QUANTIZATION OF THE ELECTROMAGNETIC FIELD IN RIEMANNIAN SPACES

By D. Kramer

Sektion Physik der Friedrich-Schiller-Universität Jena*

(*Received September 6, 1974*)

The canonical quantization procedure for the free electromagnetic field is generalized on arbitrary space-times. The metric is considered as classical background field which influences the quantized Maxwell field. Using the Heisenberg picture we get a system of first-order linear ordinary differential equations describing the time dependence of the operators. In order to compute the coefficients of this system one has to solve a 3-dimensional eigenvalue problem. It is not necessary to choose a special gauge of the potentials.

1. *Introduction*

In simple cosmological models free matter fields were canonically quantized by several authors [1–7]. Space-time symmetries and the corresponding conservation laws, generators of unitary transformations in the Hilbert space, and, especially, *static* gravitational background fields have been treated earlier [8]. In the present paper we consider the quantized electromagnetic field in an arbitrary unquantized gravitational background. To this end we give a 3-covariant formulation of Maxwell's theory [9]. Space-like hypersurfaces S are chosen and a 3-covariant (invariant) time derivative is defined. The interior metric of S and suitable boundary conditions determine the eigenfunctions (modes) of the electromagnetic field.

Comparing the quantized system with respect to distinct hypersurfaces one has to take into account essentially two effects induced by the generally time-dependent external metric field: the modification of the modes and the possible quantum transitions from one mode to the other.

The expansion of the field operators in terms of an oithonormal system which is assumed to fulfil the completeness relation leads to a quantum mechanical problem with a quadratic Hamiltonian [2]. We postulate commutation rules consistent with the equations of motion and try to define creation and annihilation operators.

* Address: Sektion Physik, Fiedrich-Schiller-Universität, Jena, Max-Wien-Platz 1, DDR.

2. Decomposition formalism

We select a family of non-intersecting space-like hypersurfaces S which have the normal unit vector n_i. With the aid of n_i and the projection tensor defined by

$$h_{ij} = g_{ij}+n_i n_j, \qquad n_i n^i = -1,$$

we decompose tensors in a generally covariant fashion, e. g.,

$$V = n_i V^i, \qquad \hat{V}_i = h_i{}^j V_j.$$

The transformation properties of a geometrical object are to be retained under an invariant derivative. For this reason we take the Lie derivative (with respect to n_i),

$$W_i = \pounds \hat{V}_i,$$

for it is well known that W_i transforms like $\hat{V}_i$. We adapt the coordinate system to the hypersurfaces,

$$S : x^4 = \text{const.} \qquad n_i = n\delta_i^4, \qquad n = \frac{1}{\sqrt{-g^{44}}}. \tag{2.1}$$

Then it is still allowed to change the spatial coordinates x^a ($a = 1$–3) separately on each hypersurface as well as the parametrization of the hypersurfaces. The admissible coordinate transformations are

$$x^{a'} = x^{a'}(x^b, x^4), \qquad x^{4'} = x^{4'}(x^4). \tag{2.2}$$

All equations should be covariant under this restricted transformation group (3-covariance). We consider *global* space-like hypersurfaces. Therefore, the group (2.2) should not be confused with the transformations which leave the frame of reference fixed and give rise to a local decomposition [10].

In the coordinate system (2.1) the metric tensor may be split as follows

$$g_{ij} = \begin{pmatrix} g_{ab} & N_a \\ N_b & -n^2+N_a N^a \end{pmatrix}, \qquad g_{ab} = h_{ab}.$$

Obviously, the following statements are valid:

1. Covariant spatial tensor components transform like 3-tensors under (2.2),

$$V_{a'} = \frac{\partial x^a}{\partial x^{a'}} V_a \qquad \hat{V}_a = V_a.$$

The components $N_a = g_{4a}$ are no 3-tensors, of course.

2. Contravariant 4-components of a tensor, multiplied by n for each index, transform like invariants under (2.2),

$$V' = V, \quad V = nV^4.$$

3. The Lie derivative applied to 3-tensors generates again 3-tensors of the same rank.

Invariant time derivatives are denoted by[1]

$$\partial_4 V \equiv -\pounds V = n^{-1}(\dot{V} - V_{,a}N^a),$$

$$\partial_4 V_a \equiv -\pounds V_a = n^{-1}(\dot{V}_a - V_{a,b}N^b - V_b N^b_{,a}).$$

It can be easily verified that

$$K_{ab} = n\begin{Bmatrix} 4 \\ ab \end{Bmatrix} = \tfrac{1}{2}\,\partial_4 g_{ab}$$

is a 3-tensor of the second rank under (2.2).

Using these rules and their evident generalization on tensors and tensor densities of arbitrary rank one obtains the elements with the aid of which physical laws can be formulated by 3-covariant equations. We emphasize the convention that in all 3-covariant equations the metric operations (covariant derivatives denoted by semicolon, moving of indices) are understood with respect to the spatial part of the metric tensor, g_{ab}, and its inverse.

We use the notations

$$g \equiv \det(g_{ab}), \qquad {}^4g \equiv \det(g_{ij}) = -n^2 g,$$

$$\varepsilon_{abc} \equiv \sqrt{g}\,\delta_{abc}, \qquad \varepsilon^{abc} \equiv \frac{1}{\sqrt{g}}\,\delta_{abc}$$

(δ_{abc} — Levi-Civita permutation symbol). For later calculations the relation

$$\partial_4 \mathscr{V}^a_{\ ,a} = \frac{1}{n}(n\partial_4 \mathscr{V}^a)_{,a}, \qquad \mathscr{V}^a = \sqrt{g}\,V^a \tag{2.3}$$

will be important.

3. *Maxwell equations*

We apply the decomposition method to the electromagnetic field and write down the Maxwell equations and the local energy-momentum balance in a totally 3-covariant form. According to the rules mentioned above we get 3-tensors

$$A_a,\ A = nA^4,$$

$$B_{ab},\ E_a = nB^4_a,$$

$$T_{ab},\ T_a = nT^4_a, \qquad T = n^2 T^{44},$$

from the potential A_i, the electromagnetic field tensor B_{ij}, and the energy-momentum tensor T_{ij}, respectively. With the abbreviations for the magnetic and electric fields (tensor densities)

$$\mathscr{B}^a = \tfrac{1}{2}\sqrt{g}\,\varepsilon^{abc}B_{bc}, \qquad \mathscr{E}^a = \sqrt{g}\,E^a$$

[1] Point denotes *partial* time derivative, $\dot{V} \equiv \partial V/\partial x^4$.

we obtain as a result of the reduction the 3-covariant Maxwell equations

$$\text{a)}\ \partial_4 \mathscr{E}^a = \sqrt{g}\,\varepsilon^{abc}\frac{1}{n}(nB_c)_{,b}, \qquad \text{c)}\ E^a_{;a} = 0,$$

$$\text{b)}\ \partial_4 \mathscr{B}^a = -\sqrt{g}\,\varepsilon^{abc}\frac{1}{n}(nE_c)_{,b}, \qquad \text{d)}\ B^a_{;a} = 0. \tag{3.1}$$

The equations not containing time derivatives are the constraints which are to be fulfilled on S. The relation $T^{ij}_{;j} = 0$ is decomposed into equations representing the generalized energy and momentum balance, respectively,

$$\partial_4 \mathscr{T} + \frac{1}{n^2}(n^2 \mathscr{T}^a)_{,a} + \tfrac{1}{2}\mathscr{T}^{ab}\partial_4 g_{ab} = 0,$$

$$\partial_4 \mathscr{T}_a + \frac{1}{n}(n\mathscr{T}^b_a)_{;b} + \mathscr{T}\frac{1}{n}n_{,a} = 0, \tag{3.2}$$

where the components of the energy-momentum tensor are given by

$$T_{ab} = E_a E_b + B_a B_b - \tfrac{1}{2} g_{ab}(E_c E^c + B_c B^c),$$

$$T_a = -\varepsilon_{abc}E^b B^c,$$

$$T = -\tfrac{1}{2}(E_a E^a + B_a B^a),$$

$$\mathscr{T}_{ab} \equiv \sqrt{g}\,T_{ab}, \qquad \mathscr{T}_a \equiv \sqrt{g}\,T_a, \qquad \mathscr{T} \equiv \sqrt{g}\,T.$$

The vector potential A_a can be uniquely written as the sum of two parts,

$$A_a = C_a + a_{,a}.$$

The transverse part is assumed to satisfy the condition

$$\left(\frac{C^a}{n}\right)_{;a} = 0, \tag{3.3}$$

and the function a is determined by

$$a = \Delta^{-1}\left(\frac{A^a}{n}\right)_{;a},$$

where Δ^{-1} is inverse to the differential operator Δ acting on some function f according to

$$\Delta f = \left[\frac{1}{n}(nf)_{,a}\right]^{;a}.$$

Under the gauge transformations of potentials, the expression

$$G \equiv A + \partial_4 a$$

is an invariant. Consequently, the gauge invariant G is completely determined by C_a.

From the equation

$$E_a = -\partial_4 C_a - \frac{1}{n}(nG)_{,a} \tag{3.4}$$

we get in view of (2.3), (3.3), and (3.1c)

$$\Delta G = \frac{1}{\sqrt{g}}[C_a \partial_4(\sqrt{g}\, g^{ab})]_{,b} - C^a \partial_4 \left(\frac{n_{,a}}{n}\right). \tag{3.5}$$

The definition of C_a has been chosen so that the right-hand member of (3.5) and hence G itself do not involve any time derivative of C_a. Thus, because of (3.4) we have

$$\frac{\partial E_b}{\partial(\partial_4 C_a)} = -\delta^a_b.$$

We take over the usual definition of the canonical momentum,

$$\pi^a = \frac{\partial \mathscr{L}}{\partial(\partial_4 C_a)} = -\sqrt{g}\, E^a,$$

$$\mathscr{L} = \tfrac{1}{2}\sqrt{g}\,(E_a E^a - B_a B^a), \tag{3.6}$$

which corresponds to the transverse vector potential C_a representing the independent dynamical degrees of freedom. Quantizing the electromagnetic field in the Minkowski space one may eliminate the scalar and longitudinal modes by using the so called Coulomb gauge, $A = 0$, $a = 0$. This particular gauge is also attainable in static gravitational fields with hypersurfaces S orthogonal to the Killing vector. In this case, after fixing $A = 0$ one has still a gauge function available whose time derivative vanishes. Because of

$$\mathscr{E}_a = -\frac{1}{n}\dot{\mathscr{A}}_a, \quad \left(\frac{\mathscr{A}^a}{n}\right)^{\cdot}_{,a} = 0, \quad \mathscr{A}^a \equiv \sqrt{g}\, A^a,$$

this is sufficient to achieve also $a = 0$. As equation (3.5) shows, it is impossible in general to take the Coulomb gauge. We stress the gauge independence of our considerations; never a special gauge is necessary.

4. *Complete orthonormal system*

We start this section with the *eigenvalue equation*

$$D^a_b u_{ka} = \{n[(nu_{ka})_{,b} - (nu_{kb})_{,a}]\}^{;a} = m_k^2 u_{kb}. \tag{4.1}$$

The index k is no tensor index, it labels the various eigenfunctions (discrete or continuous spectrum). In general the eigenvalues m_k as well as the associated eigenvectors u_{ka} depend on time; on each hypersurface S we find other solutions of (4.1). This eigenvalue equation has the following remarkable properties and consequences:

1. Provided that the eigenvalues m_k transform like n under the restricted transformation (2.2), the 3-covariance of (4.1) is evident.

2. The eigenvalue equation (4.1) follows immediately from

$$m_k v_k{}^a = \varepsilon^{abc}(n u_{kc})_{;b},$$
$$m_h u_k{}^a = \varepsilon^{abc}(n v_{kc})_{;b}. \tag{4.2}$$

In virtue of the symmetry between u_{ka} and v_{ka} the eigenvectors v_{ka} defined by (4.2) solve (4.1) in the same way as the eigenvectors u_{ka}: At least two eigenvectors belong to each eigenvalue. To interpret this degeneracy we refer to the well-known fact that in flat space-time two polarization states belong to the same momentum.

3. From the eigenvalue equation (4.1) it follows

$$u^a_{k;a} = 0, \qquad v^a_{k;a} = 0. \tag{4.3}$$

4. The differential operator D^b_a in (4.1) is self-adjoint. Therefore we have real eigenvalues

$$m_k^* \equiv m_{-k} = m_k$$

($*$ denotes complex conjugation), and the eigenvectors corresponding to distinct eigenvalues are orthogonal to each other. In the degenerate case the usual orthogonalization procedures are to be applied. We appropriately normalize the eigenvectors,

$$\int \frac{u_k^{a*} u_{k'a}}{m_k} n \sqrt{g}\, d^3x = \delta_{kk'}. \tag{4.4}$$

Then, the same relation holds also for v_{ka} in place of u_{ka}.

5. The eigenvalue equation (4.1) is invariant under the substitution

$$g_{ab} = l^2 \tilde{g}_{ab}, \qquad g^{ab} = \frac{1}{l^2} \vec{g}_{ab}, \qquad n = l\tilde{n},$$

$$u_{ka} = \frac{1}{l} \tilde{u}_{ka}, \quad v_{ka} = \frac{1}{l} \tilde{v}_{ka}, \qquad m_k = \tilde{m}_k \tag{4.5}$$

which is a manifestation of the conformal invariance of the Maxwell equations.

Postulating some of these properties we derived the form of the eigenvalue equation.

We assume the completeness of the system of eigenvectors u_{ka} in the sense that all 3-tensors of the first rank with vanishing divergence can be uniquely constructed by superposition of the eigenvectors u_{ka}. This assumption allows one to have a generalized Fourier expansion of the field tensors $n^{-1} C_a$, E_a, and B_a which are divergence-free vectors,

$$C_a = \sum q_k \frac{n}{m_k} u_{ka}, \tag{4.6}$$

$$E_a = -\sum p_k u_{ka}, \qquad B_a = \sum q_k v_{ka}. \tag{4.7}$$

In these expansions the summation (or integration) includes all real values of k which label the independent solutions of eigenvalue equation (4.1). For complex eigenvectors we use the notation

$$u^*_{ka} \equiv u_{-ka}, \qquad v^*_{ka} \equiv v_{-ka}. \tag{4.8}$$

The coefficients p_k and q_k are only functions of time coordinate x^4. This is an invariant statement with respect to the admissible coordinate transformations (2.2)

5. *Commutation rules*

Up to this point we have dealt with the classical theory. In the quantum field theory the time-dependent coefficients p_k and q_k are operators satisfying the canonical commutation rules

$$[\overset{+}{p_k}, q_{k'}] = \frac{\hbar}{i}\,\delta_{kk'}, \qquad [p_k, p_{k'}] = 0 = [q_k, q_{k'}]. \tag{5.1}$$

The electromagnetic fields E_a and B_a are observables, but the operators p_k and q_k are no Hermitian operators. For complex eigenvectors a convention similar to (4.8) is useful,

$$\overset{+}{p_k} \equiv p_{-k}, \qquad \overset{+}{q_k} \equiv q_{-k}$$

(+denotes Hermitian conjugation). For the functions w_{ka} defined by

$$w_{ka} \equiv \sqrt[4]{g}\sqrt{\frac{n}{m_k}}\,u_{ka},$$

we postulate the completeness relation

$$\sum w^{a*}_k(x) w_{kb}(x') = \delta^a_b(x, x'), \tag{5.2}$$

where the two-point function on the right-hand side of this equation has on each hypersurfaces S the property

$$w^a_k(x) = \int \delta^a_b(x, x') w^b_k(x') d^3x' \tag{5.3}$$

for all objects of the same type as w_{ka}, i. e., with

$$\left(\frac{w^a_k}{\sqrt[4]{g}\sqrt{n}}\right)_{;a} = 0.$$

The relations (5.2) and (5.3) are compatible if we take into consideration the normalization condition (4.4). Expressed by the transverse vector potential C_a and the associated canonical momentum (3.6), the commutation rules (5.1) are completely equivalent to

$$[\pi^a(x), C_b(x')] = \frac{\hbar}{i}\,\delta^a_b(x, x'), \qquad [\pi^a(x), \pi^b(x')] = 0 = [C_a(x), C_b(x')].$$

This shows us that $\delta_b{}^a(x, x')$ is the direct generalization of the transverse delta function which is important in the conventional quantum theory of the electromagnetic field.

6. *Equations of motion*

Let us insert the expansions of the electromagnetic fields into the Maxwell equations. The constraints are automatically satisfied because of (4.3), and the other set of the Maxwell equations yields the equations of motion. By means of the completeness postulate (in the restricted sense) and the relation (2.3) the expansions

$$n\partial_4(u_k{}^a\sqrt{g}) = \sum_{k'} c_{k'k}u_{k'}{}^a\sqrt{g},$$

$$n\partial_4(v_k{}^a\sqrt{g}) = \sum_{k'} d_{k'k}v_{k'}{}^a\sqrt{g}$$

are possible. From the normalization condition (4.4) we obtain for the coefficients

$$c_{kk'} = \frac{1}{m_k}\int n^2 u^*_{ka}\partial_4(u_{k'}{}^a\sqrt{g})d^3x,$$

$$d_{kk'} = \frac{1}{m_k}\int n^2 v^*_{ka}\partial_4(v_{k'}{}^a\sqrt{g})d^3x. \tag{6.1}$$

It might be remarked that the relation

$$d_{kk'} = -c^*_{k'k}$$

holds. To prove it we make use of the normalization condition (4.4), the definitions (6.1), and the formulae (4.2). In addition, we assume that the boundary of the spatial integrals is not a time function, and that the 3-space is either compact or that the asymptotic behaviour of the eigenfunctions enables us to omit certain integrals over 3-dimensional divergencies.

The structure of the Maxwell equations guided us when we defined the eigenvectors u_{ka} and v_{ka}. The definitions given above are very advantageous for the following reason: The expansion (4.7) of the magnetic field B_a follows from the expansion (4.6) of the transverse vector potential C_a by replacing essentially u_{ka} by v_{ka}. Moreover, after the insertion of the expansions of B_a and E_a into (3.1) the Maxwell equations (3.1a) and (3.1b) contain only eigenvectors u_{ka} and v_{ka}, respectively. Thus, we get the equations of motion for the operators p_k and q_k in a simple manner by utilizing the completeness and the normalization. The result is an infinite system of first-order ordinary differential equations,

$$\dot{p}_k = -m_k q_k - \sum_{k'} c_{kk'}p_{k'},$$

$$\dot{q}_k = m_k p_k + \sum_{k'} c^*_{k'k}q_{k'}. \tag{6.2}$$

By means of these equations of motion one verifies that the commutators in (5.1) do not depend on time. We conclude: If the operators p_k and q_k obey the canonical commutation rules (5.1) on one initial hypersurface S, the same statement proves right on all other hypersurfaces. The equations of motion can be written in the alternative form

$$\dot{p}_k = \frac{i}{\hbar}[H, p_k], \quad \dot{q}_k = \frac{i}{\hbar}[H, q_k],$$

where the quantum Hamiltonian H is given by the Hermitian operator

$$H = \tfrac{1}{2}\sum m_k(\overset{+}{p}_k p_k + \overset{+}{q}_k q_k) + \sum_{kk'} c_{kk'}\overset{+}{q}_k p_{k'}. \tag{6.3}$$

The infinite-dimensional matrix $c_{kk'}$ is traceless and depends, in general, on time. As a special case we consider a *stationary* metric with the time-like Killing vector k^i. We choose a coordinate system with

$$k^i = \delta^i_4, \tag{6.4}$$

and identify the 3-spaces S with the hypersurfaces x^4 = constant in this preferred coordinate system. The existence of the Killing vector gives rise to an integral of motion — the energy

$$E = \int k^i T_i^j df_j,$$

which we compare with the Hamiltonian (6.3). In a coordinate system with (6.4) both expressions are identical,

$$E = \int_{x^4=\text{constant}} T_4{}^4 \sqrt{g}\, n d^3x = H.$$

For stationary metrics the coefficients $c_{kk'}$ are

$$c_{kk'} = \int \varepsilon_{abc} v_k^{a*} u_k^c N^b \sqrt{g}\, d^3x,$$

they vanish for static gravitational fields ($N_a = 0$). Under the transformation group

$$x^{a'} = x^{a'}(x^b), \qquad x^{4'} = x^4,$$

leaving unchanged the form of the Killing vector (6.4), N_a is a 3-vector and, hence, the quantities $c_{kk'}$ are invariant coefficients.

7. *Creation and annihilation operators*

We make the linear ansatz

$$p_k = \sum_{k'} P_{kk'} a_{k'} + \sum_{k'} P^*_{-kk'} \overset{+}{a}_{k'},$$

$$q_k = \sum_{k'} Q_{kk'} a_{k'} + \sum_{k'} Q^*_{-kk'} \overset{+}{a}_{k'} \tag{7.1}$$

where the creation and annihilation operators, $\overset{+}{a}_k$ and a_k, respectively, satisfy the usual commutation relations

$$[a_k, \overset{+}{a}_{k'}] = \delta_{kk'}, \quad [a_k, a_{k'}] = 0 = [\overset{+}{a}_k, \overset{+}{a}_{k'}] \tag{7.2}$$

on each hypersurface S. Now we try to determine the coefficients $P_{kk'}$ and $Q_{kk'}$. For this purpose we form the expectation values of the components of the energy-momentum tensor,

$$t_{ab} \equiv \langle : T_{ab} : \rangle, \quad t_a \equiv \langle : T_a : \rangle, \quad t \equiv \langle : T : \rangle. \tag{7.3}$$

The symbol : : denotes the normal product with respect to a_k and a_k^+. For these expectation values the energy-momentum balance (3.3) should be fulfilled. We consider two simple realizations:

1. The operators a_k (and a_k^+) are *constant* operators,

$$\dot{a}_k = 0. \tag{7.4}$$

In order to prove that the energy-momentum balance (3.2) holds for the expectation values (7.3) in place of the classical components of T_{ij}, we use the relations

$$\sum{}' \varepsilon_{abc} v_k^b u_k^{c*} = 0,$$
$$\sum{}' (u_k^a u_{ka}^* - v_k^a v_{ka}^*) = 0. \tag{7.5}$$

(The sums include all values of k which belong to the same eigenvalue m_k). Obviously, particle production from the vacuum state induced by nonstationary gravitational fields does not occur, if a_k is independent of time. Therefore, this mathematically possible choice seems to be unsatisfactory.

2. We take over, from the usual quantum field theory, the structure of the linear ansatz (7.1),

$$p_k = -i\sqrt{\frac{\hbar}{2}}\left(a_k e^{-i\int m_k dt} - a_{-k}^+ e^{i\int m_k dt}\right),$$

$$q_k = \sqrt{\frac{\hbar}{2}}\left(a_k e^{-i\int m_k dt} + a_{-k}^+ e^{i\int m_k dt}\right). \tag{7.6}$$

The normalization condition of the eigenvectors still admits the freedom

$$u_{ka} \rightarrow u_{ka} e^{i\lambda_k(x^4)}, \qquad \lambda_k \text{ real},$$

which can be used to gauge the matrix $c_{kk'}$, so that the imaginary parts of the diagonal elements vanish.

The definition (7.6) of the creation and annihilation operators coincides with the above definition (7.4), if the coefficients $c_{kk'}$ are equal to zero. Examples are the *static* gravitational fields and the Friedman Universe. Under conformal transformations (4.5) the coefficients $c_{kk'}$ are invariant. Hence, they vanish in a *conformally flat* metric. Massless particles (photons) can not be produced from the vacuum state in the Friedman isotropic model. This conclusion is generally accepted.

The definition (7.6) guarantees the validity of the energy momentum balance (3.3) for the expectation values (7.3) only, if the condition

$$\sum_{kk'} c_{kk'}(u_k^a u_{k'a}^* - v_k^a v_{k'a}^*) = 0 \tag{7.7}$$

is fulfilled. This restriction is satisfied only in special fields with hypersurfaces S chosen in a particular way.

8. An example

Zeldovich and Starobinsky [1] investigated the quantization of the Klein-Gordon field in a metric due to Kasner,

$$ds^2 = -dt^2 + a^2dx^2 + b^2dy^2 + c^2dz^2, \; a = a(t), \; b = b(t), \; c = c(t). \tag{8.1}$$

We consider the Maxwell field in this space-time. The hypersurfaces S (t = constant) are flat 3-spaces. A complete set of eigenvectors is given by

$$u_{ka} = \bar{u}_{ka}(t) \frac{1}{(2\pi)^{3/2}} e^{ik_a x^a}, \qquad x^a : (x, y, z),$$

$$\bar{u}^*_{ka}\bar{u}_k{}^a = \frac{m_k}{abc}, \qquad k_a \bar{u}_k{}^a = 0, \qquad k_a k^a = m_k^2.$$

The same relations hold for the dual set of eigenvectors v_{ka} with

$$m_k \bar{v}_k^a = i\varepsilon^{abc} k_b \bar{u}_{kc}.$$

We split the index k into the vector k_a and the polarization index s,

$$k : (k_a, s), \qquad s = 1, 2.$$

and obtain

$$c_{k_a s k'_a s'} = 0, \qquad k_a \neq k'_a,$$

$$k_a = (0, 0, k): \quad c_{k_a s k_a s'} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \quad \gamma = \tfrac{1}{2}\left(\log \frac{b}{a}\right)^{\cdot}. \tag{8.2}$$

With the aid of (8.2) an (7.5) it can be easily shown that the condition (7.7) is fulfilled in the case under consideration. Thus, in this special model we are able to construct *regular expectation values of the energy-momentum tensor satisfying the balance equation (3.2).*

9. Summary

We have admitted an arbitrary Riemannian space and selected a family of 3-dimensional space-like hypersurfaces. We have studied the influence of the classical gravitational field on the quantized electromagnetic field. The key points are the eigenvalue equation (4.1), the equations of motion (6.2) with the associated Hamiltonian (6.3), and the commutation rules (5.1). All the equations are 3-covariant. Their explicit form depends, of course, on the choice of the hypersurfaces. Our next task is a covariant formulation of the quantized theory in the sense that all physical statements are obviously independent of the particular choice of the intermediate 3-spaces. The attempt to define creation and annihilation operators and to construct a regular expectation value of the energy-momentum tensor (source term in the Einstein-Maxwell equations) does not give satisfactory and unique results for arbitrary gravitational fields. The method of "adiabatic regularization" of the energy-momentum tensor proposed in a recent paper by Parker and Fulling [11] is also restricted to a special class of gravitational fields.

REFERENCES

[1] Ya. B. Zeldovich, A. A. Starobinsky, *JETP* **61**, 2161 (1971).
[2] N. A. Chernikov, *JETP* **53**, 1006 (1957).
[3] E. A. Tagirov, *Ann. Phys.* (USA) **76**, 561 (1973).
[4] N. A. Chernikov, E. A. Tagirov, *Ann. Inst. H. Poincaré* **A9**, 43 (1968).
[5] L. Parker, S. A. Fulling, *Phys. Rev.* **D7**, 2357 (1973).
[6] L. Parker, *Phys. Rev.* **183**, 1057 (1969).
[7] G. Börner, H. P. Dürr, *Nuovo Cimento* **A64**, 669 (1969).
[8] D. Kramer, K. H. Lotze, *Acta Phys. Pol.* **B5**, 653 (1974).
[9] C. Aragone, *Commun. Math. Phys.* **26**, 205 (1972).
[10] E. Schmutzer, *Relativistische Physik (Klassische Theorie)*, Leipzig 1968.
[11] L. Parker, A. A. Fulling, *Phys. Rev.* **D9**, 341 (1974).