociation it is possible to prove that η is related to the exchanged naturality. In the proton dissociation, however, there is no relation between η and the exchanged naturality. For instance, in the reaction $\pi p \to \pi(p\pi\pi)$ only the natural parity may be exchanged and in the process $pp \to p(p\pi\pi)$ it is impossible to separate exchanged naturalities unless either the beam or the target is polarized.

Expansion (2.3) may be easily written in the basis (3.1) if at the same time we transform the coefficients h as follows

$$h_M = \frac{1}{\sqrt{2}}(h_M^{\eta=+1} + h_M^{\eta=-1}) \tag{3.4}$$

and

$$h_{-M} = \frac{1}{\sqrt{2}}\varepsilon(-1)^{-M}(h_M^{\eta=+1} - h_M^{\eta=-1}) \quad \text{for } M > 0. \tag{3.5}$$

Before we rewrite expansion (2.3) let us remark that the amplitude (2.3) will always enter the cross section formula. This means that we are not interested in the amplitude itself, but rather in its bilinear combination summed over $\lambda_3 = \pm 1/2$ and $\lambda_a = \pm 1/2$. Therefore nothing essential changes if we perform a unitary transformation on the index λ_3. Thus we may replace everywhere $G_{\nu\lambda_3}$ by

$$G_{\nu\lambda=\pm 1} = \frac{1}{\sqrt{2}}(G_{\nu\lambda_3=\frac{1}{2}} \pm iG_{\nu\lambda_3=-\frac{1}{2}}), \tag{3.6}$$

introducing, for convenience, an integer index λ. It is easy to check directly from Eqs (2.4–2.6) that

$$G_{-\nu\lambda_3} = \varepsilon(-1)^{\nu+\lambda_3}G_{\nu-\lambda_3} \tag{3.7}$$

which after transformation (3.6) becomes

$$G_{-\nu\lambda} = \lambda\varepsilon(-1)^{\nu}G_{\nu\lambda}. \tag{3.8}$$

Now, using the above equations we can rewrite Eq. (2.3) in the form

$$\begin{aligned} f_{\lambda_a\lambda_3} = \sum_{J,\eta,\nu>0,M>0} 2\sqrt{2J+1}\,[&(\cos M\varphi \cos \nu\gamma d^{J\lambda}_{M\nu}(\vartheta) - \sin M\varphi \sin \nu\gamma d^{J-\lambda}_{M\nu}(\vartheta)) \\ &\times \sum_{Llj} h^{JLlj\eta=\lambda}_{M\lambda_a}(s, t, W, s_1, s_2)G^{JLlj}_{\nu\lambda}(s_1, s_2, W) \\ &- i(\sin M\varphi \cos \nu\gamma d^{J\lambda}_{M\nu}(\vartheta) + \cos M\varphi \sin \nu\gamma d^{J-\lambda}_{M\nu}(\vartheta)) \\ &\times \sum_{Llj} h^{JLlj\eta=-\lambda}_{M\lambda_a}(s, t, W, s_1, s_2)G^{JLlj}_{\nu\lambda}(s_1, s_2, W)], \end{aligned} \tag{3.9}$$

where we have introduced

$$d^{J\pm 1}_{a\nu}(\vartheta) = \tfrac{1}{2}(d^{J}_{a\nu}(\vartheta) \pm \varepsilon(-1)^{\nu}d^{J}_{a-\nu}(\vartheta)) \quad \text{for } \nu > 0. \tag{3.10}$$

Note that the functions defined by Eq. (3.10) are complex as $(-1)^{\nu} = \exp(i\pi\nu)$ with half-integer ν. Formula (3.9), although longer than Eq. (2.3), is much more convenient for practical calculations.

4. *Density matrices and likelihood function*

We discuss now the structure of the partial waves $h^{JLlj\eta}_{M\lambda_a}(s, t, W, s_1, s_2)$ defined in Eq. (2.3). In the Illinois partial wave program (for meson dissociation) one assumes that the function h can be factorized (cf. Hansen et al. 1974). A similar assumption can be also made for the baryon dissociation, namely

$$h^{JLlj\eta n}_{M\lambda_a}(s, t, W, s_n) = C^{JLlj n}(s, t, W) T^{JP\eta}_{M\lambda_a}(s, t, W) B^{jlLn}(W, s_n) \tag{4.1}$$

for any coupling: 1–3 ($n = 2$), 2–3 ($n = 1$) or 1–2 ($n = 3$).

The first factor C in Eq. (4.1) is assumed not to depend on λ_a, M or η and is determined by fitting the data. Its modulus squared $|C|^2$ can be interpreted as the branching ratio in the decay of the three particle system of definite spin J and parity $P = (-1)^{L+1}$ into the states of different L, l, j or n. The bilinear combination of T forms the reduced density matrix (the full density matrix is the bilinear combination of entire h)

$$\varrho^{J_1 J_2}_{M_1 M_2 \eta}(s, t, W) = \sum_{\lambda_a} T^{J_1 P_1 \eta}_{M_1 \lambda_a}(s, t, W) T^{J_2 P_2 \eta}_{M_2 \lambda_a}(s, t, W)^* \tag{4.2}$$

which is fitted by the PWA program[2]. This is the density matrix of the state J^P, describing its possible interferences with other J^P states.

The last factor B in Eq. (4.1) represents a parametrisation of the dependence of the amplitude on the Dalitz plot variables s_n, based on the two-step decay picture. Thus B is usually taken as the Breit-Wigner function centred at a mass m_0, with appropriate centrifugal barrier factors

$$B^{jlLn}(W, s_n) = \frac{p^L(W, s_n) q^l(s_n)}{s_n - m_0^2 - i m_0 \Gamma(s_n)} \tag{4.3}$$

or a phase-shift parametrisation of the two-particle subsystem

$$B^{jlLn}(W, s_n) = \frac{1}{2i} p^L(W, s_n) \left(\eta(s_n) e^{2i\delta(s_n)} - 1\right). \tag{4.4}$$

An important assumption made in the parametrisation (4.1) is that the whole dependence of the amplitude h on the Dalitz-plot variables s_1, s_2 is carried by B. In other words, neither the coefficients C nor the reduced density matrix ϱ depend on s_1, s_2. This assumption could be verified by modifying the Illinois PWA program to get coefficients C and density matrices ϱ separately for various regions of the Dalitz plot. Such a check, however, seems to be difficult at typical statistics.

The remarkable fact is that h does not depend on λ_3, the helicity of the final proton. In fact, from Eqs (2.3–2.4) we see that the function G "converts" λ_3 into ν which in turn is "converted" by the D-function in Eq. (2.3) into M. Therefore the coefficients h in the partial wave expansion (2.3) depend only on M.

We have reached the point where all the symbols in Eq. (2.3) (or better, in Eq. (3.9)) have been explained. Now all the functions are known, except for C in Eq. (4.1) and ϱ

[2] Actually the Illinois program fits the complex conjugate of ϱ.

in Eq. (4.2) which are fitted by the program. The fitting is done by maximizing the relevant part of the likelihood function (cf. Hansen et al. 1974)

$$\mathscr{L} = \sum_{i=1}^{N\text{evts}} \ln \sigma(\tau_i) - N_{\text{evts}} \int \sigma(\tau) A(\tau) d\tau, \tag{4.5}$$

where the summation runs over N_{evts} available experimental events, τ is the 7-dimensional phase space (t, W, s_1, s_2, φ, ϑ, γ), $A(\tau)$ is the acceptance function for the experiment and finally

$$\sigma(\tau) = \sum_{\lambda_a \lambda_3} |f_{\lambda_a \lambda_3}|^2 \tag{4.6}$$

is the cross section calculated from the amplitude (3.9) or (2.3).

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APPENDIX A

We present here formulae for momenta and angles in the three particle system, as shown in Fig. 1. In the following (i, j, k) is any cyclic permutation of the numbers (1, 2, 3). All the quantites are expressed by the squared effective masses of two-particle subsystems

$$s_i = (p_j + p_k)^2. \tag{A1}$$

First of all, the effective mass W of all three particles is

$$W^2 = s_1 + s_2 + s_3 - m_1^2 - m_2^2 - m_3^2. \tag{A2}$$

The momenta of particles in the (1, 2, 3) cms p_i are

$$p_i = \lambda(W^2, s_i, m_i^2)/(2W) \tag{A3}$$

with

$$\lambda(a, b, c) = (a + b - c)^2 - 4ab. \tag{A4}$$

The momentum q_i of particle j in the (j, k) cms is

$$q_i = \lambda(s_i, m_j^2, m_k^2)/(2\sqrt{s_i}). \tag{A5}$$

Now all the angles drawn in Fig. 1 can be expressed in terms of quantities (A1 – A5). We begin with the angles ϑ and χ. These angles are by definition positive and not greater

than π. Therefore to determine them it is enough to use the formulae for cosines

$$\cos \vartheta_i = (p_i^2 - p_j^2 - p_k^2)/(2p_j p_k), \tag{A6}$$

$$\cos \chi_i = \frac{\sqrt{s_i}}{4W p_i q_i} (s_k - s_j - (m_j^2 - m_k^2)(W^2 - m_i^2)/s_i). \tag{A7}$$

The most complicated are formulae for $\cos \alpha_1$ and $\cos \alpha_2$. The sines of these angles are also positive. The formula for $\cos \alpha_1$ is

$$\cos \alpha_1 = \frac{m_3^2}{4W p_3 q_1 \sqrt{s_1}} [(W^2 - s_3 - m_3^2)(s_1 - m_2^2 - m_3^2)/m_3^2 - 2(s_3 + m_2^2 - m_1^2)], \tag{A8}$$

and the formula for $\cos \alpha_2$ is obtained from Eq. (A8) by interchanging everywhere indices 1 and 2.

APPENDIX B

We quote here the properties of the eigenstates of the reflection operator Y in the $x-z$ plane (cf. Bohr 1959)

$$Y = e^{-i\pi J_y} \Pi, \tag{B1}$$

where J_y is the generator of the rotations around the y-axis and Π is the parity operator. When acting on helicity states $|J, M\rangle$ the Y operator gives (cf. Jacob and Wick 1959)

$$Y|J, M\rangle = P(-1)^{J-M}|J, -M\rangle, \tag{B2}$$

where P is the eigenvalue of the parity operator Π. We define the eigenstates of Y as

$$|J, M, \eta\rangle = \frac{1}{\sqrt{2}} (|J, M\rangle + \eta\varepsilon(-1)^M |J, -M\rangle), \quad M > 0, \tag{B3}$$

with $\varepsilon = P(-1)^{J+\frac{1}{2}}$ which may be interpreted as the naturality of the produced state, and $\eta = \pm 1$. Because in our case both J and M are half-integer, note that $(-1)^M$ is a shorthand for $\exp(i\pi M)$ and should be carefully distinguished from $(-1)^{-M} = -(-1)^M$. Then it is easy to check that

$$Y|J, M, \eta\rangle = i\eta|J, M, \eta\rangle. \tag{B4}$$

The density matrix is diagonal in η, i. e. it decomposes into two disjoint matrices. This reduces the number of relations between nonzero elements of the matrix and is, in fact, the main reason of our using the eigenstates of Y. The density matrix with definite η may be expressed in terms of ordinary helicity density matrix elements

$$\varrho^{J_1 J_2}_{M_1 M_2 \eta} = \varrho^{J_1 J_2}_{M_1 M_2} + \eta\varepsilon_2(-1)^{-M_2} \varrho^{J_1 J_2}_{M_1 -M_2} \tag{B5}$$

where $\varepsilon_2 = P_2(-1)^{J_2+\frac{1}{2}}$ and $M_1, M_2 > 0$.

We give here two examples of relation (B5).

a. $J_1 = J_2 = 1/2, P_2 = P$

The ordinary helicity matrix is

$$\begin{pmatrix} \varrho_{11} & i\varrho_{1-1} \\ -i\varrho_{1-1} & \varrho_{11} \end{pmatrix} \tag{B6}$$

and is the Y-representation it decomposes into two 1×1 matrices

$$\varrho_{11} - \varrho_{1-1} \quad \text{for} \quad \eta P = 1 \tag{B7}$$

and

$$\varrho_{11} + \varrho_{1-1} \quad \text{for} \quad \eta P = -1. \tag{B8}$$

b. $J_1 = J_2 = 3/2, P_2 = P$

The helicity density matrix is usually written in the form (cf. Gottfried and Jackson 1964)

$$\begin{pmatrix} \varrho_{33} & \varrho_{31} & \varrho_{3-1} & i\varrho_{3-3} \\ \varrho_{31}^* & \varrho_{11} & i\varrho_{1-1} & \varrho_{3-1}^* \\ \varrho_{3-1}^* & -i\varrho_{1-1} & \varrho_{11} & -\varrho_{31}^* \\ -i\varrho_{3-3} & \varrho_{3-1} & -\varrho_{31} & \varrho_{33} \end{pmatrix} \tag{B9}$$

and may be rewritten as the following two 2×2 matrices (for $\eta P = +1$ and for $\eta P = -1$)

$$\begin{pmatrix} \varrho_{33} + \eta P \varrho_{3-3}, & \varrho_{31} + i\eta P \varrho_{3-1} \\ \varrho_{31}^* - i\eta P \varrho_{3-1}^*, & \varrho_{11} - \eta P \varrho_{1-1} \end{pmatrix}. \tag{B10}$$

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