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NULL CANONICAL FORMALISM II, EINSTEIN FIELD*

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The purpose of this paper is to formulate the canonical formalism on null hypersurfaces for the Einstein field. Using the analogy between Maxwell's theory and gravitation theory, a set of the Poisson bracket relations for the null Weyl tensor components is obtained. The asymptotic properties of the theory are investigated. The Poisson bracket relations for the Bondi news-functions are computed. The Hamiltonian form of the asymptotic Einstein equations on terms of the news-functions is found.

1. *Introduction*

The conviction that quantization consists of the replacement of the Poisson bracket relations (P–BR) by commutators, stimulated the investigation of the canonical formalism in General Relativity. Such a procedure works very well in the quantization of linear field theories. Unfortunately, in non-linear theory the transition from the canonical formalism to the quantized version is not so easy. In the case of General Relativity there are additional serious troubles which are related to its complicated structure. For example, it is not clear at all what kind of dynamical variables must be chosen as canonical variables. The metric field $g_{\mu\nu}$ may be modified simply by carrying out a coordinate transformation. Such an operation does not involve any observable changes in the physics, as it corresponds only to a relabeling, under which the theory is invariant. This argument may be applied in particular to the case, where the dynamical variables are chosen to be the Christoffel symbols $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}$. Additional difficulty with the localization of gravitational energy leads to the serious problem with the "time generator". It is completely obscure which of the energy-momentum tensors should be chosen: Bel-Robinson or Landau-Lifshitz etc. In spite of these difficulties several works have been published on this subject [1, 2]. The most successful approach has been presented by Arnowitt et al. [3].

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Three essential aspects of the canonical formalism are:

1. The field equations must be of the first order in time derivatives.
2. A suitable choice of unconstrained canonical variables should be made.
3. The generators $\mathscr{K}$ of the symmetry group of the theory should induce the canonical transformation $\mathscr{F} \to \mathscr{F}'$ of the dynamical variables $\mathscr{F}$, where

$$\mathscr{F}' = \mathscr{F} + \frac{1}{1!}\{\mathscr{K}, \mathscr{F}\} + \frac{1}{2!}\{\mathscr{K}, \{\mathscr{K}, \mathscr{F}\}\} + \dots .$$

In this work we present the following procedure concerning these three requirements for an asymptotically flat space-time.

1. We use the Newman-Penrose formalism and the asymptotic form of the tetrad Einstein equations.
2. We take advantage of the formal similarity between the asymptotic form of the Einstein and Maxwell equations. We choose as canonical variables the news-functions and formulate the asymptotic canonical formalism in the same manner as in the case of the Maxwell field presented in the previous work [4].
3. We investigate the role of the Bondi-Metzner-Sachs group as the canonical transformation.

As in the Maxwell case it is possible to reconstruct step by step (not effectively) the whole canonical formalism from the asymptotic one.

2. *P–BR for gravitational dynamical variables*

The independent components of the Weyl tensor in the Newman-Penrose formalism are described by five complex tetrad components as follows [5]:

$$\begin{aligned}
\Psi_0 &= -C_{\mu\nu\varrho\sigma} l^\mu m^\nu l^\varrho m^\sigma, \\
\Psi_1 &= -C_{\mu\nu\varrho\sigma} l^\mu n^\nu l^\varrho m^\sigma, \\
\Psi_2 &= -C_{\mu\nu\varrho\sigma} \bar{m}^\mu n^\nu l^\varrho m^\sigma, \\
\Psi_3 &= -C_{\mu\nu\varrho\sigma} \bar{m}^\mu n^\nu l^\varrho n^\sigma, \\
\Psi_4 &= -C_{\mu\nu\varrho\sigma} \bar{m}^\mu n^\nu \bar{m}^\varrho n^\sigma.
\end{aligned} \tag{2.1}$$

From the "peeling off" theorem the asymptotic behavior of the tetrad Weyl components is given by:

$$\begin{aligned}
\Psi_0 &= \frac{\Psi_0^0}{r^5} + O\left(\frac{1}{r^6}\right), \\
\Psi_1 &= \frac{\Psi_1^0}{r^4} + O\left(\frac{1}{r^5}\right), \\
\Psi_2 &= \frac{\Psi_2^0}{r^3} + O\left(\frac{1}{r^4}\right),
\end{aligned}$$

$$\Psi_3 = \frac{\Psi_3^0}{r^2} + O\left(\frac{1}{r^3}\right),$$

$$\Psi_4 = \frac{\Psi_4^0}{r} + O\left(\frac{1}{r^2}\right). \tag{2.2}$$

To obtain the asymptotic Einstein equations we have followed the techniques of Newman and Unti [6]. First, one integrates the radial Bianchi identity. This solution will contain "constants" of integration. The non-radial Bianchi identity will determine the propagation of the solution off the given null hypersurface and relate the "constants" of integration to the initial data. For the time development of the tetrad functions Ψ_r, $r = 0, 1, 2, 3, 4$ we have to consider the leading term $\sigma^0 = \sigma^0(u, \Omega)$ in the shear:

$$\sigma = l_{\mu;\nu} m^\mu m^\nu = \frac{\sigma^0}{r^2} + O\left(\frac{1}{r^4}\right), \tag{2.3}$$

where $u = t - r$ and $\Omega = (\theta, \varphi)$. Finally, we obtain a set of the asymptotical Einstein equations on the null infinity $\mathscr{I}^+$:

$$\frac{\partial \Psi_3^0}{\partial u} = -\eth \Psi_4^0,$$

$$\frac{\partial \Psi_2^0}{\partial u} = -\eth \Psi_3^0 + \sigma^0 \Psi_4^0,$$

$$\frac{\partial \Psi_1^0}{\partial u} = -\eth \Psi_2^0 + 2\sigma^0 \Psi_3^0,$$

$$\frac{\partial \Psi_0^0}{\partial u} = -\eth \Psi_1^0 + 3\sigma^0 \Psi_2^0, \tag{2.4}$$

and

$$\ddot{\sigma}^0 = \frac{\partial^2 \sigma^0}{\partial u^2} = -\overline{\Psi}_4^0, \qquad \frac{\partial}{\partial u} \eth \overline{\sigma^0} = \eth \overline{\dot{\sigma}^0} = \Psi_3^0,$$

where the definitions of the angular operators $\eth$ and $\bar{\eth}$ are given in [7].

It is not possible to choose $\sigma^0 = 0$ everywhere if there is an outgoing radiation. The complex function σ^0 is of special interest in the gravitational radiation theory. It forms part of the initial data on u-const., which is used to determine the asymptotic spacetime. We may call $\ddot{\sigma}^0$ the gravitational radiation field since it represents the $1/r$ part of the Weyl tensor. Bondi and Sachs call $\dot{\sigma}^0$ a news-function since it can be used as asymptotic initial data for the gravitational radiation field. The integrals:

$$\int \dot{\sigma}^0 \overline{\dot{\sigma}^0} \, du d\Omega,$$

$$\int \dot{\sigma}^0 \overline{\dot{\sigma}^0} Y_{1m} \, du d\Omega \tag{2.5}$$

represent the energy and momentum of the gravitational radiation field [8].

At this point we make some heuristic arguments concerning the canonical formalism. In the Maxwell theory the news-functions satisfy the following P–BR [4]:

$$\{\Phi_2^0(u,\Omega),\Phi_2^0(u',\Omega')\} = \{\bar{\Phi}_2^0(u,\Omega),\bar{\Phi}_2^0(u',\Omega')\} = 0, \tag{2.6a}$$

$$\{\Phi_2^0(u,\Omega),\bar{\Phi}_2^0(u',\Omega')\} = \tfrac{1}{4}\,\delta'(u-u')\delta^{(2)}(\Omega-\Omega'), \tag{2.6b}$$

and the energy carried away by the outgoing electromagnetic radiation is represented by:

$$\mathscr{H}[\varepsilon^0] = 4\int \bar{\Phi}_2^0\Phi_2^0 du d\Omega. \tag{2.7}$$

From the formal point of view the structure of the asymptotic Einstein equation is very similar to the corresponding Maxwell equations. Five complex functions (2.1) are the analogues of the Φ's in the Maxwell theory. Differential equations (2.4), which they satisfy, are the asymptotic Bianchi identities. The algebraic and differential structure of the Maxwell and Einstein equations show the similarity between news-functions σ^0 and $2\Phi_2^0$. In both theories σ^0 and Φ_2^0 are used to determine the initial data and to represent the $1/r$ part of the radiation field. Taking into account these similarities and the form of the P–BR for Φ_2^0 (2.6) we assume the following P–BR for the gravitational news-functions:

$$\{\sigma^0(u,\Omega),\sigma^0(u',\Omega')\} = \{\overline{\sigma^0}(u,\Omega),\overline{\sigma^0}(u',\Omega')\} = 0, \tag{2.8a}$$

$$\{\sigma^0(u,\Omega),\overline{\sigma^0}(u',\Omega')\} = -\tfrac{1}{2}\,\varepsilon(u-u')\delta^{(2)}(\Omega-\Omega'). \tag{2.8b}$$

From these basic P–BR follow the additional P–BR between the asymptotic components of the Weyl tensor:

$$\begin{aligned}
&\{\Psi_4^0(u,\Omega),\Psi_4^0(u',\Omega')\} = 0,\\
&\{\Psi_4^0(u,\Omega),\bar{\Psi}_4^0(u',\Omega')\} = -\delta'''(u-u')\delta^{(2)}(\Omega-\Omega'),\\
&\{\Psi_3^0(u,\Omega),\Psi_3^0(u',\Omega')\} = 0,\\
&\{\Psi_3^0(u,\Omega),\bar{\Psi}_3^0(u',\Omega')\} = -\delta'(u-u')\eth\bar{\eth}\delta^{(2)}(\Omega-\Omega').
\end{aligned} \tag{2.9}$$

Now, we can write the asymptotic Einstein equations (2.4) in the Hamiltonian form:

$$\frac{\partial\Psi_r^0}{\partial u} = \{\Psi_r^0,\mathscr{H}\}, \qquad r = 0,1,2,3,4, \tag{2.10}$$

where the generator $\mathscr{H}$ of time translation is the Bondi-Sachs energy:

$$\mathscr{H} = \int du d\Omega \dot{\overline{\sigma^0}}\dot{\sigma}^0. \tag{2.11}$$

Using the P–BR (2.8) we get:

$$\frac{\partial\Psi_4^0}{\partial u} = \{\Psi_4^0,\mathscr{H}\} = -\int du'\dddot{\overline{\sigma^0}}\delta(u-u') = -\dddot{\overline{\sigma^0}} = \dot{\Psi}_4^0,$$

$$\frac{\partial\Psi_3^0}{\partial u} = \{\Psi_3^0,\mathscr{H}\} = \int du' d\Omega'\delta'(u-u')\eth\bar{\eth}\delta^{(2)}(\Omega-\Omega')\dot{\overline{\sigma^0}} = \eth\ddot{\overline{\sigma^0}} = -\eth\Psi_4^0.$$

Similarly we obtain:

$$\frac{\partial \Psi_2^0}{\partial u} = \{\Psi_2^0, \mathcal{H}\} = -\eth\, \Psi_3^0 + \sigma^0 \Psi_4^0,$$

$$\frac{\partial \Psi_1^0}{\partial u} = \{\Psi_1^0, \mathcal{H}\} = -\eth\, \Psi_2^0 + 2\sigma^0 \Psi_3^0,$$

$$\frac{\partial \Psi_0^0}{\partial u} = \{\Psi_0^0, \mathcal{H}\} = -\eth\, \Psi_1^0 + 3\sigma^0 \Psi_2^0.$$

We have obtained the Hamiltonian form of the asymptotic Einstein equations assuming only the P–BR between the news-functions σ^0 and $\overline{\sigma^0}$. It is also possible to compute higher order terms in the expansion of the Weyl tetrad functions in powers of $1/r$:

$$\Psi_0 = \frac{\Psi_0^0}{r^5} + \frac{\Psi_0^1}{r^6} + O\left(\frac{1}{r^7}\right),$$

$$\Psi_1 = \frac{\Psi_1^0}{r^4} + \frac{\Psi_1^1}{r^5} + O\left(\frac{1}{r^6}\right),$$

$$\Psi_2 = \frac{\Psi_2^0}{r^3} + \frac{\Psi_2^1}{r^4} + O\left(\frac{1}{r^5}\right),$$

$$\Psi_3 = \frac{\Psi_3^0}{r^2} + \frac{\Psi_3^1}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$\Psi_4 = \frac{\Psi_4^0}{r} + \frac{\Psi_4^1}{r^2} + O\left(\frac{1}{r^3}\right). \tag{2.12}$$

After simple but long calculations we obtain:

$$\dot{\Psi}_0^1 = -\bar{\eth}\eth\, \Psi_0^0 - 4\bar{\eth}(\sigma^0 \Psi_1^0),$$

$$\Psi_1^1 = \eth\, \Psi_0^0, \qquad \Psi_2^1 = \eth\, \Psi_1^0,$$

$$\Psi_3^1 = \bar{\eth}\, \Psi_2^0, \qquad \Psi_4^1 = \bar{\eth}\, \Psi_3^0.$$

From these relations it is possible to obtain the P–BR in the far zone of the null infinity. For example we obtain:

$$\{\Psi_4^1(u, \Omega), \Psi_4^1(u', \Omega')\} = 0,$$

$$\{\Psi_4^1(u, \Omega), \overline{\Psi}_4^1(u', \Omega')\} = \delta'(u-u')\,(\bar{\eth}\eth)^2 \delta^{(2)}(\Omega-\Omega'),$$

and

$$\dot{\Psi}_0^1 = \{\Psi_0^1, \mathcal{H}\} = -\bar{\eth}\eth\, \Psi_0^0 - 4\bar{\eth}(\sigma^0 \Psi_1^0).$$

In principle, it is possible to compute, step by step, all the remaining functions in the expansions (2.12). All these functions depend functionally on the news-functions σ^0 and $\overline{\sigma^0}$.

This means that we can reconstruct, step by step, (not effectively) the complete P–BR for Ψ_r functions, i.e. reconstruct the whole canonical formalism for the Einstein equations.

3. *Generators of canonical transformation*

Bondi-Metzner-Sachs discovered that the isometries, which preserve the radiation conditions and the form of the asymptotic flat metric, form a group (BMS group). The BMS group is defined by the following transformation on the θ, φ, u coordinates

$$\theta' = \theta'(\theta, \varphi),\ \varphi' = \varphi'(\theta, \varphi),$$
$$u' = K^{-1}\{u-\alpha(\theta, \varphi)\},$$

where the map $(\theta, \varphi) \to (\theta', \varphi')$ is a conformal transformation of the sphere into itself:

$$d\theta'^2+\sin^2\theta' d\varphi'^2 = K^{-2}(d\theta^2+\sin^2\theta d\varphi^2),$$

and where α is an arbitrary real function on the sphere. The particular BMS transformation for which:

$$\theta' = \theta,\ \varphi' = \varphi,\ u' = u+\alpha(\theta, \varphi)$$

is called supertranslation. If we write α in terms of spherical harmonics:

$$\alpha = \sum_{lm} a_{lm} Y_{lm}$$

and assume that $a_{ml} = 0$ for $l > 2$, then α takes the form:

$$\alpha = \varepsilon^0-\varepsilon^1 \sin\theta\cos\varphi-\varepsilon^2\sin\theta\sin\varphi-\varepsilon^3\cos\theta.$$

These 4-parameters $\varepsilon^0, \varepsilon^1, \varepsilon^2, \varepsilon^3$ correspond to the Lorentz translation on the nul infinity $\mathscr{I}^+$. Sachs has shown the following transformation for the news-function σ^0 under the BMS transformation [7]:

$$\sigma^0(u', \theta', \varphi') = K^{-1}e^{-i2\eta}\{\sigma^0(u, \theta, \varphi)-\tfrac{1}{2}\eth\eth\,\alpha(\theta, \varphi)\}, \tag{3.1}$$

where η is the angle between old and new direction on the sphere. We can easily check that the fundamental P–BR (2.8) of the news-functions are invariant under the transformation given be equation (3.1), i. e. the BMS transformations are canonical transformations.

We compute the generators $\mathscr{K}$ of the canonical transformation $\sigma^0 \to \sigma^{0\prime}$ from the relation [4]:

$$\sigma^{0\prime} = \sigma^0+\frac{1}{1!}\{\mathscr{K}, \sigma^0\}+\frac{1}{2!}\{\mathscr{K}, \{\mathscr{K}, \sigma^0\}\}+\ldots.$$

In fact, we can obtain, with the help of the P–BR (2.8), the function:

$$\mathscr{K}_1 = \int du d\Omega(\alpha\overline{\dot{\sigma}^0}\dot{\sigma}^0-\tfrac{1}{2}\,\dot{\bar{\sigma}}\eth^2\alpha-\tfrac{1}{2}\,\dot{\sigma}\,\bar{\eth}^2\alpha),$$

which generates:

$$\sigma^0 + \frac{1}{1!}\{\mathscr{K}_1, \sigma^0\} + \frac{1}{2!}\{\mathscr{K}_1, \{\mathscr{K}_1, \sigma^0\}\} + \ldots = \sigma(u-\alpha, \Omega) + \tfrac{1}{2}\eth\eth\,\alpha,$$

i. e. the supertranslation transformation. If we put $\alpha = \varepsilon^0 - \varepsilon^1 \sin\theta\cos\varphi - \varepsilon^2 \sin\theta\sin\varphi - \varepsilon^3\cos\theta$ i. e. restrict our transformation of the translation subgroup, then we obtain the Bondi-Sachs energy-momentum generators:

$$P_0 = \int du\, d\Omega\, \dot{\bar{\sigma}}^0\, \dot{\sigma}^0,$$

$$P_1 = \int du\, d\Omega\, \dot{\bar{\sigma}}^0\, \dot{\sigma}^0 \sin\theta\cos\varphi,$$

$$P_2 = \int du\, d\Omega\, \dot{\bar{\sigma}}^0\, \dot{\sigma}^0 \sin\theta\sin\varphi,$$

$$P_3 = \int du\, d\Omega\, \dot{\bar{\sigma}}^0\, \dot{\sigma}^0 \cos\theta.$$

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