NULL CANONICAL FORMALISM I, MAXWELL FIELD*

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The purpose of this paper is to formulate the canonical formalism on null hypersurfaces for the Maxwell electrodynamics. The set of the Poisson brackets relations for null variables of the Maxwell field is obtained. The asymptotic properties of the theory are investigated. The Poisson bracket relations for the news-functions of the Maxwell field are computed. The Hamiltonian form of the asymptotic Maxwell equations in terms of these news-functions is obtained.

1. Introduction

Let us consider the traditional gauge invariant canonical formalism for classical Maxwell electrodynamics. The action of the electromagnetic field has the form:

$$W = \int d^4x \mathscr{L}(x),$$

where $\mathscr{L}(x)$ is the Lagrangian density. The field equations, derived by varying the action integral with respect to $f_{\mu\nu}$ in the non-relativistic notation, have the form ($c = 1$):

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0, \tag{1.1a}$$

$$\vec{\nabla} \vec{B} = 0, \tag{1.1b}$$

$$-\frac{\partial \vec{D}}{\partial t} + \vec{\nabla} \times \vec{H} = 0, \tag{1.1c}$$

$$\vec{\nabla} \vec{D} = 0, \tag{1.1d}$$

where

$$\vec{D} = \vec{E} = (f_{01}, f_{02}, f_{03}), \quad \vec{H} = \vec{B} = -(f_{23}, f_{31}, f_{12}).$$

This non-relativistic form is related to a particular choice of space-like hypersurfaces;

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t = const., and time-like directions in the space-time. The energy-momentum tensor of the theory has the following form:

$$T^{\mu\nu} = f^{\mu\lambda}f_{\lambda}^{\nu} - g^{\mu\nu}\mathscr{L}.$$

Generators of the Poincaré transformations are obtained by integrating the energy--momentum tensor over the space-like hypersurface. The energy and momentum, for example, are given by the following formulas:

$$P^0 = \int d^3r(\vec{E}\vec{D} - \mathscr{L}), \tag{1.2a}$$

$$P^k = \int d^3r(\vec{D} \times \vec{B})^k, \quad k = 1, 2, 3. \tag{1.2b}$$

We now choose $\vec{D}(\vec{r})$ and $\vec{B}(\vec{r})$ as the canonical variables for the electromagnetic field. We assume the following Poisson bracket relations (P–BR) between these variables [1]:

$$\{B_i(\vec{r}, t), D_j(\vec{r}', t)\} = \varepsilon_{ijk}\nabla_k\delta^{(3)}(\vec{r} - \vec{r}'),$$

$$\{B_i(\vec{r}, t), B_j(\vec{r}', t)\} = \{D_i(\vec{r}, t), D_j(\vec{r}', t)\} = 0. \tag{1.3}$$

It is easy to verify that by assuming this form for the P–BR we obtain the time evolution as given by equations (1.1) from the relation:

$$\frac{d\mathscr{F}}{dt} = \{\mathscr{F}, P^0\},$$

where $\mathscr{F}$ is an arbitrary dynamical variable of the electromagnetic field. The choice of the dynamical variables, and the form of the equal time P–BR *is* related to a particular choice of space-like hypersurfaces, t = const. The relativistically invariant form of the P–BR (1.3) is:

$$\{f_{\mu\nu}(x), f_{\lambda\varrho}(y)\} = \nabla_{[\mu}g_{\nu][\lambda}\nabla_{\varrho]}D(x-y), \tag{1.4}$$

with

$$D(x) = i\int \frac{d^4p}{(2\pi)^3} e^{-ipx} \operatorname{sgn}(p_0)\delta(p^2) = \frac{1}{2\pi}\varepsilon(x^0)\delta(x^2). \tag{1.5}$$

Penrose investigated in the sixties the initial value problem in which the initial hypersurface is not space-like but null [2]. This approach has an advantage over the usual Cauchy problem in that all constraints (initial data equations (1.1b, d)) are eliminated from the theory for a large class of interacting fields which includes the Maxwell and Einstein field. Penrose has pointed out that null hypersurfaces and null directions are in fact much more convenient in calculations, and may perhaps also be regarded as more fundamental. Null hypersurfaces were applied in various branches of physics, for example, in the light--cone approach to strong interactions [3], and also in quantization on null hypersurfaces in the framework of field theory [4].

In the present work we formulate the canonical formalism on null hypersurfaces for the Maxwell electrodynamics. We transform the conventional P–BR from the initial space-like hypersurface to the desired null hypersurface, and thereby obtain an equivalent set of the P–BR for the variables defined on the null hypersurface. This set of the null P–BR for the Maxwell field has not been discussed so far.

In order to give the formulation we first consider field equations on null hypersurfaces. The Newman-Penrose tetrad formalism will be used [5]. The choice of suitable null coordinates permits one to investigate the asymptotic behaviour of the P–BR. From the set of the null P–BR we obtain the asymptotic P–BR for the so-called news-functions, introduced first by Bondi in General Relativity [6]. These P–BR for the news-functions were obtained in the framework of a Lagrangian formalism by Komar and by heuristic arguments by Sachs [7, 8]. It is possible to compute these P–BR for the news functions using the Fourier transform technique [7, 9]. (See: Appendix A). Next we shall discuss the asymptotic null canonical formalism where the news-functions are the fundamental objects. Finally we investigate the asymptotic generators of canonical transformations for the asymptotic Maxwell equations. We prove that it is possible to reconstruct, step by step, the whole theory from the asymptotic equations. Such a method may be very useful in a theory where the canonical formalism is unknown, but where we have some information about the asymptotic behavior. This procedure applied to General Relativity will be presented in the next work.

2. *Null coordinates, Maxwell equations on null hypersurfaces*

We introduce the following set of null coordinates in the Minkowski space-time; $u = t-r, r, \theta, \varphi$. Coordinates r, θ, φ are ordinary spherical coordinates in three-dimensional space. The u coordinate measures the retarded time, the surfaces u = const. are just the light cones emanating from the origin $r = 0$. The line element ds^2 can be expressed in terms of u, r, θ, φ:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = du^2+2dudr-r^2(d\theta^2+\sin^2\theta d\varphi^2),$$

with

$$x^0 = u, x^1 = r, x^2 = \theta, x^3 = \varphi.$$

Let us choose four null vectors ($l^\mu, n^\mu, m^\mu, \overline{m}^\mu$) at every point of the space-time in the following manner: l^μ is the outward null vector tangent to the cone u = const., n^μ is the inward null vector pointing towards $r = 0$, and m^μ and $\overline{m}^\mu$ are the complex vectors tangent to the two dimensional sphere defined by constant r and u. These vectors in the null frame u, r, θ, φ have the form:

$$l^\mu = \delta^\mu_1, \quad n^\mu = \delta^\mu_0-\tfrac{1}{2}\delta^\mu_1,$$

$$m^\mu = \frac{1}{\sqrt{2}\,r}\left(\delta^\mu_2+\frac{i}{\sin\theta}\delta^\mu_3\right), \quad \overline{m}^\mu = \frac{1}{\sqrt{2}\,r}\left(\delta^\mu_2-\frac{i}{\sin\theta}\delta^\mu_3\right). \tag{2.1}$$

We introduce as our field variables the tetrad components of $f_{\mu\nu}$. These are given by the following definitions [5]:

$$\begin{aligned} \Phi_0 &= f_{\mu\nu}l^\mu m^\nu, \\ \Phi_1 &= \tfrac{1}{2}f_{\mu\nu}(l^\mu n^\nu+\overline{m}^\mu m^\nu), \\ \Phi_2 &= f_{\mu\nu}\overline{m}^\mu n^\nu. \end{aligned} \tag{2.2}$$

The six real components of $f_{\mu\nu}$ are thus replaced by three complex functions Φ_0, Φ_1, and Φ_2 related to the null hypersurface. Functions Φ_0, Φ_1, and Φ_2 satisfy the tetrad Maxwell equations, which read:

$$\frac{\partial \Phi_1}{\partial r} + \frac{2\Phi_1}{r} = -\frac{\bar{\eth}}{\sqrt{2}}\Phi_0, \tag{2.3a}$$

$$\frac{\partial \Phi_2}{\partial r} + \frac{\Phi_2}{r} = -\frac{\bar{\eth}}{\sqrt{2}}\Phi_1, \tag{2.3b}$$

$$\frac{\partial \Phi_0}{\partial u} - \tfrac{1}{2}\frac{\partial \Phi_0}{\partial r} - \frac{\Phi_0}{r} = -\frac{\eth}{\sqrt{2}}\Phi_1, \tag{2.3c}$$

$$\frac{\partial \Phi_1}{\partial u} - \tfrac{1}{2}\frac{\partial \Phi_1}{\partial r} - \frac{\Phi_1}{r} = -\frac{\eth}{\sqrt{2}}\Phi_2. \tag{2.3d}$$

Definitions of angular operators $\eth$ and $\bar{\eth}$ are given in [10]. We have thus expressed the Maxwell equations in terms of null coordinates and null field variables only. This tetrad form of the Maxwell equations was introduced first by Newman and Penrose [5]. It has elegant properties, very useful in the investigation of the asymptotic behavior of the Maxwell radiation field.

3. *P–BR for electromagnetic field on null hypersurfaces*

Projecting the relation (1.4) on the suitable tetrad (2.1) we shall find the equal-time P–BR on the null hypersurface $u = \text{const}$. We get, after long and tedious calculations, with the help of the expression for D function in polar coordinates (Appendix B), the following relations:

$$\{\Phi_0(u, r, \Omega), \Phi_2(u, r', \Omega')\} = \{\Phi_0(u, r, \Omega), \Phi_1(u, r', \Omega')\} = 0,$$

$$\{\Phi_1(u, r, \Omega), \Phi_2(u, r', \Omega')\} = \{\Phi_2(u, r, \Omega), \Phi_2(u, r', \Omega')\} = 0,$$

$$\{\Phi_0(u, r, \Omega), \Phi_2(u, r', \Omega')\} = \left(-\frac{\delta'(r-r')}{2rr'} + \frac{|r-r'|}{4r^3 r'}\bar{\eth}\eth\right)\delta^{(2)}(\Omega-\Omega'),$$

$$\{\Phi_1(u, r, \Omega), \Phi_1(u, r', \Omega')\} = \left(\tfrac{1}{4}\frac{\delta'(r-r')}{rr'} + \frac{\varepsilon(r-r')}{8r^2 r'^2}\bar{\eth}\eth - \frac{\varepsilon(r-r')}{8r^3 r'}\bar{\eth}\eth\right.$$

$$\left. + \frac{|r-r'|}{8r^3 r'}\bar{\eth}\eth\right)\delta^{(2)}(\Omega-\Omega'),$$

$$\{\Phi_0(u, r, \Omega), \bar{\Phi}_0(u, r', \Omega')\} = \left(\frac{\delta'(r-r')}{rr'} - 2\frac{\delta(r-r')}{r^2 r'} + \frac{\varepsilon(r-r')}{r^3 r'}\right)\delta^{(2)}(\Omega-\Omega'),$$

$$\{\Phi_0(u, r, \Omega), \bar{\Phi}_1(u, r', \Omega')\} = \left(-\frac{\delta(r-r')}{r^2 r'} + \frac{\varepsilon(r-r')}{r^3 r'}\right)\frac{\eth}{\sqrt{2}}\delta^{(2)}(\Omega-\Omega'),$$

$$\{\Phi_0(u,r,\Omega),\bar{\Phi}_2(u,r',\Omega')\} = \frac{\varepsilon(r-r')}{4r^3r'}\bar{\eth}\bar{\eth}\delta^{(2)}(\Omega-\Omega'),$$

$$\{\Phi_1(u,r,\Omega),\bar{\Phi}_1(u,r',\Omega')\} = \left(\tfrac{1}{4}\frac{\delta'(r-r')}{rr'}+\frac{\varepsilon(r-r')}{8r^2r'^2}\bar{\eth}\eth-\tfrac{3}{8}\frac{|r-r'|}{r^3r'}\bar{\eth}\eth\right.$$
$$\left.-\tfrac{3}{8}\frac{\varepsilon(r-r')}{r^3r'}\right)\delta^{(2)}(\Omega-\Omega'),$$

$$\{\Phi_1(u,r,\Omega),\bar{\Phi}_2(u,r',\Omega')\} = -\left(\frac{\delta(r-r')}{2r^2r'}+\frac{\varepsilon(r-r')}{2r^3r'}+\frac{|r-r'|}{4r^3r'^2}\bar{\eth}\eth\right)\frac{\bar{\eth}}{\sqrt{2}}\delta^{(2)}(\Omega-\Omega'),$$

$$\{\Phi_2(u,r,\Omega),\bar{\Phi}_2(u,r',\Omega')\} = \left(\tfrac{1}{4}\frac{\delta'(r-r')}{rr'}+\tfrac{1}{2}\frac{\delta(r-r')}{r^2r'}+\frac{\varepsilon(r-r')}{4r^3r'}\right.$$
$$\left.+\frac{|r-r'|}{2r^3r'}\bar{\eth}\eth\right)\delta^{(2)}(\Omega-\Omega'). \tag{3.1}$$

We can reconstruct the tetrad Maxwell equations (2.3) from the formula

$$\frac{\partial\Phi_r}{\partial u} = \{\Phi_r, P_u\}, \qquad r = 0, 1, 2$$

using the null P–BR (3.1), where P_u is the generator of translation in "time" u. P_u is obtained by integrating the energy-momentum tensor over the null hypersurface with l^μ as normal vector:

$$P_u = \int \delta_0^\mu T_{\mu\nu}l^\nu r^2 dr d\Omega = \int (\Phi_0\bar{\Phi}_0+2\Phi_1\bar{\Phi}_1)r^2 dr d\Omega. \tag{3.2}$$

The derivation of the tetrad Maxwell equations starting from the formula and all P–BR (3.1) is an easy but tedious exercise. The structure of the P–BR given by equation (3.1) is essential to the study of the asymptotic properties of the theory.

4. *Asymptotic null canonical formalism*

Among all null hypersurfaces there is one which is distinguished by the following parametrization: $r = \infty, u, \theta, \varphi$. This hypersurface will be called the null infinity and will be denoted by $\mathscr{I}^+$. It follows from the well known "peeling off" theorem that the asymptotic behavior of the tetrad fields are given by [2]:

$$\Phi_0(u,r,\Omega) = \frac{\Phi_0^0(u,\Omega)}{r^3}+O\left(\frac{1}{r^4}\right),$$

$$\Phi_1(u,r,\Omega) = \frac{\Phi_1^0(u,\Omega)}{r^2}+O\left(\frac{1}{r^3}\right),$$

$$\Phi_2(u,r,\Omega) = \frac{\Phi_2^0(u,\Omega)}{r}+O\left(\frac{1}{r^2}\right). \tag{4.1}$$

To obtain the asymptotic Maxwell equations one integrates the "radial" equations (2.3 a, b) to obtain the asymptotic r dependence of the solution on a given hypersurface $u = \text{const}$. This solution will contain "constants" of integration, i.e. functions of θ and φ on a given null hypersurface. The non-radial field equations (2.3 c, d) will determine the propagation of the solution off the given hypersurface and relate the constant of integration to the initial data [11]. Finally, we obtain a form for the asymptotic Maxwell equations:

$$\frac{\partial \Phi_1^0}{\partial u} = -\frac{\eth}{\sqrt{2}}\Phi_2^0, \tag{4.2a}$$

$$\frac{\partial \Phi_0^0}{\partial u} = -\frac{\eth}{\sqrt{2}}\Phi_1^0, \tag{4.2b}$$

$$n\frac{\partial \Phi_0^n}{\partial u} = -\left\{\frac{\bar{\eth}\eth}{2}\Phi_0^{n-1} + (n+2)(n-1)\Phi_0^{n-1}\right\}, \quad n \geqslant 1, \tag{4.2c}$$

where

$$\Phi_0(u, r, \Omega) = \sum_{n\geqslant 1} \frac{\Phi_0^{n-1}}{r^{2+n}}.$$

The particular choice of Φ_0 corresponds to retarded multipole solutions [11]. It is a well known statement that a solution of the Maxwell equations is uniquely determined by assuming functions $\Phi_0(r, \Omega)$, $\Phi_2^0(u, \Omega)$ and $\Phi_1^0(\Omega)$ to be arbitrary functions of their arguments. $\Phi_2^0(u, \Omega)$ is called the news-function because its u dependence governs the u dependence of Φ_0 and Φ_1^0. The meaning and the physical interpretation of the news-functions have been discussed in detail in the literature [6], [12].

Assuming the news-function Φ_2^0 we can, with the help of the asymptotic Maxwell equations, compute all the remaining functions which appear in the expansion of Φ_0, Φ_1 and Φ_2 in the form of a power series in $1/r$. For example, we get:

$$\Phi_0 = \frac{\Phi_0^0}{r^3} + \frac{\Phi_0^1}{r^4} + O\left(\frac{1}{r^5}\right),$$

$$\Phi_1 = \frac{\Phi_1^0}{r^2} + \frac{(\bar{\eth}\Phi_0^0)}{\sqrt{2}\,r^3} + O\left(\frac{1}{r^4}\right),$$

$$\Phi_2 = \frac{\Phi_2^0}{r} + \frac{(\bar{\eth}\Phi_1^0)}{\sqrt{2}\,r^2} + O\left(\frac{1}{r^3}\right). \tag{4.3}$$

Following this procedure we can, with the help of equation (4.2), compute the functions Φ_0, Φ_1, and Φ_2 i.e. reconstruct the whole solution of Maxwell's equations. It follows that the whole field dynamics is contained in the news-functions. It is enough to assume only P–BR between the news-functions in order to reconstruct (not effectively, but step by step) the full canonical structure of the theory.

From the relations:

$$\{\Phi_2(u,r,\Omega),\Phi_2(u',r',\Omega')\} = \{\bar{\Phi}_2(u,r,\Omega),\bar{\Phi}_2(u',r',\Omega')\} = 0,$$

$$\{\Phi_2(u,r,\Omega),\bar{\Phi}_2(u',r',\Omega')\} = n^\nu n^\varrho \nabla_\nu \nabla_\varrho D(ur\Omega, u'r'\Omega'), \tag{4.4}$$

and using formula (B3) derived in Appendix B, we get the following P–BR for the news function $\Phi_2^0 = \lim_{r\to\infty} r\Phi_2$

$$\{\Phi_2^0(u,\Omega),\Phi_2^0(u',\Omega')\} = \{\bar{\Phi}_2^0(u,\Omega),\bar{\Phi}_2^0(u',\Omega')\} = 0,$$

$$\{\Phi_2^0(u,\Omega),\bar{\Phi}_2^0(u',\Omega')\} = \tfrac{1}{4}\,\delta'(u-u')\delta^{(2)}(\Omega-\Omega'). \tag{4.5}$$

From the relations (4.5) and (4.2) one can also obtain the following asymptotic P–BR:

$$\{\Phi_1^0(u,\Omega),\bar{\Phi}_1^0(u',\Omega')\} = \frac{\varepsilon(u-u')}{16}\,\bar{\eth}\eth\delta^{(2)}(\Omega-\Omega'),$$

$$\{\Phi_1^0(u,\Omega),\bar{\Phi}_2^0(u',\Omega')\} = -\frac{1}{4\sqrt{2}}\,\delta(u-u')\eth\delta^{(2)}(\Omega-\Omega'),$$

$$\{\Phi_0^0(u,\Omega),\bar{\Phi}_2^0(u',\Omega')\} = \tfrac{1}{16}\,\varepsilon(u-u')\eth\eth\delta^{(2)}(\Omega-\Omega'). \tag{4.6}$$

It is obvious that these relations follow from the basic P–BR of the news functions (see Eq. (4.5)). If we want to write the asymptotic Maxwell equations in Hamiltonian form, then we have to know the asymptotic generators of the Poincaré transformations. The action of the Poincaré group on the null infinity $\mathscr{I}^-$ is given by the following formula:

$$\zeta' = \frac{\kappa\zeta+\lambda}{\mu\zeta+\nu}, \qquad \det\begin{pmatrix}\kappa & \lambda\\ \mu & \nu\end{pmatrix} = 1, \qquad \zeta = \operatorname{ctg}\frac{\theta}{2}\,e^{i\varphi},$$

and

$$u' = K^{-1}\,(u+\varepsilon^0-\varepsilon^3\cos\theta-\varepsilon^1\sin\theta\cos\varphi-\varepsilon^2\sin\theta\sin\varphi),$$

where the function $K(\Omega)$ is completely defined by complex parameters $\kappa, \lambda, \mu, \nu$. Six independent real parameters of the matrix $\begin{pmatrix}\kappa\,\lambda\\ \mu\,\nu\end{pmatrix}$ and four parameters of the translation $\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3$ characterize the action of the Poincaré group on the null infinity $\mathscr{I}^-$ [7]. The Poincaré transformations induce one parameter families of news-functions $\Phi_{2\lambda}^0$ given by:

$$\Phi_{2\lambda}^0 = \Phi_2^0+\frac{\lambda}{1!}\{\mathscr{K},\Phi_2^0\}+\frac{\lambda^2}{2!}\{\mathscr{K},\{\mathscr{K},\Phi_2^0\}\}+\ldots. \tag{4.7}$$

The map $\Phi_2^0 \to \Phi_{2\lambda}^0$ is a canonical transformation whose generator is found from the equation [1]:

$$\left.\frac{d\Phi_{2\lambda}^0}{d\lambda}\right|_{\lambda=0} = \{\mathscr{K},\Phi_2^0\}. \tag{4.8}$$

In this way we can compute generators of canonical transformations given by the four parameter translation group:

$$\Omega = \Omega',$$
$$u' = u - a^{\alpha}\varepsilon_{\alpha},$$

where

$$a^{\alpha} = (-1, -\cos\theta, -\sin\theta\cos\varphi, -\sin\theta\sin\varphi).$$

Under these transformations:

$$\Phi_2^0(u - a^{\alpha}\varepsilon_{\alpha}, \Omega) = \Phi_2^0(u, \Omega) - \frac{1}{1!} a^{\alpha}\varepsilon_{\alpha} \frac{\partial}{\partial u} \Phi_2^0 + \frac{1}{2!} (a^{\alpha}\varepsilon_{\alpha})^2 \frac{\partial^2}{\partial u^2} \Phi_2^0 + \dots$$

Hence from equation (4.8) and the P–BR of the news-functions one obtains the following generators of translations:

$$\begin{aligned} \mathscr{K}[\varepsilon_0] &= 4 \int du d\Omega \Phi_2^0 \bar{\Phi}_2^0, \\ \mathscr{K}[\varepsilon_1] &= 4 \int du d\Omega \Phi_2^0 \bar{\Phi}_2^0 \sin\theta\cos\varphi, \\ \mathscr{K}[\varepsilon_2] &= 4 \int du d\Omega \Phi_2^0 \bar{\Phi}_2^0 \sin\theta\sin\varphi, \\ \mathscr{K}[\varepsilon_3] &= 4 \int du d\Omega \Phi_2^0 \bar{\Phi}_2^0 \cos\theta. \end{aligned} \tag{4.9}$$

Now, we can write the system of asymptotic Maxwell's equations (4.2) in the Hamiltonian form:

$$\frac{\partial \Phi_1^0}{\partial u} = \{\Phi_1^0, \mathscr{K}[\varepsilon_0]\} = -\frac{\eth}{\sqrt{2}} \Phi_2^0,$$

$$\frac{\partial \Phi_0^0}{\partial u} = \{\Phi_0^0, \mathscr{K}[\varepsilon_0]\} = \tfrac{1}{4} \int du' \varepsilon(u-u') \eth\eth \Phi_2^0 \Rightarrow \frac{\partial^2 \Phi_0^0}{\partial u^2} = \frac{\eth\eth}{2} \Phi_2^2,$$

and

$$\frac{\partial \Phi_2^0}{\partial u} = \{\Phi_2^0, \mathscr{K}[\varepsilon_0]\} = \frac{\partial \Phi_2^0}{\partial u}.$$

5. *Discussion*

The use of suitable null coordinates permits one to formulate the traditional, i.e. the space-like canonical formalism, in terms of null hypersurfaces and field variables related to null directions in the Minkowski space-time. It is possible to obtain a null asymptotic form for Maxwell's equations from the structure of null Maxwell's equations and the P–BR relations. It follows from the asymptotic properties of the theory that the whole theory may be described in terms of the news-functions. The asymptotic null canonical formalism is based on these fundamental properties of the news-functions, and we assume only the P–BR between them. In this work we have formulated the null canonical formalism, and we have shown how to obtain the null canonical formalism based on the news-functions. In principle, if we know the asymptotical null canonical formalism, then it is possible, using equations (4.2), to reconstruct the whole theory. The curious feature of the null hypersurface approach is the fact that one-half of the amount of information which is required in the usual Cauchy problem is apparently sufficient here. For example the news-functions Φ_2^0 for the free Maxwell field gives us two real numbers, while for a space-like hypersurface we require four real numbers per point (six $\vec{\boldsymbol{D}}$, $\vec{\boldsymbol{B}}$ minus two for the constraints $\vec{\nabla}\vec{\boldsymbol{B}} = 0 = \vec{\nabla}\vec{\boldsymbol{D}}$). Hence the null canonical formalism is simpler because it requires only the P–BR between the news-functions.

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APPENDIX A

Now, we will show that the Fourier transform of the news-function is given by the Fourier component of the full electromagnetic field $f_{\mu\nu}$. Fourier transform of the electromagnetic field is given by the following formula [1]:

$$F_{\mu\nu} = \tfrac{1}{2}(f_{\mu\nu}+\check{f}_{\mu\nu}) = \int \frac{d^3k}{2(2\pi)^3 k} e_{\mu\nu}(k)\,[f(k, +1)e^{-ikt+i\vec{k}\vec{r}} + f^*(k, -1)e^{ikt-i\vec{k}\vec{r}}],$$

where $\check{f}_{\mu\nu}$ denote the dual tensor of the field.

We now expand the plane wave in a series of the spherical Bessel functions:

$$e^{i\vec{k}\vec{r}} = 4\pi \sum_{lm} i^l j_l(kr) Y_{lm}(\hat{k}) Y^*_{lm}(\hat{r}).$$

Inserting the asymptotic values for the Bessel functions we get on a future cone the following formula:

$$F_{\mu\nu} \sim \frac{e_{\mu\nu}}{r} \sum_{lm} \int dk[f_{lm}(k, +1)e^{-iku}Y_{lm} + f^*_{lm}(k, -1)e^{iku}Y^*_{lm}],$$

where $u = t-r$. From the Goldberg-Kerr theorem [13] it follows that

$$F_{\mu\nu} \sim \frac{N_{\mu\nu}}{r} + \frac{III_{\mu\nu}}{r^2} + O\left(\frac{1}{r^3}\right),$$

where the function $N_{\mu\nu}$ corresponds to the Newman-Penrose tetrad function Φ_2. Then we have

$$\Phi_2^0(u, \Omega) \sim \sum_{lm} \int dk[f_{lm}(k, +1)e^{-iku}Y_{lm} + f^*_{lm}(k, -1)e^{iku}Y^*_{lm}].$$

Thus $f_{\mu\nu}$ and Φ_2^0 determine each other uniquely.

APPENDIX B

We can expand the function D (1.5) in a series of spherical harmonics

$$D(x-y) = \sum_{ll'} \sum_{mm'} f_{ll'mm'} Y_{lm}(\hat{x}) Y^*_{l'm'}(\hat{y}), \tag{B1}$$

where

$$f_{ll'mm'} = \frac{1}{2\pi} \int d\Omega_x d\Omega_y \delta((x-y)^2) \varepsilon(x^0 - y^0) Y^*_{lm}(\hat{x}) Y_{l'm'}(\hat{y}).$$

Using the rotation formula [14]:

$$Y^*_{lm}(\hat{x}) = \sum_k D^l_{mk}(\varphi_y, \theta_y, 0) Y^*_{lk}(u),$$

where $\hat{u}$ are the angles between the $\hat{x}$ and $\hat{y}$ directions, and D^l_{mk} is the usual Wigner rotation function, we find

$$f_{ll'mm'} = \frac{1}{2\pi} \int d\Omega_x d\Omega_y \delta((t-t')^2 - r^2 - r'^2 - 2rr' \cos \theta_u) \varepsilon(t-t') \sum_k D^l_{mk}(\varphi_y, \theta_y, 0)$$

$$\times Y^*_{lm}(\hat{u}) Y_{l'm'}(\hat{y}).$$

Using the orthonormality condition for Wigner functions:

$$\int d\Omega D^{*j'}_{m'k'} D^j_{mk} = \frac{4\pi}{2j+1} \delta_{jj'} \delta_{mm'} \delta_{kk'}$$

and the fact that [14]

$$D^{*l}_{m0}(\varphi, \theta, 0) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\Omega),$$

we obtain

$$f_{ll'mm'} = \varepsilon(t-t') \delta_{ll'} \delta_{mm'} \frac{1}{2rr'} P_l \left(\frac{r^2 + r'^2 - (t-t')^2}{2rr'} \right),$$

where P_l are the Legendre polynomials.

If we now introduce retarded-time variables, then we obtain

$$D(ur\Omega, ur'\Omega') = \sum_{lm} \varepsilon(u-u'+r-r') \frac{1}{2rr'} P_l \left(\frac{r^2 + r'^2 - (u-u'+r-r')^2}{2rr'} \right) Y_{lm}(\Omega) Y^*_{lm}(\Omega'). \tag{B2}$$

From this formula we can find:

$$\lim_{r,r' \to \infty} rD(ur\Omega, u'r'\Omega')r' = \frac{\varepsilon(u-u')}{2} \sum_{lm} Y_{lm}(\Omega) Y^*_{lm}(\Omega') = \frac{\varepsilon(u-u')}{2} \delta^{(2)}(\Omega - \Omega'). \tag{B3}$$

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