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THE GAUSS-CODAZZI EQUATIONS AND FIELD EQUATIONS FOR A SPECIAL CASE OF NULL HYPERSURFACES IN THE THEORY OF GRAVITATION

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It is shown that on every null hypersurface a linear connection exists which is both metric and integrable, but not symmetric in general. Using this connection the Gauss-Codazzi equations are derived in the case when the null hypersurface admits this connection to be symmetric. Then these equations are very simple and are used to prove that the Einstein constraint equations impose no restrictions on the inner geometry of those particular null hypersurfaces.

1. Introduction

In recent years, the use of null hypersurfaces in general relativity has become an increasingly powerful tool in the study of gravitational radiation and of initial value problems for field equations. The investigation of general null hypersurfaces is still in its initial stages, however, and the differential geometry of those manifolds is not a well developed theory. For any Riemannian hypersurface imbedded in a Riemannian space the fundamental Gauss-Codazzi equations hold. Regarding the space-time V_4 as an A_4 and the null surface $\overset{*}{V}_3$ as a three-dimensional space imbedded in the A_4, Lemmer [1] has obtained "the generalized Gauss-Codazzi equations for a rigged A_{n-1} in A_n" (A_n is a space with a symmetric linear connection defined on it). These equations contain the two second fundamental tensors of the surface. Moreover, the connection on the surface is not metrical, therefore this appıoach seems to us to be unsatisfactory.

In this paper the Gauss-Codazzi equations are presented in the case when a null hypersurface is "totally reducible" (see below). Our connection is metrical and we use only one second fundamental tensor of the surface.

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2. Teleparallelism — a new linear connection on a null hypersurface

Consider a null hypersurface $\overset{*}{V}_3$ in a Riemannian space V_4 with the line element

$$ds^2 = \overset{*}{g}_{ik}(u^l)du^i du^k, \quad i, k = 1, 2, 3.$$

The metric is degenerate and rank $(\overset{*}{g}_{ik}) = 2$.

Because of the degeneracy of the metrics the Christoffel symbols of second kind do not exist and a linear connection can be introduced in many ways.

It seems natural that the proper connection of the surface ought to be metrical and ought to be defined in a purely geometrical manner. There is a connection satisfying these conditions — it is teleparallelism, introduced by Weitzenböck [2] and studied by Einstein in his unified field theory [3].

We shall build teleparallelism with the aid of the metric. Every symmetric and degenerate tensor in a three-dimensional space can be presented in the form

$$\overset{*}{g}_{ik} = -\sum_{A=2}^{3} \overset{A}{v}_i \overset{A}{v}_k, \tag{1}$$

where $\overset{A}{v}_i$, $A = 2, 3$ are a pair of linearly independent covariant vectors.

For given metrics we solve Eqs (1) for unknown vectors $\overset{A}{v}_i$. These two vectors are determined up to rotations in the plane spanned on the vectors $\overset{A}{v}_i$. We take as a solution any pair satisfying (1). Then we define a contravariant vector $\underset{1}{w}^k$ as a null vector of the metrics

$$\overset{*}{g}_{ik} \underset{1}{w}^k = 0,$$

which is fixed up to an arbitrary scalar factor.

The vector $\underset{1}{w}^k$ is orthogonal to $\overset{A}{v}_i$

$$\underset{1}{w}^i \overset{A}{v}_i = 0.$$

When the vectors $\overset{A}{v}_i$ are chosen, we define a pair of contravariant vectors $\underset{A}{w}^i$ by relations

$$\overset{*}{g}_{ik} \underset{A}{w}^k = -\overset{A}{v}_i.$$

It follows from the linear independence of vectors $\overset{A}{v}_i$ that

$$\underset{A}{w}^i \overset{B}{v}_i = \delta^B_A.$$

At last we introduce a covariant vector $\overset{1}{v}_i$ as a solution of equations

$$\overset{1}{v}_i \underset{a}{w}^i = \delta^1_a, \quad a = 1, 2, 3,$$

In this manner we have built up, on the null surface, a covariant triad $\{\overset{a}{v}_i\}$ and a contravariant triad $\{\underset{a}{w}^i\}$ satisfying relations

$$\overset{a}{v}_i \underset{b}{w}^i = \delta^a_b \tag{2}$$

and hence

$$\overset{a}{v}_i \underset{a}{w}^k = \delta^k_i, \qquad a, b = 1, 2, 3. \tag{2a}$$

Having determined orthonormal triads in $\overset{*}{V}_3$ we may introduce a quantity

$$\Gamma^i_{kl} \equiv \underset{a}{w}^i \partial_k \overset{a}{v}_l \tag{3}$$

(summation convention over index a is assumed). In fact, this quantity transforms like an affine connection and is called the teleparallelism. This connection has several useful properties.

First it can be easily shown that the vectors $\underset{a}{w}_i$ and $\overset{a}{v}_i$ are absolutely parallel with respect to it

$$\overset{*}{\nabla}_i \underset{a}{w}^k \equiv \partial_i \underset{a}{w}^k + \Gamma^k_{in} \underset{a}{w}^n = \overset{*}{\nabla}_i \overset{a}{v}_k = 0. \tag{4a}$$

As a result of it, the connection is metrical: $\overset{*}{\nabla}_i \overset{*}{g}_{kl} \equiv 0$. Then it may be proved that the connection is integrable — its curvature tensor vanishes identically

$$\overset{*}{R}_{ikl}{}^m \equiv 2\partial_{[i}\Gamma^m_{k]l} + 2\Gamma^m_{[i|n|}\Gamma^n_{k]l} = 0. \tag{4b}$$

Hence, any space equippped with teleparallelism is an affine-flat space. It is obvious from Eq. (3) that this connection is not symmetric in general.

3. Differentiation within the null surface

The method presented here is based upon [4]. The hypersurface $\overset{*}{V}_3$ is described in V_4 by equations

$$x^\alpha = x^\alpha(u^1, u^2, u^3).$$

The connecting quantities

$$B^\alpha_i \equiv \frac{\partial x^\alpha}{\partial u^i} \tag{5}$$

are mixed tensors — the covariant vectors in $\overset{*}{V}_3$ and contravariant vectors in V_4. As contravariant vectors the B^α_i constitute a set of 3 vectors tangent to $\overset{*}{V}_3$. Another set of tangent to $\overset{*}{V}_3$ vectors is obtained by projecting of the vectors $\underset{a}{w}^i$ into imbedding space

$$\underset{a}{\omega}^\alpha \equiv B^\alpha_i \underset{a}{w}^i. \tag{6}$$

The metric $\overset{*}{g}_{ik}$ of $\overset{*}{V}_3$ is related to the metric $g_{\alpha\beta}$ of V_4 by means of the connecting quantities

$$\overset{*}{g}_{ik} = g_{\alpha\beta}B_i^\alpha B_k^\beta. \tag{7}$$

Then the vectors $\underset{a}{\omega}^\alpha$ satisfy the equations

$$g_{\alpha\beta}\underset{1}{\omega}^\alpha\underset{1}{\omega}^\beta = g_{\alpha\beta}\underset{1}{\omega}^\alpha\underset{A}{\omega}^\beta = 0, \qquad g_{\alpha\beta}\underset{A}{\omega}^\alpha\underset{B}{\omega}^\beta = -\delta_{AB}, \tag{8}$$

where $A, B = 2, 3$. The two sets of the vectors tangent to $\overset{*}{V}_3$ are interrelated by (6) and by

$$B_i^\alpha = \overset{a}{v}_i\underset{a}{\omega}^\alpha. \tag{9}$$

The null vector $\underset{1}{\omega}^\alpha$ can be always chosen as a gradient-field of the isotropic hypersurface $\overset{*}{V}_3$. Then the fourth vector is added to the triad $\{\underset{a}{\omega}^\alpha\}$ to form a linearly independent tetrad at points of $\overset{*}{V}_3$, it is determined by relations

$$g_{\alpha\beta}m^\alpha m^\beta = 0, \tag{10}$$

$$g_{\alpha\beta}m^\alpha\underset{a}{\omega}^\beta = \delta_a^1. \tag{11}$$

Since the null vector m^α is not orthogonal to $\underset{1}{\omega}^\alpha$ and is not tangent to the hypersurface, then it is directed off the $\overset{*}{V}_3$. This vector defines the rigging of any isotropic hypersurface. For the vectors B_i^α and m^α one has

$$g_{\alpha\beta}m^\alpha B_i^\beta = \overset{1}{v}_i. \tag{12}$$

For any mixed tensor given at points of the $\overset{*}{V}_3$ the covariant derivative within the null hypersurface with respect to u^l (denoted by $\overset{*}{\nabla}_l$) involves both $\left\{ {\alpha \atop \mu\lambda} \right\}$ and Γ^i_{kl}. For instance

$$\overset{*}{\nabla}_l T^{\alpha i}_{\beta k} = \frac{\partial}{\partial u^l} T^{\alpha i}_{\beta k} + \left\{ {\alpha \atop \mu\nu} \right\} T^{\nu i}_{\beta k} B^\mu_l - \left\{ {\nu \atop \mu\beta} \right\} T^{\alpha i}_{\nu k} B^\mu_l + \Gamma^i_{ls} T^{\alpha s}_{\beta k} - \Gamma^s_{lk} T^{\alpha i}_{\beta s}. \tag{13}$$

From (13) one gets immediately

$$\overset{*}{\nabla}_i B^\mu_k - \overset{*}{\nabla}_k B^\mu_i = 2\Gamma^n_{[ki]} B^\mu_n \equiv 2S_{ki}{}^n B^\mu_n. \tag{14}$$

For a pure vector in the V_4 the formula (13) yields

$$\overset{*}{\nabla}_l T^\alpha = B^\mu_l \nabla_\mu T^\alpha, \tag{15}$$

where ∇_μ denotes the covariant differentiation in the V_4 with respect to $\left\{ {\alpha \atop \mu\lambda} \right\}$. If the process of covariant differentiation is applied twice in succession we get, after alternation over the indices of differentiation, an expression containing the $S_{ik}{}^{l}$ and the Riemann tensor $R_{\alpha\beta\mu\lambda}$ of the V_4. For instance ([5])

$$(\overset{*}{\nabla}_i\overset{*}{\nabla}_k-\overset{*}{\nabla}_k\overset{*}{\nabla}_i)T^{\alpha l}_{\beta m} = R^{\alpha}_{\sigma\lambda\mu}T^{\sigma l}_{\beta m}B^{\lambda}_{i}B^{\mu}_{k}-R^{\sigma}_{\beta\lambda\mu}T^{\alpha l}_{\sigma m}B^{\lambda}_{i}B^{\mu}_{k}-2S_{ik}{}^{n}\overset{*}{\nabla}_nT^{\alpha l}_{\beta m}. \tag{16}$$

Here we have used the fact that the curvature tensor $\overset{*}{R}_{ikl}{}^{m}$ of the connection Γ^{i}_{kl} given by (4b), is zero everywhere.

4. Derivatives of the vectors of the fundamental tetrad

We want to express the covariant derivatives of the tetrad $\{m^\alpha, \underset{a}{\omega}{}^\alpha\}$ in terms of these vectors. For general isotropic hypersurfaces the teleparallelism cannot be symmetric. In this paper we shall consider these particular hypersurfaces for which this connection can be symmetric

$$\Gamma^{l}_{[ik]} \equiv S_{ik}{}^{l} = 0.$$

In this case $\partial_{[i}\overset{a}{v}_{k]} = 0$ and the vectors $\overset{a}{v}_i$ are gradient-fields and those hypersurfaces are called totally reducible. First we calculate the derivative $\overset{*}{\nabla}_iB^{\alpha}_{k}$. By differentiation of (7) one gets

$$\overset{*}{\nabla}_lg_{ik} = g_{\alpha\beta}B^{\beta}_{k}\overset{*}{\nabla}_lB^{\alpha}_{i}+g_{\alpha\beta}B^{\alpha}_{i}\overset{*}{\nabla}_lB^{\beta}_{k} = 0,$$

since $\overset{*}{\nabla}_l\overset{*}{g}_{\alpha\beta} = B^{\mu}_{l}\nabla_\mu g_{\alpha\beta} = 0.$

Cyclic permutation of indices $k, l, i,$ yields two subsequent similar equations. It follows from these three equations that

$$g_{\alpha\beta}B^{\alpha}_{i}\overset{*}{\nabla}_kB^{\beta}_{l} = 0. \tag{17}$$

Since the contravariant vector $\overset{*}{\nabla}_iB^{\alpha}_{k}$ is orthogonal to all vectors B^{α}_{k}, which are tangent to the $\overset{*}{V}_3$, it is parallel to the only orthogonal to the surface vector $\underset{1}{\omega}{}^\alpha$

$$\overset{*}{\nabla}_iB^{\alpha}_{k} = \Omega_{ik}\underset{1}{\omega}{}^\alpha. \tag{18}$$

The Ω_{ik} is the second fundamental tensor field of the null hypersurface. For the symmetric connection we have

$$\overset{*}{\nabla}_iB^{\alpha}_{k} = \overset{*}{\nabla}_kB^{\alpha}_{i} \tag{19}$$

and hence $\Omega_{ik} = \Omega_{ki}$.

Transvecting (18) with $\underset{a}{w}{}^k$ and the use of (4a) and (6) gives

$$\overset{*}{\nabla}_i\underset{a}{\omega}{}^\alpha = \underset{a}{w}{}^k\Omega_{ik}\underset{1}{\omega}{}^\alpha. \tag{20}$$

From (11) one gets

$$\Omega_{ik} = m_\alpha \overset{*}{\nabla}_i B_k^\alpha = m_\alpha \overset{a}{v}_i \overset{*}{\nabla}_k \underset{a}{\omega}^\alpha. \tag{21}$$

It is useful to study the properties of the Ω_{ik}. The vector m^α is fixed by formulae (10) and (11). Moreover, it can be shown following [6] that the relations for m^α and $\underset{1}{\omega}^\alpha$ hold

$$m^\alpha \nabla_\alpha \underset{1}{\omega}_\lambda = m^\alpha \nabla_\lambda \underset{1}{\omega}_\alpha = A \underset{1}{\omega}_\lambda, \tag{22}$$

where A is a scalar field. The property of the $\underset{1}{\omega}^\alpha$ of being a gradient-field has been assumed here. Using (21) and (22) we get

$$\underset{1}{w}^n \Omega_{ni} = m_\alpha \overset{*}{\nabla}_i \underset{1}{\omega}^\alpha = m_\alpha B_i^\mu \nabla_\mu \underset{1}{\omega}^\alpha = B_i^\mu A \underset{1}{\omega}_\mu,$$

$$\underset{1}{w}^n \Omega_{ni} = 0 \tag{23}$$

The tensor Ω_{ki} is hence symmetric and degenerate. Next we calculate $\overset{*}{\nabla}_i m^\alpha$. We differentiate covariantly (12) and make use of (18)

$$0 = g_{\alpha\beta} m^\alpha \Omega_{ik} \underset{1}{\omega}^\beta + g_{\alpha\beta} B_k^\beta \overset{*}{\nabla}_i m^\alpha,$$

then

$$\Omega_{ik} = -g_{\alpha\beta} B_k^\beta \overset{*}{\nabla}_i m^\alpha. \tag{24}$$

At last differentiation of (10) gives

$$g_{\alpha\beta} m^\alpha \overset{*}{\nabla}_i m^\beta = 0.$$

The vector m^α is orthogonal to $\overset{*}{\nabla}_i m^\alpha$ and hence the vector $\overset{*}{\nabla}_i m^\alpha$ is a linear combination of m^α and $\underset{A}{\omega}^\alpha$

$$\overset{*}{\nabla}_i m^\alpha = a_i m^\alpha + b_i \underset{2}{\omega}^\alpha + c_i \underset{3}{\omega}^\alpha.$$

This formula we put in the expression (24) for Ω_{ik} and using identities

$$g_{\alpha\beta} B_k^\alpha \underset{A}{\omega}^\beta = -\overset{A}{v}_k,$$

we obtain

$$\Omega_{ik} = -a_i \overset{1}{v}_k + b_i \overset{2}{v}_k + c_i \overset{3}{v}_k.$$

It follows from (23) that $a_i = 0$. Thus we have obtained the expressions for the covariant derivatives of the tetrad $\{m^\alpha, \underset{a}{\omega}^\alpha)\}$ in terms of that tetrad itself

$$\overset{*}{\nabla}_i m^\alpha = \Omega_{ik}(\underset{2}{w}^k \underset{2}{\omega}^\alpha + \underset{3}{w}^k \underset{3}{\omega}^\alpha), \tag{25}$$

$$\overset{*}{\nabla}_i \underset{3}{\omega}^\alpha = \underset{a}{w}^k \Omega_{ik} \underset{1}{\omega}^\alpha. \tag{26}$$

5. The Gauss-Codazzi equations

Taking the vectors m^α, $\underset{a}{\omega}^\alpha$ as unknown functions we seek for integrability conditions of Eqs (25) and (26). After differentiating of (25) with respect to u^l and after alternating over indices i and l we have

$$(\overset{*}{\nabla}_i \overset{*}{\nabla}_l - \overset{*}{\nabla}_l \overset{*}{\nabla}_i) m^\alpha = (\underset{2}{w}^k \underset{2}{\omega}^\alpha + \underset{3}{w}^k \underset{3}{\omega}^\alpha)(\overset{*}{\nabla}_i \Omega_{lk} - \overset{*}{\nabla}_l \Omega_{ik}). \tag{27}$$

The left-hand side of (27) equals $R^\alpha_{\beta\lambda\mu}\, m^\beta B^\lambda_i B^\mu_l$. Contraction of (27) with m_α gives identically zero, contraction with $g_{\alpha\sigma} B^\sigma_h$ yields

$$-R_{\alpha\beta\lambda\mu} m^\alpha B^\beta_h B^\lambda_i B^\mu_l = -(\underset{2}{w}^k \overset{2}{v}_h + \underset{3}{w}^k \overset{3}{v}_h)(\overset{*}{\nabla}_i \Omega_{lk} - \overset{*}{\nabla}_l \Omega_{ik}).$$

The use of the orthonormality condition (2a) and of (23) simplifies the right-hand side of this equation to

$$-(\overset{*}{\nabla}_i \Omega_{lh} - \overset{*}{\nabla}_l \Omega_{ih}).$$

Thus we have derived the Codazzi equations

$$R_{\alpha\beta\lambda\mu} m^\alpha B^\beta_h B^\lambda_i B^\mu_k = 2\overset{*}{\nabla}_{[i} \Omega_{k]h}. \tag{28}$$

The same procedure is repeated for equation (26). We get

$$(\overset{*}{\nabla}_i \overset{*}{\nabla}_l - \overset{*}{\nabla}_l \overset{*}{\nabla}_i)\underset{a}{\omega}^\alpha = \underset{a}{w}^k \underset{1}{\omega}^\alpha (\overset{*}{\nabla}_i \Omega_{lk} - \overset{*}{\nabla}_l \Omega_{ik}).$$

We transvect this equation with the vector $\overset{a}{v}_h g_{\alpha\beta} B^\beta_r$, as a result we obtain the Gauss equations

$$R_{\alpha\beta\lambda\mu} B^\alpha_i B^\beta_k B^\lambda_l B^\mu_m = 0. \tag{29}$$

6. The Einstein constraint equations

The Gauss-Codazzi equations (28) and (29) are so simple that we can use them in the study of field equations. Let the hypersurface $\overset{*}{V}_3$ be given by

$$x^0 \overset{*}{=} 0, \quad x^i \overset{*}{=} u^i.$$

In this case we have $B_i^\alpha = \delta_i^\alpha$, hence $\overset{*}{g}_{ik} \overset{*}{=} g_{ik}$. The coordinates u^i on the surface can be always chosen in such a way that the vector $\underset{1}{w}^k$ takes the form

$$\underset{1}{w}^k \overset{*}{=} \delta_1^k.$$

Then we have $g_{1i} \overset{*}{=} 0$, $\underset{1}{\omega}^\alpha \overset{*}{=} \delta_1^\alpha$, $\underset{1}{\omega}_\alpha \overset{*}{=} \partial_\alpha x^0 = \delta_\alpha^0$.

It is well known that the four field equations

$$G_\alpha^0 \equiv R_\alpha^0 - \tfrac{1}{2}\delta_\alpha^0 R = 0$$

do not contain any $g_{\alpha\beta,00}$ derivatives and therefore they constitute the constraint equations for the quantities $g_{\alpha\beta}$ and $g_{\alpha\beta,\lambda}$. In the above system of coordinates these equations take the form

$$-2G_0^0 \overset{*}{=} g^{ik}(2R_{0ik1} + g^{lm}R_{likm}) = 0, \tag{30}$$

$$G_i^0 \overset{*}{=} R_{01i1} + g^{lm}R_{li1m} = 0. \tag{31}$$

Now we take into account the Gauss-Codazzi equations, which are now very simple

$$R_{iklm} \overset{*}{=} 0, \qquad R_{0hik} \overset{*}{=} 2\overset{*}{\nabla}_{[i}\Omega_{k]h}. \tag{32}$$

By substitution of (32) into (31) we obtain

$$G_0^0 \overset{*}{=} g^{ik}\overset{*}{\nabla}_1\Omega_{ik} = 0, \tag{33}$$

$$G_i^0 \equiv 0. \tag{34}$$

We have used here the property that $\underset{1}{w}^k\Omega_{ik} \overset{*}{=} \Omega_{i1} = 0$. The inner geometry of the null surface is described by its metrics and therefore by teleparallelism. On the other side the tensor Ω_{ik} gives the position of the surface in the imbedding space. Thus we obtain the following theorem: If the Gauss-Codazzi equations are satisfied for a totally reducible null hypersurface, then the field constraint equations do not impose any restrictions on the inner geometry of that surface.

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REFERENCES

[1] G. Lemmer, *Nuovo Cimento* 37, 1659 (1965).
[2] R. Weitzenböck, *Invariantentheorie*, Groningen 1923, pp. 408.
[3] A. Einstein, *Sitz. Preuss. Akad.*, 217–221 (1928).
[4] P. Rashevsky, *Riemannian Geometry and Tensor Calculus*, Moskva 1967 (in Russian).
[5] J. Schouten, *Ricci-calculus*, Springer Verlag 1954, p. 140.
[6] P. Jordan, J. Ehlers, R. Sachs, *Akad. Wiss. Lit.*, Mainz 1961, No. 1.