NON-LINEAR ELECTRODYNAMICS IN THE NEWMAN--PENROSE FORMALISM

By J. Bičák

Department of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, Prague*

and J. Slavík**

Institute of Theoretical Physics, Warsaw University***

(Received September 30, 1974)

An approach to non-linear electrodynamics by means of the spin-coefficient formalism of Newman and Penrose is presented. The field equations are rewritten in the Newman-Penrose form, and their spherically symmetric solutions are discussed. An approximation procedure, suitable for treating radiation problems in non-linear electrodynamics, is then suggested on the basis of an analysis of the Maxwell electrodynamics with sources. Since the field equations are non-linear, wave tails will in general develop. This is illustrated in detail on an example of the approximate solution, representing a radiating dipole in the zero approximation. Conserved quantities, analogous to those discovered by Newman and Penrose in Maxwell's and Einstein's theories are found for a large class of non-linear theories of electrodynamics. Their number depends on the choice of a particular theory — it is greater than, or equal to 16 for theories satisfying the correspondence principle with Maxwell's theory.

1. Introduction

It has been suggested by Dirac in 1964 [1] that in regions near to charges one may have to modify Maxwell's theory so as to make it into a non-linear electrodynamics. One now expects that very strong electromagnetic fields also occur in the vicinity of neutron stars or, possibly, black holes, so that it may be interesting to study non-linear effects also with an astrophysical motive.[1]

---

* Address: Department of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Prague 2, Czechoslovakia.

** This paper is based partly on the Diploma thesis submitted by this author to the Faculty of Mathematics and Physics in Prague.

*** Address: Institute of Theoretical Physics, Warsaw University, Hoża 69, 00-681 Warsaw, Poland.

[1] Z. Białynicka-Birula and I. Białynicki-Birula [2] were led by this motive, for example.

It is well-known that, in many respects, non-linear electrodynamics offers the best classical solution to the self-energy problem of charged particles. From a more practical point of view, however, it is important that a phenomenological theory of electrodynamics, which describes vacuum polarizational effects, must be non-linear in any case. As demonstrated by Stehle and De Baryshe [3], theories with Lagrangians similar to the Heisenberg-Euler effective Lagrangian (the weak-field expansion of the Lagrangian of the Born-Infeld theory is of this form, for example) are more accurate classical approximations of quantum electrodynamics than Maxwell's theory in the case of fields with high intensities at a fixed frequency. Moreover, it is not clear whether even a low-frequency limit leads to Maxwell's theory for arbitrary intensities.

In this paper we shall study some problems of non-linear electrodynamics in the framework of the Newman-Penrose (NP) formalism [4] in flat spacetime. We are led by successful applications of the NP formalism in general relativity, in particular in the analyses of gravitational radiation, of the radiative properties of test fields on given curved background spacetimes, etc. For example, it has been demonstrated [5] that electromagnetic radiation (described by Maxwell's theory) propagating on the curved background spacetime surrounding a collapsing star, backscatters, and wave tails develop. Radiation scattering also occurs in the approximate solutions of the Einstein-Maxwell theory [6]. It will be shown explicitly in the following how wave tails arise even in flat spacetime, provided that the electrodynamics is non-linear.

The NP formalism also led to the discovery of new conservation laws in the Einstein theory [7, 8] and in the Einstein-Maxwell theory [9], the physical content of which is still not well understood. We shall find the analogues of the Newman-Penrose conserved quantities for a large class of non-linear theories of electrodynamics. This will enable us to see how the number of conserved quantities is influenced by the non-linearity of a particular theory.

The best known non-linear electrodynamics was developed by Born and Infeld [10]. The Lagrangian of the Born-Infeld theory has the form

$$L = b^2[(1+2b^{-2}F+b^{-4}G^2)^{1/2}-1],$$

where the constant b with the dimension of the electromagnetic field may be called the "absolute field", F and G are the invariants of the field. (In Born's original theory the Lagrangian $L = b^2[(1+2b^{-2}F)^{\frac{1}{2}}-1]$ was used.) Although in comparison with other non-linear theories the Born-Infeld electrodynamics has some attractive features (see, for example [11]), there are situations in which other theories are preferable (cf. the extensive text on non-linear electrodynamics by Plebański [12]). Furthermore, the expansion of the phenomenological Lagrangian of quantum electrodynamics does not exactly coincide with the expansion of the Born-Infeld Lagrangian. Therefore, following Plebański, we shall not restrict ourselves to a particular form of the Lagrangian, but rather, all Lagrangians depending on the two invariants of the electromagnetic field, and satisfying the correspondence principle with Maxwell's theory will be considered.

Some questions studied in this paper have been investigated in the NP formalism by Chellone [13, 14] and in the Debever self-dual formalism, by Porter [15]. However, these authors restrict themselves to a particular type of Lagrangian; the NP formalism has actually only been applied to Born's theory. (Moreover, as we shall indicate in the following, the approximation procedure, developed by Chellone, is not entirely consistent and some of the conserved quantities are not given.)

In Section 2 the relevant relations of the NP formalism are briefly summarized. The NP form of the field equations of non-linear electrodynamics and their spherically symmetric solutions are given in Section 3. Before the discussion of the NP conservation laws (Section 5) and of the approximation method for treating radiation problems in non-linear electrodynamics (Section 6), it is helpful to describe the procedure of solving the equations of Maxwell electrodynamics with sources in the NP formalism (Section 4). In Section 7 the approximation method, developed in Section 6, is applied to the problem of the radiating dipole and, finally, some open problems are indicated in Section 8.

2. *The Newman-Penrose formalism*

At any point in curved spacetime we introduce a complex null tetrad $l^\mu, n^\mu, m^\mu, \overline{m}^\mu$ ($\overline{m}^\mu$ being the complex conjugate of m^μ) such that the only non-vanishing scalar products are

$$l_\nu n^\nu = 1, \qquad m_\nu \overline{m}^\nu = -1.$$

Instead of using six real components of the Maxwell field tensor $F_{\mu\nu}$, we will describe the electromagnetic field by three independent complex quantities

$$\begin{aligned} \Phi_0 &= F_{\mu\nu} l^\mu m^\nu, \\ \Phi_1 &= \tfrac{1}{2} F_{\mu\nu}(l^\mu n^\nu + \overline{m}^\mu m^\nu), \\ \Phi_2 &= F_{\mu\nu} \overline{m}^\mu n^\nu. \end{aligned} \tag{2.1}$$

In order to be able to rewrite the field equations into the NP form, we define the invariant differential operators by

$$D = l^\mu \frac{\partial}{\partial x^\mu}, \quad \delta = m^\mu \frac{\partial}{\partial x^\mu}, \quad \bar{\delta} = \overline{m}^\mu \frac{\partial}{\partial x^\mu}, \quad \Delta = n^\mu \frac{\partial}{\partial x^\mu}, \tag{2.2}$$

and introduce twelve spin coefficients as follows:

$$\begin{aligned} \kappa &= l_{\mu;\nu} m^\mu l^\nu, & \nu &= -n_{\mu;\nu} \overline{m}^\mu n^\nu, \\ \varrho &= l_{\mu;\nu} m^\mu \overline{m}^\nu, & \mu &= -n_{\mu;\nu} \overline{m}^\mu m^\nu, \\ \sigma &= l_{\mu;\nu} m^\mu m^\nu, & \lambda &= -n_{\mu;\nu} \overline{m}^\mu \overline{m}^\nu, \\ \tau &= l_{\mu;\nu} m^\mu n^\nu, & \pi &= -n_{\mu;\nu} \overline{m}^\mu l^\nu, \\ \beta &= \tfrac{1}{2}(l_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu} \overline{m}^\mu m^\nu), & \alpha &= -\tfrac{1}{2}(n_{\mu;\nu} l^\mu \overline{m}^\nu - \overline{m}_{\mu;\nu} m^\mu \overline{m}^\nu), \\ \varepsilon &= \tfrac{1}{2}(l_{\mu;\nu} n^\mu l^\nu - m_{\mu;\nu} \overline{m}^\mu l^\nu), & \gamma &= -\tfrac{1}{2}(n_{\mu;\nu} l^\mu n^\nu - \overline{m}_{\mu;\nu} m^\mu n^\nu). \end{aligned} \tag{2.3}$$

If spacetime is flat, it is convenient to introduce the null polar coordinates $x^0 \equiv u = t-r$, $x^1 \equiv r$, $x^2 \equiv \theta$, $x^3 \equiv \varphi$, and to choose the null tetrad adapted to the coordinate system — l^μ as the outward pointing null vector tangent to the null cone, n^μ as the inward null vector, and m^μ, $\bar{m}^\mu$ as vectors tangent to the sphere r = const, u = const. Then the line-element takes the form

$$ds^2 = du^2+2dudr-r^2(d\theta^2+\sin^2\theta d\varphi^2),$$

the components of the null tetrad become

$$l^\mu = \delta^\mu_1, \quad n^\mu = \delta^\mu_0-\tfrac{1}{2}\delta^\mu_1,$$

$$m^\mu = \frac{1}{\sqrt{2}\,r}\left[\delta^\mu_2+\frac{i}{\sin\theta}\delta^\mu_3\right], \tag{2.4}$$

and only four spin coefficients do not vanish:

$$\varrho = -\frac{1}{r}, \quad \mu = -\frac{1}{2r}, \quad \alpha = -\frac{1}{2\sqrt{2}\,r}\cot\theta, \quad \beta = \frac{1}{2\sqrt{2}\,r}\cot\theta. \tag{2.5}$$

The null tetrad is not determined uniquely. Defining the spin weight s of a quantity η by the requirement that $\eta \to \eta' = e^{is\chi}\eta$ under the transformation $m^\mu \to m^{\mu'} = e^{i\chi}m^\mu$, we introduce the differential operator $\eth$ and the complex conjugate operator $\bar{\eth}$ (see [16]):

$$\eth = -(\sin\theta)^s\left[\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right](\sin\theta)^{-s}\eta,$$

$$\bar{\eth} = -(\sin\theta)^s\left[\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right](\sin\theta)^{-s}\eta. \tag{2.6}$$

This angular operator enables us to form a complete set of the orthonormal functions ${}_sY_{lm}$ (spin-s spherical harmonics) for quantities of the spin-weight s:

$$\begin{aligned}{}_sY_{lm}(\theta,\varphi) &= [(l-s)!/(l+s)!]^{1/2}\eth^s Y_{lm}(\theta,\varphi), \quad \text{for} \quad 0 \leqslant s \leqslant l,\\ &= (-1)^s[(l+s)!/(l-s)!]^{1/2}\bar{\eth}^{-s}Y_{lm}(\theta,\varphi), \quad \text{for} \quad -l \leqslant s \leqslant 0;\end{aligned} \tag{2.7}$$

here ${}_0Y_{lm} = Y_{lm}$ are ordinary spherical harmonics and s, l, m are integrals, $l = 0, 1, 2, \ldots$, $|m| \leqslant l$, $|s| \leqslant l$; (the ${}_sY_{lm}$ are not defined for $|s| > l$, or $|m| > l$). From the definition of the spin-s spherical harmonics we deduce the useful relations

$${}_s\bar{Y}_{lm} = (-1)^{m+s}{}_{-s}Y_{l-m}, \tag{2.8}$$

$$\eth\,{}_sY_{lm} = [(l-s)(l+s+1)]^{1/2}{}_{s+1}Y_{lm}, \tag{2.9}$$

$$\bar{\eth}\,{}_sY_{lm} = -[(l+s)(l-s+1)]^{1/2}{}_{s-1}Y_{lm}, \tag{2.10}$$

$$\bar{\eth}\eth\,{}_sY_{lm} = -(l-s)(l+s+1)\,{}_sY_{lm}. \tag{2.11}$$

Note also that any quantity η with the spin weight s satisfies

$$(\bar{\eth}\eth-\eth\bar{\eth})\,\eta = 2s\eta. \tag{2.12}$$

Provided that ξ and ζ have the spin weight $-l-1$ and $l+1$ respectively, one can derive

$$\int {}_sY_{lm}\eth^{l-s+1}\xi d\Omega = 0 = \int {}_s\bar{Y}_{lm}\bar{\eth}^{l-s+1}\zeta d\Omega, \tag{2.13}$$

$d\Omega = \sin\theta d\theta\, d\varphi$, integrations being carried over a sphere, $r = \text{const}$. Further, regarding (2.11) it follows that

$$\int {}_s\bar{Y}_{lm}\bar{\eth}\eth\eta d\Omega = -(l-s)(l+s+1)\int {}_s\bar{Y}_{lm}\eta d\Omega, \tag{2.14}$$

with η having the spin weight s. The last relation we will need in the following, is the expansion of the product of two spin spherical harmonics [17],

$$ {}_{s_1}Y_{l_1m_1}(\theta,\varphi)_{s_2}Y_{l_2m_2}(\theta,\varphi)$$

$$= \sum_l \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}\right]^{1/2} {}_sY_{lm}(\theta,\varphi)\,\langle l_1, l_2; m_1, m_2|l, m\rangle\,\langle l_1, l_2; -s_1, -s_2|l, -s\rangle,$$

where $\langle l_1, l_2; m_1, m_2|l, m\rangle$ is the Clebsch-Gordan coefficient, $m = m_1+m_2$, $s = s_1+s_2$, $|l_1-l_2| \leqslant l \leqslant l_1+l_2$. The preceding relation will in fact only be used for $m_1 = m_2 = 0$; it is convenient to write it in the form

$$ {}_{s_1}Y_{l_10\,s_2}Y_{l_20} = \sum_l (-1)^{s_1+s_2}\frac{1}{\sqrt{4\pi}}\left[(2l_1+1)(2l_2+1)(2l+1)\right]^{1/2}$$

$$\times \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l \\ -s_1 & -s_2 & s_1+s_2\end{pmatrix} {}_{s_1+s_2}Y_{l0}, \tag{2.15}$$

where the Clebsch-Gordan coefficients are replaced by the Wigner 3-j symbols, defined by

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3\end{pmatrix} = (-1)^{l_1-l_2-m_3}(2l_3+1)^{-1/2}\langle l_1, l_2; m_1, m_2|l_3, -m_3\rangle.$$

3. The field equations of non-linear electrodynamics in the Newman-Penrose form and their spherically symmetric solutions

For the time being we describe the electromagnetic field by the Maxwell tensor $F_{\mu\nu}$, formed from a potential, $F_{\mu\nu} = A_{\nu;\mu}-A_{\mu;\nu} = A_{\nu,\mu}-A_{\mu,\nu}$. Equivalently, we assume the equations

$$F^{*\mu\nu}{}_{;\nu} = 0, \tag{3.1}$$

where $F^*_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu}{}^{\varrho\sigma}F_{\varrho\sigma}$, to be satisfied. In a general case Lagrangian L of non-linear electrodynamics is supposed to be the scalar function of the invariants $F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $G = \frac{1}{4}F_{\mu\nu}F^{*\mu\nu}$. The field equations derived in a standard way from the variational principle take the form

$$P^{\mu\nu}{}_{;\nu} = 0, \tag{3.2}$$

where $P^{\mu\nu} = L_F F^{\mu\nu} + L_G F^{*\mu\nu}$; henceforth, we use the abbreviations $L_F = \partial L/\partial F$, $L_G = \partial L/\partial G$, $L_{FF} = \partial^2 L/\partial F^2$, etc. Since only $F^{\mu\nu}$ are considered as fundamental variables, both (3.2) and (3.1) must be taken into account. In general we will not restrict the form of L except for the requirements which guarantee that there exists a correspondence with the Maxwell theory (where $L = F$). For weak fields, when F and G are small, we require that $L = F + O(F^2, G^2)$, i. e. $L_F(0, 0) = 1$, $L_G(0, 0) = 0$. If we wish our theory to be invariant under reflections, we must assume $L = L(F, G^2)$, because G is a pseudo-invariant.

Now we want to rewrite the equations (3.2) and (3.1) in terms of the Φ's given by (2.1), the differential operators D, δ, Δ by (2.2), and the spin coefficients by (2.3). This can be accomplished conveniently by translating the tensor equations first into the spinor form, and, thereafter, with the help of basic spinors (a "dyad") corresponding to the null tetrad l^μ, n^μ, m^μ, $\bar{m}^\mu$, by going over into the NP-form. This procedure, similar to that described in detail by Newman and Penrose [4] for the Einstein equations, yields the following system:

$$\begin{aligned} &2L_{\mathrm{I}}[D\Phi_1 - \bar{\delta}\Phi_0 - (\pi - 2\alpha)\Phi_0 - 2\varrho\Phi_1 + \kappa\Phi_2] \\ &= L_{\mathrm{II}}[\Phi_0\bar{\delta}\bar{\mathscr{A}} - \Phi_1 D\bar{\mathscr{A}}] + L_{\mathrm{III}}[\Phi_0\bar{\delta}\mathscr{A} - \Phi_1 D\mathscr{A}] \\ &\quad + L_{\mathrm{II}}[\bar{\Phi}_0\delta\mathscr{A} - \bar{\Phi}_1 D\mathscr{A}] + \bar{L}_{\mathrm{III}}[\bar{\Phi}_0\delta\bar{\mathscr{A}} - \bar{\Phi}_1 D\bar{\mathscr{A}}], \end{aligned} \tag{3.3}$$

$$\begin{aligned} &2L_{\mathrm{I}}[D\Phi_2 - \bar{\delta}\Phi_1 + \lambda\Phi_0 - 2\pi\Phi_1 - (\varrho - 2\varepsilon)\Phi_2] = L_{\mathrm{II}}[\Phi_1\bar{\delta}\bar{\mathscr{A}} - \Phi_2 D\bar{\mathscr{A}}] \\ &+ L_{\mathrm{III}}[\Phi_1\bar{\delta}\mathscr{A} - \Phi_2 D\mathscr{A}] + L_{\mathrm{II}}[\bar{\Phi}_0\Delta\mathscr{A} - \bar{\Phi}_1\bar{\delta}\mathscr{A}] + \bar{L}_{\mathrm{III}}[\bar{\Phi}_0\Delta\bar{\mathscr{A}} - \bar{\Phi}_1\bar{\delta}\bar{\mathscr{A}}], \end{aligned} \tag{3.4}$$

$$\begin{aligned} &2L_{\mathrm{I}}[\delta\Phi_1 - \Delta\Phi_0 - (\mu - 2\gamma)\Phi_0 - 2\tau\Phi_1 + \sigma\Phi_2] = L_{\mathrm{II}}[\Phi_0\Delta\bar{\mathscr{A}} - \Phi_1\delta\bar{\mathscr{A}}] \\ &+ L_{\mathrm{III}}[\Phi_0\Delta\mathscr{A} - \Phi_1\delta\mathscr{A}] + L_{\mathrm{II}}[\bar{\Phi}_1\delta\mathscr{A} - \bar{\Phi}_2 D\mathscr{A}] + \bar{L}_{\mathrm{III}}[\bar{\Phi}_1\delta\bar{\mathscr{A}} - \Phi_2 D\bar{\mathscr{A}}], \end{aligned} \tag{3.5}$$

$$\begin{aligned} &2L_{\mathrm{I}}[\delta\Phi_2 - \Delta\Phi_1 + \nu\Phi_0 - 2\mu\Phi_1 - (\tau - 2\beta)\Phi_2] = L_{\mathrm{II}}[\Phi_1\Delta\bar{\mathscr{A}} - \Phi_2\delta\bar{\mathscr{A}}] + \\ &+ L_{\mathrm{III}}[\Phi_1\Delta\mathscr{A} - \Phi_2\delta\mathscr{A}] + L_{\mathrm{II}}[\bar{\Phi}_1\Delta\mathscr{A} - \bar{\Phi}_2\bar{\delta}\mathscr{A}] + \bar{L}_{\mathrm{III}}[\bar{\Phi}_1\Delta\bar{\mathscr{A}} - \bar{\Phi}_2\bar{\delta}\bar{\mathscr{A}}]; \end{aligned} \tag{3.6}$$

here

$$\begin{aligned} \mathscr{A} &= \tfrac{1}{2}(F + G) = \Phi_0\Phi_2 - \Phi_1^2, \\ L_{\mathrm{I}} &= L_F, \quad L_{\mathrm{II}} = L_{FF} - L_{GG}, \\ L_{\mathrm{III}} &= L_{FF} + 2L_{FG} + L_{GG}, \\ \bar{L}_{\mathrm{III}} &= L_{FF} - 2L_{FG} + L_{GG}. \end{aligned} \tag{3.7}$$

The system (3.3)–(3.6) represents the NP form of the free-field equations of non-linear electrodynamics in a curved spacetime. (Thus, it also describes how the electromagnetic field is influenced by — possibly its own — gravitational field.) For $L = F$ the NP form of the Maxwell equations in general relativity is retrieved (cf. Eqs (A1) in [4]).

Hereafter, however, we shall confine ourselves to the study of the electromagnetic field in flat spacetime. Then, as mentioned in Section 2, it is convenient to introduce null polar coordinates and the null tetrad (2.4), so that the only non-vanishing spin coefficients are given by (2.5). The field equations (3.3)–(3.6) simplify into the following form:

$$
\begin{aligned}
&2L_{\mathrm{I}}\left[\partial_r\Phi_1+\frac{2}{r}\Phi_1+\frac{1}{\sqrt{2}\,r}\bar{\eth}\Phi_0\right]\\
&=-L_{\mathrm{II}}\left[\frac{1}{\sqrt{2}\,r}\Phi_0\bar{\eth}\bar{\mathscr{A}}+\Phi_1\partial_r\bar{\mathscr{A}}\right]-L_{\mathrm{III}}\left[\frac{1}{\sqrt{2}\,r}\Phi_0\bar{\eth}\mathscr{A}+\Phi_1\partial_r\mathscr{A}\right]\\
&\quad-L_{\mathrm{II}}\left[\frac{1}{\sqrt{2}\,r}\bar{\Phi}_0\eth\mathscr{A}+\bar{\Phi}_1\partial_r\mathscr{A}\right]-\bar{L}_{\mathrm{III}}\left[\frac{1}{\sqrt{2}\,r}\bar{\Phi}_0\eth\bar{\mathscr{A}}+\bar{\Phi}_1\partial_r\bar{\mathscr{A}}\right],
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&2L_{\mathrm{I}}\left[\partial_r\Phi_2+\frac{1}{r}\Phi_2+\frac{1}{\sqrt{2}\,r}\bar{\eth}\Phi_1\right]\\
&=-L_{\mathrm{II}}\left[\frac{1}{\sqrt{2}\,r}\Phi_1\bar{\eth}\bar{\mathscr{A}}+\Phi_2\partial_r\bar{\mathscr{A}}\right]-L_{\mathrm{III}}\left[\frac{1}{\sqrt{2}\,r}\Phi_1\bar{\eth}\mathscr{A}+\Phi_2\partial_r\mathscr{A}\right]\\
&\quad+L_{\mathrm{II}}\left[\bar{\Phi}_0\left(\partial_u-\frac{1}{2}\partial_r\right)\mathscr{A}+\frac{1}{\sqrt{2}\,r}\bar{\Phi}_1\bar{\eth}\mathscr{A}\right]+\bar{L}_{\mathrm{III}}\left[\bar{\Phi}_0\left(\partial_u-\frac{1}{2}\partial_r\right)\bar{\mathscr{A}}+\frac{1}{\sqrt{2}\,r}\bar{\Phi}_1\bar{\eth}\bar{\mathscr{A}}\right],
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&-2L_{\mathrm{I}}\left[\left(\partial_u-\frac{1}{2}\partial_r-\frac{1}{2r}\right)\Phi_0+\frac{1}{\sqrt{2}\,r}\eth\Phi_1\right]\\
&=L_{\mathrm{II}}\left[\Phi_0\left(\partial_u-\frac{1}{2}\partial_r\right)\mathscr{A}+\frac{1}{\sqrt{2}\,r}\Phi_1\eth\bar{\mathscr{A}}\right]+L_{\mathrm{III}}\left[\Phi_0\left(\partial_u-\frac{1}{2}\partial_r\right)\mathscr{A}+\frac{1}{\sqrt{2}\,r}\Phi_1\eth\mathscr{A}\right]\\
&\quad-L_{\mathrm{II}}\left[\frac{1}{\sqrt{2}\,r}\bar{\Phi}_1\eth\mathscr{A}+\bar{\Phi}_2\partial_r\mathscr{A}\right]-L_{\mathrm{III}}\left[\frac{1}{\sqrt{2}\,r}\bar{\Phi}_1\eth\bar{\mathscr{A}}+\bar{\Phi}_2\partial_r\bar{\mathscr{A}}\right],
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&-2L_{\mathrm{I}}\left[\left(\partial_u-\frac{1}{2}\partial_r-\frac{1}{r}\right)\Phi_1+\frac{1}{\sqrt{2}\,r}\eth\Phi_2\right]\\
&=L_{\mathrm{II}}\left[\Phi_1\left(\partial_u-\frac{1}{2}\partial_r\right)\bar{\mathscr{A}}+\frac{1}{\sqrt{2}\,r}\Phi_2\eth\bar{\mathscr{A}}\right]+L_{\mathrm{III}}\left[\Phi_1\left(\partial_u-\frac{1}{2}\partial_r\right)\mathscr{A}+\frac{1}{\sqrt{2}\,r}\Phi_2\eth\mathscr{A}\right]\\
&\quad+L_{\mathrm{II}}\left[\bar{\Phi}_1\left(\partial_u-\frac{1}{2}\partial_r\right)\mathscr{A}+\frac{1}{\sqrt{2}\,r}\bar{\Phi}_2\bar{\eth}\mathscr{A}\right]+L_{\mathrm{III}}\left[\bar{\Phi}_1\left(\partial_u-\frac{1}{2}\partial_r\right)\bar{\mathscr{A}}+\frac{1}{\sqrt{2}\,r}\bar{\Phi}_2\bar{\eth}\bar{\mathscr{A}}\right].
\end{aligned}
\tag{3.11}
$$

Although our equations look rather cumbersome even in flat spacetime, there exist simple cases, in which the solutions can easily be found.

Let us first observe that null-field solutions, characterized by $F = G = 0$ (i. e. $\mathscr{A} = 0$), are common to all theories of the electromagnetic field, based on Lagrangians depending of F and G. (Any non-linear electrodynamics thus admits the plane wave of the Maxwell theory, for example.) Going back to (3.3)–(3.6) we see that the same is also true on a curved background.

Now we turn to spherically symmetric solutions in flat spacetime. The stationarity of the field will not be assumed — we will be able to prove it in all physically plausible theories, as a consequence of the field equations (cf. the Birkhoff theorem in general relativity).

Writing the Φ's in terms of the field strengths $\boldsymbol{E}$ and $\boldsymbol{B}$, it can easily be seen that the assumption of spherical symmetry implies the vanishing of Φ_0 and Φ_2. Therefore, the equations (3.8)–(3.11) reduce to two equations for $\Phi_1 = \Phi_1(u, r)$:

$$2L_{\mathrm{I}}\left(\partial_r\Phi_1 + \frac{2}{r}\Phi_1\right) = -L_{\mathrm{II}}\Phi_1\partial_r\bar{\mathscr{A}} - L_{\mathrm{III}}\Phi_1\partial_r\mathscr{A} - L_{\mathrm{II}}\bar{\Phi}_1\partial_r\mathscr{A} - \bar{L}_{\mathrm{III}}\bar{\Phi}_1\partial_r\bar{\mathscr{A}}, \tag{3.12}$$

$$2L_{\mathrm{I}}\partial_u\Phi_1 = L_{\mathrm{II}}\Phi_1\partial_u\bar{\mathscr{A}} + L_{\mathrm{III}}\Phi_1\partial_u\mathscr{A} + L_{\mathrm{II}}\bar{\Phi}_1\partial_u\mathscr{A} + \bar{L}_{\mathrm{III}}\bar{\Phi}_1\partial_u\bar{\mathscr{A}}. \tag{3.13}$$

(To get the last equation, we multiplied (3.11) by $\frac{1}{2}$, and subtracted it from (3.8).) Taking these equations complex conjugated and comparing them with the unconjugated form, we obtain

$$\partial_r(\Phi_1 - \bar{\Phi}_1) + \frac{2}{r}(\Phi_1 - \bar{\Phi}_1) = 0,$$

and

$$\partial_u(\Phi_1 - \bar{\Phi}_1) = 0,$$

so that

$$\Phi_1 = \varphi + i\frac{Q}{2r^2}, \tag{3.14}$$

where φ is a real function of u and r, and Q is a real constant.

It is now convenient to go over from F and G to the real variables $\mathscr{E}$ and $\mathscr{B}$ given by[2]

$$F = \tfrac{1}{2}(\mathscr{B}^2 - \mathscr{E}^2),\ G = -i\mathscr{E}\mathscr{B}; \tag{3.15}$$

from here, conversely,

$$\mathscr{E} = (|F+G| - F)^{1/2}, \quad \mathscr{B} = \operatorname{sgn}(iG)\,(|F+G| + F)^{1/2}.$$

By straightforward calculations, we can express Φ_1 in terms of $\mathscr{E}$ and $\mathscr{B}$,

$$\Phi_1 = \tfrac{1}{2}(\mathscr{E} + i\mathscr{B}), \tag{3.16}$$

and, thereafter, also the form of Eqs. (3.12) and (3.13):

$$L_{\mathscr{E}\mathscr{E}}\partial_r\mathscr{E} + L_{\mathscr{E}}\frac{2}{r} - L_{\mathscr{E}\mathscr{B}}\frac{2\mathscr{B}}{r} = 0, \tag{3.17}$$

$$L_{\mathscr{E}\mathscr{E}}\partial_u\mathscr{E} = 0. \tag{3.18}$$

[2] The introduction of $\mathscr{E}$ and $\mathscr{B}$ proved to be convenient also in other circumstances investigated by Plebański [12]. In the spherically symmetric case $\Phi_1 = \frac{1}{2}(\mathscr{E} + i\mathscr{B}) = \frac{1}{2}(\boldsymbol{E} + i\boldsymbol{B})\,\boldsymbol{C}$, where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the usual field strengths, $\boldsymbol{C}$ is a unit radial vector.

Since, by (3.14) and (3.16), $\mathscr{B} = Qr^{-2}$, (3.17) yields $\partial_r(r^2 L_{\mathscr{E}}) = 0$, from which

$$L_{\mathscr{E}} = \frac{q(u)}{r^2}, \tag{3.19}$$

$q(u)$ being an arbitrary real function of u.

We may now distinguish three cases:

1. Purely electric solutions ($\mathscr{B} = 0$)

a) $L_{\mathscr{E}\mathscr{E}}(\mathscr{E}, \mathscr{B} = 0) \neq 0$. (3.18) and (3.19) imply $q = \text{const}$. The solution is given implicitly by $L_{\mathscr{E}}(\mathscr{E}, 0) = qr^{-2}$. From this it can be easily observed that only non-polynomial Lagrangians can lead to well-behaved fields in the origin. In particular, in the Born-Infeld theory $\mathscr{E} = q(r^4+r_0^4)^{-1/2}$, where $r_0 = (|q|b^{-1})^{1/2}$.

b) $L_{\mathscr{E}\mathscr{E}}(\mathscr{E}, 0) = 0$ identically. Then $L(\mathscr{E}, 0) = K_1 + K_2\mathscr{E}$, K_1 and K_2 being constant. If $K_2 = 0$, then the field equations do not impose any condition on $\mathscr{E}$. However, the respective Lagrangians do not satisfy the correspondence principle. If $K_2 \neq 0$, no solution exists.

2. Purely magnetic solutions ($\mathscr{E} = 0$)

(3.18) holds identically and (3.17) reduces to $L_{FG}(F, 0) = 0$. Every Lagrangian invariant with respect to reflections may contain only even powers of G, thus leading to purely magnetic solutions.

3. "Mixed" solutions ($\mathscr{E} \neq 0$, $\mathscr{B} \neq 0$)

a) $L_{\mathscr{E}\mathscr{E}} \neq 0$. (3.18) and (3.19) yield $L_{\mathscr{E}}(\mathscr{E}, \mathscr{B}) = qr^{-2}$, q being a constant. In the Born-Infeld theory, the spherically symmetric solution representing both electric and magnetic monopoles has the form

$$\mathscr{E} = \frac{q}{r^2}\left(1 + \frac{Q^2}{b^2r^4}\right)^{1/2}\left[\left(1 - \frac{Q^2}{b^2r^4}\right)^2 + \left(1 - \frac{Q^2}{b^2r^4}\right)\frac{q^2}{b^2r^4}\right]^{-1/2},$$

$$\mathscr{B} = \frac{Q}{r^2}.$$

b) $L_{\mathscr{E}\mathscr{E}} = 0$ identically. Eq. (3.17) leads to pathological Lagrangians of the form $L(\mathscr{E}, \mathscr{B}) = \mathscr{K}_1(\mathscr{B}) + \mathscr{K}_2(\mathscr{B})\mathscr{E}$, where $\mathscr{K}_1$ and $\mathscr{K}_2$ are arbitrary functions. If $\mathscr{K}_2(\mathscr{B}) = K_2\mathscr{B}$, with $K_2 = \text{const.}$, $\mathscr{E}$ can be an arbitrary function of u and r. For other choices of $\mathscr{K}_2(\mathscr{B})$, no solution for $\mathscr{E}$ exists.

Summarizing, we see that non-linear electrodynamics does not imply the static character of spherically symmetric solutions in general (cf. 1b and 3b) but in physically reasonable cases, in which the correspondence principle with the Maxwell theory is satisfied, the spherical symmetric solutions have to be static modulo the field equations.

In order to be able to study other solutions of the equations (3.8)–(3.11) we have to turn to approximation methods. Before analyzing the approximation technique used in this paper, it will be useful to discuss briefly the Maxwell electrodynamics with sources in the framework of the NP formalism.

4. The Maxwell electrodynamics with sources

In a manner similar to going over from the tensor equations (3.1), (3.2) to the NP form (3.8)–(3.11) in the case of vacuum non-linear electrodynamics, we can find the NP form of the equations of the inhomogeneous Maxwell electrodynamics in flat space. Applying Heaviside's units, we obtain the system

$$\partial_r \Phi_1 + \frac{2}{r}\Phi_1 = -\frac{1}{\sqrt{2}\,r}\bar{\eth}\Phi_0 - J_0, \tag{4.1}$$

$$\partial_r \Phi_2 + \frac{1}{r}\Phi_2 = -\frac{1}{\sqrt{2}\,r}\bar{\eth}\Phi_1 - \bar{J}_1, \tag{4.2}$$

$$\left(\partial_u - \frac{1}{2}\partial_r - \frac{1}{2r}\right)\Phi_0 = -\frac{1}{\sqrt{2}\,r}\eth\Phi_1 + J_1, \tag{4.3}$$

$$\left(\partial_u - \frac{1}{2}\partial_r - \frac{1}{r}\right)\Phi_1 = -\frac{1}{\sqrt{2}\,r}\eth\Phi_2 + J_2, \tag{4.4}$$

where $J_0 = \frac{1}{2}J_\mu l^\mu$, $J_1 = \frac{1}{2}J_\mu m^\mu$, and $J_2 = \frac{1}{2}J_\mu n^\mu$ are (up to $\frac{1}{2}$) the null-tetrad components of the four-current J^μ. In this section we assume these source terms to be given. (Note that these terms have opposite signs than used conventionally, so that the charge of the electron is positive.)

If we assume the usual asymptotical conditions $\Phi_0 \sim O(r^{-3})$, $\bar{\eth}\Phi_0 \sim O(r^{-3})$, $\bar{\eth}\bar{\eth}\Phi_0 \sim O(r^{-3})$ (cf. [18]), which correspond to imposing the outgoing radiation condition, Eqs (4.1) and (4.2) may be integrated to yield

$$\Phi_1 = \frac{1}{r^2}\Phi_1^0(u,\theta,\varphi) - \frac{1}{\sqrt{2}\,r^2}\int_\infty^r r'\bar{\eth}\Phi_0 dr' - \frac{1}{r^2}\int_\infty^r r'^2 J_0 dr', \tag{4.5}$$

$$\Phi_2 = \frac{1}{r}\Phi_2^0(u,\theta,\varphi) - \frac{1}{\sqrt{2}\,r}\int_\infty^r \bar{\eth}\Phi_1 dr' - \frac{1}{r}\int_\infty^r r'\bar{J}_1 dr', \tag{4.6}$$

where Φ_1^0, Φ_2^0 are functions of integration. For the solutions to be well-behaved we have to require that

$$J_0 \sim O(r^{-4}), \quad \bar{\eth}J_0 \sim O(r^{-4}), \quad J_1 \sim O(r^{-3}); \tag{4.7}$$

this is, of course, fulfilled for insular sources. From (4.5) and (4.6), it is immediately seen that $\Phi_1 \sim O(r^{-2})$ and $\Phi_2 \sim O(r^{-1})$ — the statement of the "peeling-off" theorem well-known from the vacuum linear theory [18]. (Owing to the conditions (4.7) its proof does not require the insular character of the source.)

As a consequence of (4.5) and (4.6) we may write (4.3) and (4.4) in terms of Φ_0, Φ_1^0 and Φ_2^0 only. After some rearrangements (using (2.12) for $s = 0$) we obtain

$$\partial_u\Phi_0 - \frac{1}{2}\partial_r\Phi_0 - \frac{1}{2r}\Phi_0 - \frac{1}{2r^3}\eth\bar{\eth}\int_\infty^r r'\Phi_0 dr'$$

$$= -\frac{1}{\sqrt{2}\,r^3}\eth\Phi_1^0 + \frac{1}{\sqrt{2}\,r^3}\eth\int_\infty^r r'^2 J_0 dr' + J_1, \tag{4.8}$$

and

$$\frac{1}{r^2}\left(\partial_u\Phi_1^0 + \frac{1}{\sqrt{2}}\eth\Phi_2^0\right)$$

$$-\frac{1}{\sqrt{2}\,r^2}\bar{\eth}\int_\infty^r\left[r'\partial_u\Phi_0 - \frac{1}{2}\partial_{r'}(r'\Phi_0) - \frac{1}{2}\frac{\eth\bar{\eth}}{r'^2}\int_\infty^{r'} r''\Phi_0 dr''\right.$$

$$\left. + \frac{1}{\sqrt{2}\,r'^2}\eth\Phi_1^0 - \frac{1}{\sqrt{2}\,r'^2}\eth\int_\infty^{r'} r''^2 J_0 dr'' - r'J_1\right]dr'$$

$$-\frac{1}{r^2}\int_\infty^r\left[r'^2\partial_u J_0 - \frac{1}{2}\partial_{r'}(r'^2 J_0)\right]dr' - \frac{1}{\sqrt{2}\,r^2}\int_\infty^r r'[\eth\bar{J}_1 + \bar{\eth}J_1]dr' - J_2 = 0. \tag{4.9}$$

Regarding (4.8) multiplied by r, the last equation reduces to

$$\frac{1}{r^2}\left(\partial_u\Phi_1^0 + \frac{1}{\sqrt{2}}\eth\Phi_2^0\right)$$

$$-\frac{1}{r^2}\int_\infty^r\left[r'^2\partial_u J_0 - \frac{1}{2}\partial_{r'}(r'^2 J_0)\right]dr' - \frac{1}{\sqrt{2}\,r^2}\int_\infty^r r'[\eth\bar{J}_1 + \bar{\eth}J_1]dr' - J_2 = 0. \tag{4.10}$$

This equation can be simplified further if we realize that the continuity equation in the NP formalism reads

$$\left(\partial_u - \frac{1}{2}\partial_r - \frac{1}{r}\right)J_0 + \frac{1}{\sqrt{2}\,r}(\bar{\eth}J_1 + \eth\bar{J}_1) + \left(\partial_r + \frac{2}{r}\right)J_2 = 0, \tag{4.11}$$

so that (after integration)

$$J_2 = \frac{1}{r^2}J_2^0(u,\theta,\varphi) - \frac{1}{r^2}\int_\infty^r r'^2\left(\partial_u - \frac{1}{2}\partial_{r'} - \frac{1}{r'}\right)J_0 dr'$$

$$-\frac{1}{\sqrt{2}\,r^2}\int_\infty^r r'(\bar{\eth}J_1 + \eth\bar{J}_1)dr', \tag{4.12}$$

where J_2^0 is an integration function. As a result, (4.10) yields

$$\partial_u \Phi_1^0 + \frac{1}{\sqrt{2}} \eth \Phi_2^0 = J_2^0. \tag{4.13}$$

Now, instead of the original system (4.1)–(4.4), we adopt the equivalent system of Eqs. (4.5), (4.6), (4.8), and (4.13).

The integro-differential Eq. (4.8) can be solved by the expansions in spin-s spherical harmonics (cf. [19] for weak gravitational fields). At first we assume

$$\begin{aligned} \Phi_0 &= \sum g_{0lm\ 1}Y_{lm}, \quad \Phi_1^0 = \sum g_{1lm\ 0}^0 Y_{lm}, \\ J_0 &= \sum h_{0lm\ 0}Y_{lm}, \quad J_1 = \sum h_{1lm\ 1}Y_{lm}, \end{aligned} \tag{4.14}$$

and introduce new quantities x_{lm} by the Couch substitution [20]

$$g_{0lm} = r^{l-1}D^{l+1}\left(\frac{x_{lm}}{r^l}\right), \quad D = \partial_r; \tag{4.15}$$

(the substitution is suitable owing to our asymptotic conditions). Then, with the help of (4.15) and of the relation $D\{r^{l+1}D^l(F/r)\} = r^l D^{l+1}F$ (proof by induction), Eq. (4.8) can be adjusted to read

$$\partial_u x_{lm} - \tfrac{1}{2} D x_{lm} = r^l \underbrace{\int \dots \int}_{l+1} H_{lm}(dr)^{l+1},$$

where H_{lm} is given by

$$r^{l-1}H_{lm} = -\frac{1}{r^3}\left[\frac{l}{2}(l+1)\right]^{1/2} g_{1lm}^0 + \frac{1}{r^3}\left[\frac{l}{2}(l+1)\right]^{1/2} \int_\infty^r r'^2 h_{0lm} dr' + h_{1lm}. \tag{4.16}$$

Changing the variables u and r into u and $v = -u - 2r$, we arrive at

$$x_{lm} = \int_{u_0}^{u} [r^l \underbrace{\int \dots \int}_{l+1} H_{lm}(dr)^{l+1}] du' + x_{lm}^0(v). \tag{4.17}$$

(While integrating we put all functions of integration equal to zero, because they do not contribute to g_{0lm}.) Now it is simple to solve the remaining Eqs. (4.5), (4.6), and (4.13). Knowing $\Phi_2^0(u, \theta, \varphi)$ — i. e. the "news function" (cf. [18]) — and $\Phi_1^0(u = u_0, \theta, \varphi)$, Eq. (4.13) determines Φ_1^0 at all times $u > u_0$. From Eq. (4.8) and (4.17), the function Φ_0 can be found at all times $u > u_0$ provided that $\Phi_0(u = u_0, r, \theta, \varphi)$ is given. Finally, Eqs. (4.5) and (4.6) yield Φ_1 and Φ_2.

In particular, the solution representing the incoming 2^l-pole can be found by taking $\Phi_2^0 = \Phi_1^0(u = u_0) = 0$, so that, by (4.13), $\Phi_1^0 = 0$ and, therefore, $H_{lm} = 0$. The resulting field has the form

$$\Phi_A = 2^{\frac{1-A}{2}} \left[\frac{(l+A-1)!}{(l-A+1)!}\right]^{1/2} r^{l-1} D^{l+1-A} \left[\frac{x_{lm}(v)}{r^{l+A}}\right]_{1-A} Y_{lm}, \quad A = 0, 1, 2. \tag{4.18}$$

By performing the inversion of time we can obtain the outgoing 2^l-pole field:

$$\Phi_A = 2^{\frac{1-A}{2}}\left[\frac{(l-A+1)!}{(l+A-1)!}\right]^{1/2} r^{l-1} d^{l-1+A}\left[\frac{x_{lm}(u)}{r^{l+2-A}}\right]{}_{1-A}Y_{lm}, \tag{4.19}$$

where $d \equiv -2\partial_u+\partial_r$. Both (4.18) and (4.19) will be needed in § 7.

5. The Newman-Penrose conservation laws

In this section we wish to discuss the analogues of the conserved quantities discovered by Newman and Penrose [7], [8] for linear zero rest-mass fields and for the gravitational field in asymptotically flat spacetimes. In order to see how many conservation laws of this type a particular non-linear theory of electromagnetic field yields, we will keep the form of the Lagrangian as general as possible.

At first let us cast the non-linear field equations (3.8)–(3.11) into "linear" form (4.1)–(4.4), in which, of course, the "source terms" J_0, J_1, $\bar{J}_1$, J_2 will now involve the field itself. To simplify Eqs (3.8)–(3.11), we multiply them by the factor $\frac{1}{2}\lambda$. In some cases, this factor can be chosen in such a way that the terms $\lambda L_{\rm I}$, $\lambda L_{\rm II}$, and $\lambda L_{\rm III}$ are polynomial in F and G so that the resulting form of the field equations is polynomial in the Φ's, although the Lagrangian L may be non-polynomial. For example, this can be done for Lagrangians like $L = (P_1/P_2)^\alpha$, $L = \exp\,(P_1/P_2)^\alpha$, or $L = \log\,(P_1/P_2)$, in which P_1 and P_2 are polynomials in F and G, and α is an arbitrary real number. (In particular, $\lambda = [1+(2/b^2)F+(1/b^4)\,G^2]^{3/2}$ in the Born-Infeld theory.) Hereafter, we shall restrict ourselves to Lagrangians of this kind and, moreover, to all Lagrangians which lead to power series (in F and G) for $\lambda L_{\rm I}$, $\lambda L_{\rm II}$, and $\lambda L_{\rm III}$. These Lagrangians will then be determined by the expansion coefficients in these series.

Now we formally write $(\lambda/2)\,2L_{\rm I} = \lambda L_{\rm I}-1+1$ and transfer the terms with $(\lambda L_{\rm I}-1)$ in (3.8)–(3.11) to the right-hand sides. Realizing that for any Lagrangian of our type, satisfying the correspondence principle ($L_F(0,0) = 1$), there exists such a λ that the term $\lambda L_{\rm I}-1$ starts, at least, with linear terms in F and G, we convert (3.8)–(3.11) into the form of (4.1)–(4.4), where the J's contain, at least, terms cubic in the Φ's.

To find NP conserved quantities we need explicit expressions for J_0 and J_1 only. These read as follows:

$$\begin{aligned} J_0 &= (\lambda L_{\rm I}-1)\left[\partial_r\Phi_1+\frac{2}{r}\Phi_1+\frac{1}{\sqrt{2}\,r}\bar{\eth}\Phi_0\right] \\ &+\frac{1}{2}\lambda\left\{L_{\rm II}\left[\Phi_1\partial_r\bar{\mathscr{A}}+\frac{1}{\sqrt{2}\,r}\Phi_0\bar{\eth}\bar{\mathscr{A}}\right]+L_{\rm III}\left[\Phi_1\partial_r\mathscr{A}+\frac{1}{\sqrt{2}\,r}\Phi_0\bar{\eth}\mathscr{A}\right]\right. \\ &\left.+L_{\rm II}\left[\bar{\Phi}_1\partial_r\mathscr{A}+\frac{1}{\sqrt{2}\,r}\bar{\Phi}_0\bar{\eth}\mathscr{A}\right]+\bar{L}_{\rm III}\left[\bar{\Phi}_1\partial_r\bar{\mathscr{A}}+\frac{1}{\sqrt{2}\,r}\bar{\Phi}_0\bar{\eth}\bar{\mathscr{A}}\right]\right\}, \end{aligned} \tag{5.1}$$

$$J_1 = -(\lambda L_{\mathrm{I}}-1)\left[\partial_u\Phi_0-\frac{1}{2}\partial_r\Phi_0-\frac{1}{2r}\Phi_0+\frac{1}{\sqrt{2}\,r}\bar{\eth}\Phi_1\right]$$

$$-\frac{1}{2}\lambda\left\{L_{\mathrm{II}}\left[\Phi_0\left(\partial_u-\frac{1}{2}\partial_r\right)\bar{\mathscr{A}}+\frac{1}{\sqrt{2}\,r}\Phi_1\bar{\eth}\bar{\mathscr{A}}\right]+L_{\mathrm{III}}\left[\Phi_0\left(\partial_u-\frac{1}{2}\partial_r\right)\mathscr{A}+\frac{1}{\sqrt{2}\,r}\Phi_1\bar{\eth}\mathscr{A}\right]\right.$$

$$\left.-L_{\mathrm{II}}\left[\bar{\Phi}_2\partial_r\bar{\mathscr{A}}+\frac{1}{\sqrt{2}\,r}\bar{\Phi}_1\bar{\eth}\bar{\mathscr{A}}\right]-\bar{L}_{\mathrm{III}}\left[\bar{\Phi}_2\partial_r\mathscr{A}+\frac{1}{\sqrt{2}\,r}\bar{\Phi}_1\bar{\eth}\mathscr{A}\right]\right\}. \tag{5.2}$$

($\bar{J}_1$ and J_2 may be found analogously. Note, however, that $\bar{J}_1$ is not identically equal to the complex conjugate of J_1; the equality holds for the solutions of the field equations.) Assuming now $\Phi_0 \sim O(r^{-3})$, $\partial_r\Phi_0 \sim O(r^{-4})$, $\eth\Phi_0 \sim O(r^{-3})$, $\bar{\eth}\eth\Phi_0 \sim O(r^{-3})$ (outgoing radiation condition) and the expansions of Φ_1 and Φ_2 in the inverse powers[3] of r, the inspection of Eqs. (4.1) and (4.2) (with the J's found as indicated) reveals that $\Phi_1 \sim O(r^{-2})$, $\Phi_2 \sim O(r^{-1})$. Consequently, for the asymptotic behaviour of the J's we get $J_0 \sim O(r^{-7})$, $J_1 \sim O(r^{-6})$, $\bar{J}_1 \sim O(r^{-6})$ and $J_2 \sim O(r^{-7})$. We may thus write

$$\Phi_0 = \sum_{n=0}^{\infty}\frac{\Phi_0^n(u,\theta,\varphi)}{r^{3+n}},\qquad \Phi_1 = \sum_{n=0}^{\infty}\frac{\Phi_1^n(u,\theta,\varphi)}{r^{2+n}},\qquad \Phi_2 = \sum_{n=0}^{\infty}\frac{\Phi_2^n(u,\theta,\varphi)}{r^{1+n}},$$

$$J_0 = \sum_{n=0}^{\infty}\frac{J_0^n(u,\theta,\varphi)}{r^{7+n}},\qquad J_1 = \sum_{n=0}^{\infty}\frac{J_1^n(u,\theta,\varphi)}{r^{6+n}},$$

$$\bar{J}_1 = \sum_{n=0}^{\infty}\frac{\bar{J}_1^n(u,\theta,\varphi)}{r^{6+n}},\qquad J_2 = \sum_{n=0}^{\infty}\frac{J_2^n(u,\theta,\varphi)}{r^{6+n}}. \tag{5.3}$$

Since the coefficient with r^{-2} in the expansion of J_2 vanishes, Eq. (4.13) takes the form

$$\dot{\Phi}_1^0 = -\frac{1}{\sqrt{2}}\,\eth\Phi_2^0, \tag{5.4}$$

with the dot denoting ∂_u. Multiplying this equation by ${}_0\bar{Y}_{00}$ and integrating over the sphere, we obtain (using (2.13) for $l = 0$)

$$\frac{d}{du}\int {}_0\bar{Y}_{00}\Phi_1^0 d\Omega = -\frac{1}{\sqrt{2}}\int {}_0\bar{Y}_{00}\eth\Phi_2^0 d\Omega = 0.$$

Thus, in any non-linear theory under consideration, the conservation law for the total charge, $\int {}_0\bar{Y}_{00}\Phi_1^0 d\Omega = \text{const.}$, holds in the same form as in the Maxwell theory. (The charge is complex: (electric)$+i$("magnetic").)

---

[3] It is actually sufficient to assume the expansions only up to r^{-N}, with N being a suitable integer.

In order to derive the NP conservation laws, we turn to Eq. (4.8) in which J_0 and J_1 are given by (5.1) and (5.2), respectively. The substitution of expansions (5.3) into (4.8) yields

$$\sum_{n=0}^{\infty} \frac{\dot{\Phi}_0^n}{r^{3+n}} + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(n+2)\Phi_0^n}{r^{4+n}} + \frac{1}{\sqrt{2}}\frac{\eth\Phi_1^0}{r^3} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\eth\bar{\eth}\Phi_0^n}{(n+1)r^{4+n}}$$

$$= -\frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\frac{\eth J_0^n}{(n+4)r^{7+n}} + \sum_{n=0}^{\infty}\frac{J_1^n}{r^{n+6}}.$$

After rearranging the terms and using (2.12) we arrive at

$$\dot{\Phi}_0^0 = -\frac{1}{\sqrt{2}}\eth\Phi_1^0, \tag{5.5}$$

and

$$(n+1)\dot{\Phi}_0^{n+1} = -\frac{1}{2}[\bar{\eth}\eth\Phi_0^n + n(n+3)\Phi_0^n] - \frac{1}{\sqrt{2}}\eth J_0^{n-3} + (n+1)J_1^{n-2}, \tag{5.6}$$

where $n \geqslant 0$ with formal definitions $J_0^{-3} = J_0^{-2} = J_0^{-1} = J_1^{-2} = J_1^{-1} = 0$. Eq. (5.5) does not lead to any conservation law. Multiplying (5.6) by ${}_1\bar{Y}_{lm}$ and integrating over the sphere (using (2.14)), we obtain the relation

$$\frac{d}{du}\int (n+1)\,{}_1\bar{Y}_{lm}\Phi_0^{n+1}d\Omega = \frac{1}{2}[(l-1)(l+2)-n(n+3)]\int {}_1\bar{Y}_{lm}\Phi_n^0 d\Omega$$

$$+(n+1)\int {}_1\bar{Y}_{lm}J_1^{n-2}d\Omega - \frac{1}{\sqrt{2}}\int {}_1\bar{Y}_{lm}\eth J_0^{n-3}d\Omega. \tag{5.7}$$

Provided that $l = n+1$, the term in the square bracket vanishes and, furthermore, if $n = 0, 1$, the remaining terms on the right-hand side vanish, too. Therefore, any non-linear electrodynamics satisfying our requirements on the Lagrangian yields eight conservation laws

$$\frac{d}{du}\int {}_1\bar{Y}_{1m}\Phi_0^1 d\Omega = 0, \qquad m = -1, 0, 1,$$

and

$$\frac{d}{du}\int {}_1\bar{Y}_{2m}\Phi_0^2 d\Omega = 0, \qquad m = -2, -1, 0, 1, 2.$$

Since our restrictions on the form of the Lagrangian seem to be rather weak, we conjecture that any physically reasonable (satisfying the correspondence principle) non-linear electrodynamics possesses at least 16 real conserved quantities, in addition to the total charge. Owing to the "source terms" in (5.7), it is rather improbable that we get any further conservation law of this type in a general case. For example, we have 16 conserved quantities in the Born theory, and the same number in the Born-Infeld theory. (This is in agreement with the results of Porter [15] who, in a somewhat different way, derived 16 conserved quantities in the Born-Infeld theory, whereas Chellone [14] only gives the first 6 quantities in the Born theory without mentioning the other 10 quantities.)

However, there are non-linear theories, satisfying the correspondence principle, in which "source terms" vanish up to a greater order in r^{-1}, so that more conservation laws exist. To learn this more specifically, we expand $\lambda L_{\text{I}}-1$, λL_{II}, and λL_{III} in powers of F and G and denote the degree of the lowest powers by p, q, and s, respectively. It can then be easily proved that there exist at least $2(N-2)(N-4)$ real conserved quantities where $N = \min(3+4p, 6+4q, 6+4s)$; $N \geqslant 6$ because $p \geqslant 1$ due to the correspondence principle. For example, Lagrangians of the form $L = F + \sum_{i,j=1}^{\infty} a_{ij} F^i G^j$ with $a_{ij} = 0$ for $i+j \leqslant k$ lead to $16k(2k-1)$ conserved quantities.

Unfortunately, concerning a physical interpretation of the conserved quantities we cannot say more than for the gravitational field in general relativity. The quantities seem to characterize both incoming radiation (wave tails considered in § 7) and the multipole moments of source. For (linear) test fields on the Schwarzschild background Bardeen and Press [21] argued that the NP quantities may have a lesser physical meaning than originally expected. However, the problem appears far from being settled. In non-linear electrodynamics a smaller number of exact solutions is available than in general relativity. For example, we are not aware of any exact static solution for which the quantities would not vanish.

6. *Approximation method*

In this section we describe an iterative procedure of constructing both stationary and radiative approximative solutions of the field equations of non-linear electrodynamics satisfying the correspondence principle. This procedure suggests itself immediately when the field equations are written in the form of Eqs (4.1)–(4.4), with J_0 and J_1 given by (5.1) and (5.2), respectively, $\bar{J}_1$, J_2 being expressed analogously.

Assume $\Phi_2^0(u, \theta, \varphi)$, $\Phi_1^0(u = u_0, \theta, \varphi)$, and $\Phi_0(u = u_0, r, \theta, \varphi)$ to be given ($\Phi_0 \sim O(r^{-3})$ at large values of r). As shown in Section 4, these data uniquely determine the exact solution of the Maxwell equations — our zero approximation. Inserting this solution into the J's in (4.1)–(4.4), we obtain the equations in the form of the Maxwell equations with the sources. Now it may happen that the J's found in this manner will not satisfy the continuity equation, so that the solution of (4.1)–(4.4) will not exist. This will in fact be true whenever the factor λ (introduced in the preceding section) is a non-constant function of F and G. To investigate it, let us turn back to the field equations in the original tensor form (3.2). In accordance with (4.1)–(4.4) (cf. also (3.8)–(3.11)), Eq. (3.2) can be written as $F^{\mu\nu}_{,\nu} = -J^{\mu} \equiv -[\lambda P^{\mu\nu}_{,\nu} - \lambda L_G \overset{*}{F}{}^{\mu\nu}_{,\nu} - F^{\mu\nu}_{,\nu}]$. From here it is seen that only if $\lambda = \text{const.}$, $J^{\mu}_{,\mu} = 0$ for arbitrary $F^{\mu\nu}$'s (i. e. Φ's) satisfying the potential condition (3.1). ($P^{\mu\nu}_{,\nu}$ does not vanish for approximate solutions.) However, we easily avoid this shortcoming by solving the system (4.5), (4.6), (4.8) and (4.13), instead of (4.1)–(4.4), or (4.5), (4.6), (4.8), and

(4.10). (Note that $J_2^0 = 0$ in (4.13) owing to (5.3).) In Section 4 we have seen that Eqs. (4.5), (4.6), (4.8) and (4.13) determine unique solution for given initial data — this will be the first-order approximation. Higher-order approximations can be obtained by repeating the procedure. Of course, the solution found in this way will also satisfy the system (4.5), (4.6), (4.8) and (4.13) with $\lambda = 1$, in which case the continuity equation holds identically. Therefore, independently of λ, both the continuity equation and Eq. (4.10) (i. e. also (4.4)) will be satisfied.

Multiplication by a non-constant λ is not necessary for Lagrangians given as power series in F and G. Moreover, λ will often be of the form $\lambda = 1+$(small terms quadratic in the Φ's)$+\ldots$, so that $J^{\mu}_{,\mu} = 0$ with sufficient accuracy in every step of iteration.

An iterative method for constructing solutions in the Born theory was suggested and applied by Chellone [13]. However, the problem of the choice of a consistent system of equations, supplemented by initial data, was not examined and, in fact, some of Chellone's results representing the first-order approximation do not satisfy all field equations considered[4].

7. *Wave tails and transients*[5]

The approximation method, described in the last section, was first applied to static spherically symmetric solutions because, in this case, a comparison with the exact form of solutions was possible. Assuming the Lagrangians to be power series in F and G, $L = \sum a_{ij} F^i G^j$, the second-order approximation yields

$$\Phi_1 = \frac{q}{2r^2} + \frac{q^3}{8r^6} A + \frac{q^5}{64r^{10}} B + O\left(\frac{1}{r^{14}}\right),$$

where q is the charge, and the constants A, B are determined by the coefficients a_{ij}. These solutions deviate from the exact ones by the terms $O(r^{-14})$.

In order to exhibit the physical effects of non-linearities on the propagation of waves we investigated the first-order solutions which, in the zero order, represented an axially symmetric radiating dipole. (Lagrangians were assumed to satisfy our requirements as described in § 5). The null approximation for the Φ's reads as follows:

$$\Phi_0 = f_0 \, {}_1Y_{10}, \quad \Phi_1 = f_1 \, {}_0Y_{10}, \quad \Phi_2 = f_2 \, {}_{-1}Y_{10}, \tag{7.1}$$

where

$$f_0 = 2ar^{-3}, \quad f_1 = -2\dot{a}r^{-2} - 2ar^{-3}, \quad f_2 = 2\ddot{a}r^{-1} + 2\dot{a}r^{-2} + \tfrac{1}{2} ar^{-3},$$

$a(u)$ being a dipole moment and the dot denoting ∂_u. Using (2.15) for the product of the Y's, we obtain J_0 and J_1 in the form

$$J_0 = h_{01} \, {}_0Y_{10} + h_{03} \, {}_0Y_{30} + h_{05} \, {}_0Y_{50} + \ldots,$$

$$J_1 = h_{11} \, {}_1Y_{10} + h_{13} \, {}_1Y_{30} + h_{15} \, {}_1Y_{50} + \ldots.$$

---

[4] This can be observed if the solution (6.19) given by Chellone is substituted into his Eq. (6.2).

[5] The detailed calculations the results of which are summarized in this section are given in J. Slavík Diploma thesis, Department of Theoretical Physics, Charles University 1972 (unpublished).

The functions h depending on u and r, and on the type of non-linear theory under consideration, are rather involved and will not be given here. The terms with ${}_0Y_{(2l+1)0}$ and ${}_1Y_{(2l+1)0}$ for $l \geqslant 3$ are omitted, so that $2^{(2l+1)}$-poles (with $l \geqslant 3$) are neglected. Denoting (cf. (4.16) and (4.17))

$$F_1(u, r) = r \iint [r^{-3} \int_\infty^r r'^2 h_{01} dr' + h_{11}] (dr)^2,$$

$$F_3(u, r) = r^3 \iiiint [\sqrt{6}\, r^{-5} \int_\infty^r r'^2 h_{03} dr' + h_{13} r^{-2}] (dr)^4,$$

$$F_5(u, r) = r^5 \iiiiiint [\sqrt{15}\, r^{-7} \int_\infty^r r'^2 h_{05} dr' + h_{15} r^{-4}] (dr)^6,$$

and

$$x_1 = \int_{-\infty}^u F_1(u', v) du', \qquad x_3 = \sqrt{6} \int_{-\infty}^u F_3(u', v) du',$$

$$x_5 = \sqrt{15} \int_{-\infty}^u F_5(u', v) du', \qquad (v = -u - 2r),$$

we get the first approximation in the following final form:

$$\Phi_0 = f_0\, {}_1Y_{10} + D^2(x_1 r^{-1})_1 Y_{10} + \frac{r^2}{\sqrt{6}} D^4(x_3 r^{-3})_1 Y_{30}$$

$$+ \frac{r^4}{\sqrt{15}} D^6(x_5 r^{-5})_1 Y_{50}, \tag{7.2}$$

$$\Phi_1 = f_1\, {}_0Y_{10} + D(x_1 r^{-2})_0 Y_{10} + r^2 D^3(x_3 r^{-4})_0 Y_{30}$$

$$+ r^4 D^5(x_5 r^{-6})_1 Y_{50} - r^{-2} \int_\infty^r r'^2 h_{01} dr'\, {}_0Y_{10}$$

$$- r^{-2} \int_\infty^r r'^2 h_{03} dr'\, {}_0Y_{30} - r^{-2} \int_\infty^r r'^2 h_{05} dr'\, {}_0Y_{50},$$

$$\Phi_2 = f_2\, {}_{-1}Y_{10} + x_1 r^{-3}\, {}_{-1}Y_{10} + \sqrt{6}\, r^2 D^2(x_3 r^{-5})_{-1} Y_{30}$$

$$+ \sqrt{15}\, r^4 D^4(x_5 r^{-7})_{-1} Y_{50} + r^{-1} \int_\infty^r [r'^{-2} \int_\infty^{r'} r''^2 h_{01} dr''] dr'\, {}_{-1}Y_{10}$$

$$+ \sqrt{6}\, r^{-1} \int_\infty^r [r'^{-2} \int_\infty^{r'} r''^2 h_{03} dr''] dr'\, {}_{-1}Y_{30} + \sqrt{15}\, r^{-1} \int_\infty^r [r'^{-2} \int_\infty^{r'} r''^2 h_{05} dr''] dr'\, {}_{-1}Y_{50}$$

$$+ r^{-1} \int_\infty^r r' \bar{h}_{11} dr'\, {}_{-1}Y_{10} + r^{-1} \int_\infty^r r' \bar{h}_{13} dr'\, {}_{-1}Y_{30} + r^{-1} \int_\infty^r r' \bar{h}_{15} dr'\, {}_{-1}Y_{50}.$$

Although we did not give the explicit structure of the h's, we may now interpret the individual terms appearing in the Φ's. Supposing that the dipole moment $a(u)$ is non-vanishing only in a small range of u, $u \in (-\varepsilon, +\varepsilon)$, then from the field equations we see that the J's and thus also the h's are non-zero only for $u \in (-\varepsilon, +\varepsilon)$; the same is also true of F_1, F_3 and F_5. However, the quantities x_1, x_3, and x_5 vanishing only for $u < -\varepsilon$ become complicated functions of u and v for $u \in (-\varepsilon, +\varepsilon)$, and are the functions of v only for $u > +\varepsilon$. The first terms on the right-hand sides of (7.2) are the zero-order solutions (cf. (7.1)). The next three terms represent incoming radiation (dipole, octupole, 2^5-pole), which may be considered to be caused by the scattering of the outgoing pulse by itself due to the non-linearity of the field equations. These are the wave tails. The remaining terms, with a structure resembling the outgoing dipole, octupole, and 2^5-pole, are non-vanishing only during the "broadcasting period" when $u \in (-\varepsilon, +\varepsilon)$. These are the transients.[6]

8. *Discussion*

Although one may expect the existence of a tail of electromagnetic radiation in generic cases of non-linear electrodynamics, situations might arise in which (at least) the first-order correction would contain no tail. A surprising result of this type was obtained by Couch and Torrence [22] for a quadrupole gravitational wave imploding from infinity towards the origin, and then re-exploding back to infinity. In order to investigate an analogous problem in non-linear electrodynamics one may use the approximation method of Section 6, starting with an imploding and re-exploding dipole wave as the zero approximation. In particular, one may ask whether, in the first-order approximation, the self-interaction of the wave would not lead to a tail only for a special choice of Lagrangian.

Concerning the Newman-Penrose conserved quantities, it is well known that they are essentially trivial in linear theories. In order to see their physical meaning in a full non-linear theory, it might be helpful to analyze them in a suitable example of non-linear electrodynamics.

The NP formalism should also be useful in studying the interaction of a non-linear electromagnetic field with a gravitational field, in particular, when treating radiation problems (cf., for example, the study of radiation scattering in the Einstein-Maxwell theory [6]). Not even spherically symmetric solutions, however, have been analyzed in detail, though some results were obtained by means of the standard tensor formalism [23, 24].

Thanks are due to Professor J. Kvasnica and Dr. J. Niederle for providing us with useful literature on angular momentum in quantum mechanics.

REFERENCES

[1] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Yeshiva University, Belfer Graduate School of Science, New York 1964.
[2] Z. Białynicka-Birula, I. Białynicki-Birula, *Phys. Rev.* **D2**, 2341 (1970).
[3] P. Stehle, P. G. DeBaryshe, *Phys. Rev.* **152**, 1135 (1966).

[6] In [13] some transient terms were lost due to the approximation procedure adopted there in.

[4] E. T. Newman, R. Penrose, *J. Math. Phys.* **3**, 566 (1962).
[5] R. H. Price, *Phys. Rev.* **D5**, 2439 (1972).
[6] W. E. Couch, W. H. Halliday, *J. Math. Phys.* **12**, 2170 (1971).
[7] E. T. Newman, R. Penrose, *Phys. Rev. Lett.* **15**, 231 (1965).
[8] E. T. Newman, R. Penrose, *Proc. R. Soc. (London)* **A305**, 175 (1968).
[9] A. R. Exton, E. T. Newman, R. Penrose, *J. Math. Phys.* **10**, 1566 (1969).
[10] M. Born, L. Infeld, *Proc. R. Soc. (London)* **A144**, 425 (1934).
[11] G. Boillat, *J. Math. Phys.* **11**, 941 (1970).
[12] J. Plebański, *Lectures on Non-Linear Electrodynamics*, Nordita, Copenhagen 1970.
[13] D. S. Chellone, *J. Phys.* **A4**, 184 (1971).
[14] D. S. Chellone, *J. Phys.* **A5**, 1545 (1972).
[15] J. R. Porter, *Proc. Camb. Philos. Soc.* **72**, 319 (1972).
[16] J. N. Goldberg, A. J. Macfarlane, E. T. Newman, F. Rohrlich, E. C. G. Sudarshan, *J. Math. Phys.* **8**, 2155 (1967).
[17] A. Held, E. T. Newman, R. Posadas, *J. Math. Phys.* **11**, 3145 (1970).
[18] A. I. Janis, E. T. Newman, *J. Math. Phys.* **6**, 902 (1965).
[19] R. J. Torrence, A. I. Janis, *J. Math. Phys.* **8**, 1355 (1967).
[20] W. E. Couch, R. J. Torrence, A. I. Janis, E. T. Newman, *J. Math. Phys.* **9**, 484 (1968).
[21] J. M. Bardeen, W. H. Press, *J. Math. Phys.* **14**, 7 (1973).
[22] W. E. Couch, R. J. Torrence, *J. Math. Phys.* **11**, 2096 (1970).
[23] A. Peres, *Phys. Rev.* **122**, 273 (1961).
[24] R. Pellicer, R. J. Torrence, *J. Math. Phys.* **10**, 1718 (1969).