SPHERICAL SYMMETRY IN RAINICH'S THEORY

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A time-dependent, spherically symmetric solution of Rainich's field equations is derived when the field is null. It is concluded that in this case the Already Unified Field Theory and the Einstein-Maxwell theory are not equivalent.

1. Introduction

Some time ago, Lynch and the present author (Ref. [1]) criticised Rainich's Already Unified Field Theory (Refs [2, 3]) by showing explicitly the strong relation it implies between electric field and inertial/gravitational mass. The question arises whether this criticism applies necessarily also to the Einstein-Maxwell theory. This is the case, of course, if there exists a one to one correspondence between the two theories but such correspondence has been established only for non-null Rainich fields (e. g. Ref. [3]). One way of investigating this problem is to consider whether Birkhoff's Theorem is valid for Rainich's field equations. Birkhoff's Theorem has been discussed by numerous authors, notably by Bonnor (Ref. [4]) and, for an Einstein-Maxwell field, by Hoffman (Ref. [5]).

As a matter of fact, a time-dependent solution was found by Bonnor and was rejected because of curious geometrical situation it seemed to portray. However, Bonnor's solution requires a non-zero cosmological constant which plays no part in Rainich's theory.

We shall consider as field equations, Rainich's algebraic relations

$$R = 0,$$
$$R_{\mu}^{\ \alpha}R_{\alpha\nu} = \tfrac{1}{4}\, g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}, \tag{1}$$
$$R_{44} > 0,$$

which follow directly from the structure of Maxwell's energy-stress-momentum tensor

$$E_{\mu\nu} = f_{\mu}^{\ \alpha}f_{\alpha\nu} + \tfrac{1}{4}\, g_{\mu\nu}g_{\alpha\beta}g^{\alpha\beta}. \tag{2}$$

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Throughout this work Greek indices are assumed to go from 1 to 4, $R_{\mu\nu}$ is the Ricci tensor, $R = R^{\alpha}_{\ \alpha} = g^{\mu\nu}R_{\mu\nu}$, $g_{\mu\nu}$ is the metric tensor and $f_{\mu\nu}$, the skew symmetric, electromagnetic intensity tensor. We shall not require Misner-Wheeler differential relations which guarantee that $f_{\mu\nu}$ should represent a Maxwell field, because the case we shall find particularly interesting is the null case when

$$R_{\alpha\beta}R^{\alpha\beta} = 0. \tag{3}$$

We shall solve Eqs (1) in a spherically symmetric Riemann space which is formally time--dependent.

2. *Spherical symmetry*

Bonnor proves (Ref. [4]) that a spherically symmetric metric necessarily has the form

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4}$$

where ν and λ are at least twice continuously differentiable function of r and of t only. His proof is independent of the field equations so that we can assume (4) to be the appropriate form also of a Rainich space. We write down, for the sake of completeness, expressions for the Christoffel brackets $\left\{ {\lambda \atop \mu\nu} \right\}$ and the corresponding components of the Ricci tensor

$$R_{\mu\nu} = -\left\{ {\sigma \atop \mu\nu} \right\}_{,\sigma} + \left\{ {\sigma \atop \mu\sigma} \right\}_{,\nu} + \left\{ {\sigma \atop \mu\varrho} \right\}\left\{ {\varrho \atop \sigma\nu} \right\} - \left\{ {\sigma \atop \mu\nu} \right\}\left\{ {\varrho \atop \sigma\varrho} \right\}.$$

Let

$$\lambda_1, \nu_1 \equiv \frac{\partial}{\partial r}(\lambda, \nu), \qquad \lambda_4, \nu_4 \equiv \frac{\partial}{\partial t}(\lambda, \nu) \text{ etc.}$$

Then the non-zero brackets are

$$\left\{ {1 \atop 11} \right\} = \tfrac{1}{2}\lambda_1, \quad \left\{ {1 \atop 14} \right\} = \tfrac{1}{2}\lambda_4, \quad \left\{ {1 \atop 22} \right\} = \left\{ {1 \atop 33} \right\}\operatorname{cosec}^2\theta = -re^{-\lambda},$$
$$\left\{ {1 \atop 44} \right\} = \tfrac{1}{2}\nu_1 e^{\nu-\lambda}, \quad \left\{ {2 \atop 12} \right\} = \left\{ {3 \atop 13} \right\} = r^{-1}, \quad \left\{ {2 \atop 33} \right\} = -\sin\theta\cos\theta,$$
$$\left\{ {3 \atop 23} \right\} = \cot\theta, \quad \left\{ {4 \atop 44} \right\} = \tfrac{1}{2}\nu_4, \quad \left\{ {4 \atop 14} \right\} = \tfrac{1}{2}\nu_1, \quad \left\{ {4 \atop 11} \right\} = \tfrac{1}{2}\lambda_4 e^{\lambda-\nu}. \tag{4a}$$

Similarly, the non-vanishing components of the Ricci tensor are

$$R_{11} = -\tfrac{1}{2}(\lambda_4 e^{\lambda-\nu})_4 + \tfrac{1}{2}\nu_{11} + \tfrac{1}{4}\lambda_4^2 e^{\lambda-\nu} - \tfrac{1}{4}\lambda_4\nu_4 e^{\lambda-\nu} - \tfrac{1}{2}\lambda_1(2/r + \tfrac{1}{2}\nu_1),$$
$$R_{22} = R_{33}\operatorname{cosec}^2\theta = (re^{-\lambda})_1 + \tfrac{1}{2}re^{-\lambda}(\lambda_1 + \nu_1) - 1,$$
$$R_{44} = -\tfrac{1}{2}(\nu_1 e^{\nu-\lambda})_1 + \tfrac{1}{2}\lambda_{44} + \tfrac{1}{4}\lambda_4^2 + \tfrac{1}{4}\nu_1^2 e^{\nu-\lambda} - \tfrac{1}{4}\lambda_4\nu_4 - \tfrac{1}{2}\nu_1(\tfrac{1}{2}\lambda_1 + 2/r)e^{\nu-\lambda},$$
$$R_{14} = -\lambda_4/r. \tag{4b}$$

If λ is independent of $t(\lambda_4 = 0$, so that $R_{14} = 0)$ we revert to the general relativistic case of the Birkhoff-Hoffman theorem. Hence, we assume in the sequel that

$$R_{14} \neq 0. \tag{5}$$

3. *Rainich's field equations*

It can be shown easily that the first and second of the Eqs (1) imply that

$$R_{14}(g^{11}R_{11}+g^{44}R_{44}) = 0.$$

Hence, because of the assumption (5),

$$g^{11}R_{11}+g^{44}R_{44} = 0, \tag{6}$$

while the two remaining equations become

$$R_{22} = 0, \tag{7}$$

and

$$R_{11} = \pm\sqrt{-g_{11}g^{44}}\,R_{14}. \tag{8}$$

We have also the nullity condition (3). Written out in full the above equations become

$$e^{-\nu}(\lambda_{44}+\tfrac{1}{2}\lambda_4{}^2-\tfrac{1}{2}\lambda_4\nu_4)-e^{-\lambda}(\nu_{11}+\tfrac{1}{2}\nu_1{}^2-\tfrac{1}{2}\lambda_1\nu_1-(\lambda_1-\nu_1)/r = 0,$$

$$\lambda_1-\nu_1 = 2(1-e^{\lambda})/r,$$

$$e^{-\nu}(\lambda_{44}+\tfrac{1}{2}\lambda_4{}^2-\tfrac{1}{2}\lambda_4\nu_4)-e^{-\lambda}(\nu_{11}+\tfrac{1}{2}\nu_1{}^2-\tfrac{1}{2}\lambda_1\nu_1-2\lambda_1/r) = \pm 2\lambda_4 e^{-(\lambda+\nu)/2}/r.$$

It is by no means certain that these equations are compatible. If they are then

$$(\lambda_1+\nu_1)e^{(\lambda+\nu)/2} = \pm 2\lambda_4 e^{\lambda}, \tag{9}$$

so that Eqs (7) and (9) form a system of first order differential equations. Differentiating them with respect to t and r and eliminating λ_{14} and ν_{14} (as well as λ_{11}) we recover Eq. (6). It follows that Eqs (6), (7) and (8) are compatible and all we have to do is to solve (7) and (9) simultaneously.

4. *A similarity solution*

Let

$$e^{\lambda} = u/v \quad \text{and} \quad e^{\nu} = uv.$$

Eqs (7) and (9) become

$$u_1 = \pm(u/v)_4 \text{ and } u = (rv)_1.$$

Eliminating u between them we get

$$(rv)_{11} = \pm(rv_1/v)_4. \tag{10}$$

Suppose that v is a function of

$$z = r^m t^n,$$

where m and n are constants only. Eq. (10) then becomes

$$(m+1)v' + mzv'' = \pm(n/v^2)(r/t)(vv' + z(vv'' - v'^2)), \tag{11}$$

dashes denoting differentiation with respect to z. A "similarity" solution therefore is possible providing we choose

$$m+n = 0. \tag{12}$$

The simplest case results if we take

$$m = -1, \quad n = +1, \quad z = t/r. \tag{13}$$

Then

$$z^2 v'' = \mp \frac{d}{dz}(zv'/v). \tag{14}$$

Let us further choose the upper sign. The equation we wish to solve is

$$z^2 v'' + \frac{d}{dz}(zv'/z) = 0. \tag{15}$$

If this equation possesses a continuous solution, z can be regarded as a function of v. Let us then write

$$v = \exp(y), \quad zv = \exp(x), \quad \text{and} \quad \frac{dy}{dx} = Y/(1+Y).$$

An elementary calculation shows that

$$\frac{1+Y}{Y(1-Y)} \frac{dY}{dx} = \frac{\exp(x)}{1+\exp(x)}, \tag{16}$$

whence

$$\frac{dy}{dx} = \tfrac{1}{2}(1 \mp \sqrt{b/(b+2)}), \tag{17}$$

where

$$b = k/(1+\exp(x)),$$

and k is a constant.

If, finally, we write

$$1 + \tfrac{1}{2}k = K^{-2},$$

the complete solution (through which v can be expressed in terms of z) becomes

$$y = B + \tfrac{1}{2}(1 - \sqrt{1-K^2})\, x \mp \ln(1 + \sqrt{1+K^2 \exp(x)}), \tag{18}$$

B being a constant of integration. We should observe that K cannot vanish since the case $K = 0$ corresponds to $z = \text{const}$. Substituting back in terms of v and z, we have

$$v^{1-p} = CzP(1+\sqrt{1+K^2zv})^{\mp 1},$$

where

$$\tfrac{1}{4}K^2 = p(1-p).$$

Another way of writing the complete similarity solutions is to put

$$\frac{dy}{dx} = v'/v = f,$$

say, then

$$rv'/v = f/(f-1),$$

and

$$\exp(\lambda) = (2f-1)/(f-1), \ \exp(\nu) = ((2f-1)/(f-2))\, v^2.$$

In the next two sections we shall investigate the nature of the solution obtained above.

5. *Killing equations*

We consider first the problem of hidden symmetries of the space characterised by Eq. (18). Fortunately, it is not necessary to exhibit the explicit solution. The Killing equations are

$$\xi_{1,1}-\tfrac{1}{2}\lambda_1\xi_1-\tfrac{1}{2}\lambda_4 e^{\lambda-\nu}\xi_4 = 0, \tag{19a}$$

$$\xi_{1,4}+\xi_{4,1}-\lambda_4\xi_1-\nu_1\xi_4 = 0, \tag{19b}$$

$$\xi_{4,4}-\nu_1 e^{\nu-\lambda}\xi_1-\nu_4\xi_4 = 0, \tag{19c}$$

$$\xi_{1,2}+\xi_{2,1}-\frac{2}{r}\xi_2 = 0, \tag{19d}$$

$$\xi_{1,3}+\xi_{3,1}-\frac{2}{r}\xi_3 = 0, \tag{19e}$$

$$\xi_{2,2}+re^{-\lambda}\xi_1 = 0, \tag{19f}$$

$$\xi_{2,3}+\xi_{3,2}-2\xi_3\cot\theta = 0, \tag{19g}$$

$$\xi_{3,3}+re^{-\lambda}\sin^2\theta\xi_1+\sin\theta\cos\theta\xi_2 = 0, \tag{19h}$$

$$\xi_{2,4}+\xi_{4,2} = 0, \tag{19i}$$

$$\xi_{3,4}+\xi_{4,3} = 0, \tag{19j}$$

ξ_μ being the components of a Killing vector.

Because of their form, Eqs (19) imply that the φ dependence of every ξ_μ is the same, separable, and at most exponential.

$$e^{k\varphi}, \tag{20}$$

where k is a constant. Thus, writing

$$(\xi_1 = re^{\lambda}y_1, \quad \xi_2 = r^2 y_2, \quad \xi_3 = r^2 y_3) \text{ times } e^{k\varphi},$$

Eq. (19d) to (19h) become

$$re^{\lambda}y_{1,2}+y_{2,1} = 0, \tag{21a}$$

$$kre^{\lambda}y_1+y_{3,1} = 0, \tag{21b}$$

$$y_{2,2}+y_1 = 0, \tag{21c}$$

$$ky_2+y_{3,2}-2y_3\cot\theta = 0, \tag{21d}$$

$$ky_3+\sin^2\theta y_1+\sin\theta\cos\theta y_2 = 0. \tag{21e}$$

Eq. (21b) gives

$$kre^{\lambda}y_{1,2}+y_{3,12} = 0,$$

so that from (21a)

$$-ky_{2,1}+y_{3,12} = 0.$$

Also, from (21d)

$$ky_{2,1}+y_{3,21}-2y_{3,1}\cot\theta = 0.$$

Hence, assuming continuity,

$$(y_{3,1})_{,2}-(y_{3,1})\cot\theta = 0,$$

or

$$y_3 = f(r,t)\sin\theta+g(r,t)+h(\theta,t), \tag{22}$$

where, by (21c)

$$fre^{\lambda}+f_{,1} = 0. \tag{23}$$

With the help of (22) we readily find that the only possible solution of Eqs (21) is given by

$$ky_2 = f\cos\theta, \quad ky_1 = +f\sin\theta, \quad y_3 = f\sin\theta$$

with (from 21e)

$$(k^2+1)f\sin\theta = 0. \tag{24}$$

Thus $k = \pm i$.

The resulting real Killing vectors are

$$(re^{\lambda}f \sin\theta \sin\varphi,\ r^2 f \cos\theta \sin\varphi,\ r^2 f \sin\theta \cos\varphi,\ r^2 f_{,4} \sin\theta \sin\varphi),$$

$$(re^{\lambda}f \sin\theta \cos\varphi,\ r^2 f \cos\theta \cos\varphi,\ -r^2 f \sin\theta \sin\varphi,\ r^2 f_{,4} \sin\theta \cos\varphi),$$

ξ_4 being determined from (19i) and (19j).

In considering Eqs (19a) to (19c) it is clearly sufficient to take

$$\xi_1 = re^{\lambda}f, \quad \xi_4 = r^2 f_{,4}.$$

With the help of (23) we can eliminate f to get

$$2(e^{\lambda} - r^2 e^{\lambda} + 2 - \nu_1 r)(1 + \tfrac{1}{2} r\lambda_1 - r^2 e^{\lambda}) = r\lambda_4^2 e^{\lambda-\nu}, \tag{25}$$

which is incompatible with the field Eqs (7) and (9) in the case of the similarity solution (18). Hence ξ_μ cannot be φ dependent. If we eliminate $\xi_{\mu,3}$ terms from Eqs (19) we readily find that

$$\xi_1 = -r^{-1}e^{\lambda}f(r, t)\cos\theta, \quad \xi_2 = f(r, t)\sin\theta, \quad \xi_3 = r^2 \sin^2\theta,$$
$$\xi_4 = f_{,4}\cos\theta,$$

where

$$rf_{,1} = (2 - e^{\lambda})f. \tag{26}$$

With (26), Eqs (19a) to (19c) give again a condition

$$2r^2 e^{\lambda+\nu}(2 - 2e^{\lambda} - \nu_1) = e^{\lambda} - 1 - \tfrac{1}{2} r\lambda_1, \tag{27}$$

incompatible with the solution (18). This time however we can put

$$f = 0$$

to obtain a non-trivial Killing vector

$$(0, 0, r^2 \sin^2\theta, 0), \tag{28}$$

which is space-like. Consequently we have proved that solution (18) does not admit any time-like Killing vectors.

6. *The Maxwell condition*

The Rainich field we have been considering is null (Eq. (3)) and this implies in turn that $f_{\mu\nu}$ is partially null:

$$f_{\alpha\beta}f^{\alpha\beta} = 0. \tag{29}$$

It follows that the field tensor is of the form

$$f_{\mu\nu} = \begin{Bmatrix} 0 & \cdot & \cdot & 0 \\ \cdot & 0 & 0 & \cdot \\ \cdot & 0 & 0 & \cdot \\ 0 & \cdot & \cdot & 0 \end{Bmatrix}, \tag{30}$$

and that it is to be determined from

$$R_{\mu\nu} = f_{\mu}^{\alpha} f_{\alpha\nu}. \tag{31}$$

In fact $f_{\mu\nu}$ is not determined uniquely by Eq. (31) since Rainich's algebraic equations are satisfied by any f_{12}, f_{13} fulfilling the conditions

$$\begin{aligned} -R_{11} &= g^{22} f_{12}^2 + g_2^{33} f_{13}^2, \\ f_{24}^2 = -g'' g_{44} f_{12}^2, &\quad f_{34}^2 = -g'' g_{44} f_{13}^2, \\ R_{14} &= g^{22} g_{12} f_{24}. \end{aligned} \tag{32}$$

For a null field the Misner-Wheeler criterion that $f_{\mu\nu}$ should satisfy Maxwell equations breaks down and we must look elsewhere to ensure that there exists a four-vector φ_{μ} such that

$$f_{\mu\nu} = \varphi_{\nu,\mu} - \varphi_{\mu,\nu},$$

or

$$\sum_{\substack{\text{cyclic} \\ \lambda\mu\nu}} f_{\mu\nu,\lambda} = 0. \tag{33}$$

In the case of our field geometry (metric (4), Christoffel brackets (4a)) however, we can prove a remarkable result. If the only nonzero components of $f_{\mu\nu}$ are f_{12} and f_{24} (so that $f_{13} = f_{34} = 0$) and

$$f_{24} = \sqrt{-g^{11} g_{44}}\, f_{12}, \tag{34}$$

the condition that $f_{\mu\nu}$ should satisfy the second set of Maxwell's equations (33) is equivalent to a Bianchi identity. To prove this we observe that under the above conditions only one of Eqs (33) survives and can be written as

$$f_{12;4} + f_{24;1} + f_{41;2} = 0,$$

or

$$f_{12;4} + f_{24;1} + \left\{ {2 \atop 12} \right\} f_{24} = 0.$$

Multiplying this by f_{12} and using (34), the equation can be written in the form

$$f_{12;4}^2 + \sqrt{-g_{44} g^{11}}\, f_{12;1}^2 + 2 \left\{ {2 \atop 12} \right\} \sqrt{-g_{44} g^{11}}\, f_{12}^2 = 0. \tag{35}$$

The contracted Bianchi identities for a Rainich field are

$$R_{\mu\ ;\nu}^{\ \nu} = 0,$$

and in particular, when $\mu = 1$, and (4a) give the Christoffel brackets

$$g^{11} R_{11;1} + g^{44} R_{14;4} - \left[g^{22} \left\{ {1 \atop 22} \right\} + g^{33} \left\{ {1 \atop 33} \right\} \right] R_{11} = 0. \tag{36}$$

It is now a simple matter to verify, using

$$R_{11} = -g^{22}f_{12}^2$$

and Rainich's algebraic Eqs (7), that Eqs (36) and (35) are in fact identical. Since the Ricci tensor satisfies Bianchi identities the field $f_{\mu\nu}$ determined by (31) and the above additional requirement is a Maxwell field.

7. Conclusions

The solution obtained above is of the form

$$ds^2 = g(z)dt^2 - f(z)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{17}$$

It can be seen easily that the transformation

$$q = \ln t, \qquad p = z^{-1},$$

maps the above metric into

$$ds^2 = e^{2q}(u(p)dt^2 + 2v(p)dqdp - w(p)dq^2 - p^2(d\theta^2 + \sin^2\theta d\varphi^2)).$$

After a further transformation

$$q = T - \int^{q} (v/u)dq, \qquad R = p\exp(-\int^{2} (v/u)dq),$$

the metric takes the final form

$$ds^2 = e^{2T}(A(R)dT^2 - B(R)dR^2 - R^2(d\theta^2 + \sin^2\theta d\varphi^2)). \tag{18}$$

It might have been possible to assume this form of the metric from the start except that its feasibility depends strictly on the existence of a similarity solution of Rainich's field equations.

In view of the results obtained in Section 5, we conclude that the similarity solution (18) cannot be transformed into a time independent solution of Rainich's equations. It follows that Birkhoff's theorem does not hold in the case of a null, spherically symmetric Rainich field. To this extent at any rate, Rainich's Already Unified Field Theory and the Einstein-Maxwell theory are not equivalent.

We may observe also that a spherically symmetric electromagnetic field should have f_{14} and f_{24} as its only non-zero components (corresponding to parallel electric and magnetic vectors in the $\hat{r}$ direction). According to Section 6, however, the electromagnetic field corresponding to a non-trivial solution of Rainich's equations such as our similarity solution (18) consists of transverse and mutually orthogonal electric and magnetic vectors. Thus our solution cannot correspond to a spherically symmetric electromagnetic field which is the source of an Einstein-Maxwell geometry. Of course, the spherically symmetric field is necessarily non-null and this is an invariant condition.

As a final remark, we may note that choice of the positive sign in Eq. (14) gives $e^x/(e^x - 1)$ on the right hand side of (16) and, consequently, a singularity at $x = 0$ (or $v = z^{-1}$). Such a singular surface is difficult to understand from a physical point of view.

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