A CLASS OF COSMOLOGICAL MODELS WITH TORSION AND SPIN*

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Homogeneous models of the Universe filled with a spinning fluid are studied in the framework of the Einstein-Cartan theory of gravitation. It is assumed that the models admit a group of motions simply transitive on three-surfaces orthogonal to the world lines of the substratum. For certain group types, the field equations are partially integrated. The models of the Bianchi types I, VII_0, V are shown to be non-singular, provided the influence of spin exceeds that of shear, and an equation of state satisfies some physically reasonable conditions.

1. *Introduction*

The Einstein-Cartan theory of gravitation accepts as a model of spacetime a non-Riemannian four dimensional differential manifold with a metric tensor g_{ab} and a linear connection ω_{ab} compatible with the metric. The torsion of spacetime is related to the spin of matter in such a way that the field equations in a vacuum remain the same as in the classical General Relativity. The history and the present state of the theory is presented in Hehl's article [1]. We use its recent formulation given by Trautman [2].

Kopczyński [3] found the first cosmological models with spin. They are non-singular under some reasonable assumptions. Physical properties of the non-singular universes were examined by Trautman [4] and later by Stewart and Hajiček [5]. Further solutions with torsion were obtained by Tafel [6]. In this paper we investigate the non-rotating cosmological models with spin for several Bianchi types. In particular the influence of an equation of state on some properties of the models is considered.

Let us discuss the notation. For the units used, $G = 1/8\pi$, $c = 1$. Indices a, b, c, d, run from 0 to 3 and i, j, k, l run from 1 to 3. η_{abcd} is a completely antisymmetric pseudotensor such that $\eta_{0123} = |\det [g_{ab}]|^{1/6}$. $[M_{ab}]$ denotes matrix with the components M_{ab}, and $[M^a]$ denotes one-column matrix with the components M^a. The transposition of any matrix M is denoted by M^T. (θ^a) is a basis of 1-forms. The vector basis (e_a) is dual with

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respect to (θ^a). If f_A is a tensor field, its ordinary and covariant derivative in direction e_a is $\partial_a f_A$, $\nabla_a f_A$, respectively. The relations $\omega_{ab} = \Gamma_{abc}\theta^c$, $d\theta^a = -\frac{1}{2}\gamma^a{}_{bc}\,\theta^b \wedge \theta^c$ define coefficients Γ_{abc} and $\gamma^a{}_{bc}$.

The curvature 2-form $\Omega^a{}_b$ and the curvature tensor $R^a{}_{bcd}$ of the connection $\omega^a{}_b$ are defined as follows

$$\Omega^a{}_b = \tfrac{1}{2} R^a{}_{bcd}\theta^c \wedge \theta^d := d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{1.1}$$

The contraction of the curvature tensor gives the generalized Ricci tensor

$$R_{ab} := R_a{}^c{}_{bc} = \partial_c\Gamma^c{}_{ab} - \partial_b\Gamma^c{}_{ac} + \Gamma^c{}_{dc}\Gamma^d{}_{ab} - \Gamma^c{}_{ad}\Gamma^d{}_{cb} - \Gamma^c{}_{ad}\gamma^d{}_{cb}. \tag{1.2}$$

The torsion 2-form Θ^a is covariant derivative of the basis (θ^a),

$$\Theta^a = \tfrac{1}{2} Q^a{}_{bc}\theta^b \wedge \theta^c := d\theta^a + \omega^a{}_b \wedge \theta^b. \tag{1.3}$$

Tensor $Q^a{}_{bc}$ is called the torsion tensor.

The condition of metricity of the connection $Dg_{ab} = 0$, and (1.3) imply that

$$\Gamma_{abc} = \tfrac{1}{2}(\partial_c g_{ab} - \partial_a g_{bc} + \partial_b g_{ca}) - \tfrac{1}{2}(\gamma_{abc} + \gamma_{bca} - \gamma_{cab}) - \tfrac{1}{2}(Q_{abc} + Q_{bca} - Q_{cab}). \tag{1.4}$$

If we denote the canonical energy-momentum tensor and the spin tensor of matter by t_{ab} and S_{abc}, respectively, the field equations, derived from the variational principle analogous to that used in the classical Einstein theory, can be written as

$$R_{ab} - \tfrac{1}{2} R^c{}_c g_{ab} = t_{ab}. \tag{1.5}$$

$$Q_{abc} + 2g_{a[b}Q^d{}_{c]d} = S_{abc}. \tag{1.6}$$

In this article we consider the Weyssenhoff model of matter, for which

$$t_{ab} = u_a h_b - p g_{ab}, \qquad u^a u_a = 1. \tag{1.7}$$

$$S_{abc} = u_a S_{bc}, \qquad S_{(bc)} = 0, \qquad u^a S_{ab} = 0, \tag{1.8}$$

where u^a is the vector of four-velocity, p is the pressure, and h_b is the enthalpy vector of the fluid. Moreover, we assume that

there exists a group of motions simply transitive on three-surface orthogonal to the velocity. (1.9)

Such models, in the framework of the classical Einstein theory, were investigated recently in systematic way by Ellis and MacCallum [7].

The assumption (1.9) implies

$$u^b\nabla_b u^a = 0,$$

$$\tilde{\nabla}_{[a}u_{b]} = 0,$$

where $\tilde{\nabla}$ denotes covariant derivative with respect to the Riemannian connection given by (1.4) without torsion components. From the Bianchi identities it results that t_{ab} reduces to the energy-momentum tensor for a perfect fluid,

$$t_{ab} = (\varepsilon+p)u_a u_b - p g_{ab}, \tag{1.10}$$

where ε is the energy density in the local rest frame of matter.[1] The Bianchi identities show further that the enthropy and the density of spin are conserved,

$$u^a \nabla_a \varepsilon + \theta(\varepsilon+p) = 0, \tag{1.11}$$

$$\nabla_a(Su^a) = 0. \tag{1.12}$$

The expansion tensor θ_{ab}, the expansion θ, and the shear σ are defined in the usual way [8], and the spin vector S^a and the density of spin S by the relations

$$S_a := -\tfrac{1}{2}\,\eta_{abcd}u^b S^{cd}, \qquad S := (S_a S^a)^{1/2}. \tag{1.13}$$

2. *The field equations*

Following the approach of Ellis and MacCallum [7] we introduce an orthonormal tetrad (e_a) with the time-like vector e_0 chosen as the velocity vector u, i.e. such that

$$u^i = \delta^i{}_0. \tag{2.1}$$

The basis of 1-forms is (θ^a), where $\theta^0 = dt$. The coefficients $\gamma^a{}_{bc}$ depend on time t only and are required to satisfy the Jacobi identities. $\gamma^a{}_{bc}$ can be decomposed in the following way:

$$\gamma_{ij0} = \theta_{ij} + \tfrac{1}{2}\, S_{ij} + \varepsilon_{ijk}\Omega^k, \tag{2.2}$$

$$\gamma^i{}_{jk} = n^{il}\varepsilon_{ljk} - \delta^i{}_j a_k + \delta^i{}_k a_j, \tag{2.3}$$

where $\varepsilon_{ijk} := \eta_{0ijk}$, $n^{[il]} = 0$. Ω^k is the local angular velocity of the triad (e_i) with respect to the Fermi-propagated rest frame of matter. Since the spin of light wave applied to

TABLE I

Classification of three-parameter groups following Behr [9]

Group types	a	n_1	n_2	n_3
I	0	0	0	0
II	0	+	0	0
VII_c	0	+	+	0
VI_0	0	+	−	0
IX	0	+	+	+
VIII	0	+	+	−
V	+	0	0	0
IV	+	0	0	+
VII_h	+	0	+	+
VI_h	+	0	+	−

[1] The metric tensor has the signature $(+1, -1, -1, -1)$.

experimental description of the non-rotating orthonormal frame, is zero, then such frame is defined by the equation $\tilde{\Omega}_i = 0$, where $\tilde{\Omega}_i = \Omega_i - \frac{1}{2}S_i$ is the Riemannian angular velocity. The coefficients n^{ij} and a_i behave as tensors under time-dependent rotations of the frame (e_i). Using the Jacobi identities it can be shown that with respect to the frame (e_i^*), consisting of the eigenvectors of n^{ij}, the vector a_i has only one non-zero component, say $a_1 = a$, $a_2 = a_3 = 0$. The classification of models with respect to the value of a and signs of the eigenvalues n_i of matrix $[n_{ij}]$ is given in Table I. The non-trivial Jacobi identities are written out in Appendix I with the tetrad (e_i^*). It follows from them that the group type cannot change in a continuous way.

A direct calculation, using (1.2), (1.4), (1.6), leads to the following field equations holding in the frame (e_a)[2]:

$$\tfrac{1}{3}\theta^2 - \sigma^2 - \tfrac{1}{2}R^* + \tfrac{1}{4}S^2 = t_{00} = \varepsilon, \tag{2.4}$$

$$-3\theta_i^{\ j}a_j + \theta a_i + \varepsilon_{ijk}n^{kl}\theta_l^{\ j} + \tfrac{1}{2}S_i^{\ j}a_j + \tfrac{1}{4}n_i^{\ j}\varepsilon_{jkl}S^{kl} = t_{(0i)} = 0, \tag{2.5}$$

$$-\dot{\theta}_{ij} - \theta\theta_{ij} + 2\theta^k{}_{(i}\varepsilon_{j)kl}\Omega^l - 2n^k{}_{(i}\varepsilon_{j)kl}a^l - 2n_{ik}n^k{}_j$$
$$+ n_k^{\ k}n_{ij} + \delta_{ij}(-\dot{\theta} - \tfrac{2}{3}\theta^2 - \sigma^2 - \tfrac{1}{2}R^* - 4a_ka^k + \tfrac{1}{4}S^2) = t_{(ij)} = p\delta_{ij}, \tag{2.6}$$

$$\dot{S}_{ij} + \theta S_{ij} + 2S^k{}_{[i}\varepsilon_{j]kl}\Omega^l = -2t_{[ij]} = 0. \tag{2.7}$$

R^* is the curvature scalar of the hypersurfaces $t = \text{const.}$,

$$R^* = -6a_ia^i + n_{ij}n^{ij} - \tfrac{1}{2}(n_i^{\ i})^2 = 6a^2 + \tfrac{1}{2}(n_1^2 + n_2^2 + n_3^2) - n_1n_2 - n_1n_3 - n_2n_3. \tag{2.8}$$

It may be negative only for the type IX models.

For further considerations it is convenient to introduce a positive function $R(t)$ and the function $F(t)$, defined by the equations

$$3R^{-1}\dot{R} := \theta, \tag{2.9}$$

$$F := \int_{t_0}^{t} R^{-3}(t')dt'. \tag{2.10}$$

$R(t)$ is determined up to a factor by a positive constant.

The field equations can be divided into two parts. The first of them represents the maximum system of the equations without the energy density and the pressure; they are written out in Appendix I. The second part consists of the generalized Friedmann equations

$$3R^{-2}\dot{R}^2 - \sigma^2 - \tfrac{1}{2}R^* + \tfrac{1}{4}S^2 = \varepsilon, \tag{2.11}$$

$$-2R^{-1}\ddot{R} - R^{-2}\dot{R}^2 - \sigma^2 + \tfrac{1}{6}R^* + \tfrac{1}{4}S^2 = p. \tag{2.12}$$

We use the notation $\mu := (\frac{1}{4}S^2 - \sigma^2)R^6$ if this quantity is constant.

In Sections 3 to 9 we try to integrate the equations without assuming any equation of state, and to represent all quantities as explicit functions of R and F. In some cases the equations written out in Appendix I constitute a complete system. Then the solution

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[2] The dot denotes the differentiation with respect to the time.

depends on arbitrary constants, but does not depend on arbitrary functions. Therefore the equation of state is determined up to a constant provided $\theta \neq 0$.[3]

Next we examine particular group types under the assumption $S \neq 0$.

3. *Type I*

Since $n_1 = n_2 = n_3 = 0$ we have the freedom of the most general rotation of (e_i^*), which we use to obtain $\theta_{ij} = 0\,(i \neq j)$. Eqs. (I.5) read $(\theta_{ii} - \theta_{jj})\Omega_k = 0$, where (ijk) is a permutation of (123). Using (I.5) or the freedom of rotation in the e_i^*/e_j^* plane in the case $\theta_{ii} = \theta_{jj}$, we find that triad (e_i^*) is Fermi-propagated, i.e. $\Omega_l = 0$ for any l. Eqs (I.6)–(I.8) give $\theta_{ii} = -R^{-1}\dot{R} - c_i R^{-3}$, $S_i = 2b_i R^{-3}$ where b_i, c_i are constants and $c_1 + c_2 + c_3 = 0$. The spin and the exterior derivative of the basis forms may be written in the matrix notation as

$$[S_{ij}] = R^{-3}(M^{\mathrm{T}} - M), \tag{3.1}$$

$$d[\theta^{*i}] = dt \wedge (R^{-1}\dot{R} + R^{-3}M)\,[\theta^{*i}], \tag{3.2}$$

where

$$M = \begin{pmatrix} c_1 & -b_3 & b_2 \\ b_3 & c_2 & -b_1 \\ -b_2 & b_1 & c_3 \end{pmatrix}, \; c_1 + c_2 + c_3 = 0. \tag{3.3}$$

The field equations reduce to the Friedmann equations

$$3R^{-2}\dot{R}^2 + \mu R^{-6} = \varepsilon, \tag{3.4}$$

$$-2R^{-1}\ddot{R} - R^{-2}\dot{R}^2 + \mu R^{-6} = p, \tag{3.5}$$

where

$$\mu = -\tfrac{1}{2}\,\mathrm{Tr}\, M^2, \tag{3.6}$$

If we assume an equation of state $p = p(\varepsilon)$, then (3.4), (3.5) constitute the complete system of equations with respect to the functions R, ε.

We are interested in the description of spacetime in local coordinates. For type I there exist coordinates x^i such that $[\theta^{*i}] = RX(t)[dx^i]$, where X is an unknown matrix. With the frame (dx^i)

$$[g_{ij}] = -R^2 X^{\mathrm{T}} X, \tag{3.7}$$

$$[S_{ij}] = R^{-1} X^{\mathrm{T}}(M^{\mathrm{T}} - M)X. \tag{3.8}$$

Eq. (3.2) is equivalent to

$$\dot{X} = R^{-3}MX. \tag{3.9}$$

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[3] In this case the Eq. (1.11) implies either $\varepsilon \neq$ const., then $\varepsilon = \varepsilon(t)$, $p = p(t)$ is the parametric equation of state, or $\varepsilon =$ const. and $p = -\varepsilon$.

TABLE II

Case	Λ	E_Λ	μ	Transformations H	Number of parameters
(a)	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\frac{1}{2}\lambda+\Delta & 0 \\ 0 & & -\frac{1}{2}\lambda-\Delta \end{pmatrix}$	$e^{-\frac{1}{2}\lambda F}\cdot\begin{pmatrix} e^{\frac{3}{2}\lambda F} & 0 & 0 \\ 0 & e^{\Delta F} & 0 \\ 0 & 0 & e^{-\Delta F} \end{pmatrix}$	$-\frac{3}{4}\lambda^2-\Delta^2$	$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$	5
(b)	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\frac{1}{2}\lambda & 0 \\ 0 & -\Delta & -\frac{1}{2}\lambda \end{pmatrix}$	$e^{-\frac{1}{2}\lambda F}\cdot\begin{pmatrix} e^{\frac{3}{2}\lambda F} & 0 & 0 \\ 0 & \cos(\Delta F) & \sin(\Delta F) \\ 0 & -\sin(\Delta F) & \cos(\Delta F) \end{pmatrix}$	$-\frac{3}{4}\lambda^2+\Delta_2$	$\begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & -C & B \end{pmatrix}$	5
(c)	$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\frac{1}{2}\lambda & 1 \\ 0 & 0 & -\frac{1}{2}\lambda \end{pmatrix}$	$e^{-\frac{1}{2}\lambda F}\cdot\begin{pmatrix} e^{\frac{3}{2}\lambda F} & 0 & 0 \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix}$	$-\frac{3}{4}\lambda^2$	$\begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & 0 & B \end{pmatrix}$	4
(*d*)	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & F & \frac{1}{2}F^2 \\ 0 & 1 & F \\ 0 & 0 & 1 \end{pmatrix}$	0	$\begin{pmatrix} A & B & C \\ 0 & A & B \\ 0 & 0 & A \end{pmatrix}$	3

According to the algebraic classification of M we obtain four types of solutions. The matrix M can be represented in the form $M = A\Lambda A^{-1}$; Λ is given in Table II, and A is a non-singular real matrix depending on M. Substituting $E_\Lambda := A^{-1}X$ in (3.9) we obtain

$$\dot{E}_\Lambda = R^{-3}\Lambda E_\Lambda. \tag{3.10}$$

It is enough to find only one solution of (3.10) since different solutions can be transformed one into another by change of the coordinates. From (3.7) and (3.8) it results that

$$[g_{ij}] = -R^2 E_\Lambda^{\mathrm{T}} B E_\Lambda, \tag{3.11}$$

$$[S_{ij}] = R^{-1} E_\Lambda^{\mathrm{T}}(\Lambda^{\mathrm{T}}B - B\Lambda)E_\Lambda = R^{-3}([g_{ij}]\Lambda - \Lambda^{\mathrm{T}}[g_{ij}]), \tag{3.12}$$

where

$$B = A^{\mathrm{T}}A. \tag{3.13}$$

Now it is not difficult to show that the parameters occurring in Λ may take all values, and the matrix B is only restricted by general assumptions concerning g_{ij}.

The metric and spin given by (3.11), (3.12) and Table II, where R satisfies the Friedmann Eqs (3.4), (3.5), and B is a positively defined, symmetric, constant matrix 3×3, are the general type I solution written in coordinates.

Table II enumerates the affine transformations of coordinates H, which can be compensated by the change $B \to H^{\mathrm{T}}BH$. Using them it is possible to obtain

$R^6 = -\det[g_{ij}]$. R corresponds then to the scale factor in the Robertson-Walker models. The last column of Table II indicates the number of independent parameters of solution.

The case (a) contains all solutions without spin, they occur when $B = \mathbf{1}$ and are given by (3.15) with $b = 0$.

The relations between components of M on one hand and A, B on the other become simple when the spin vector is a shear eigenvector, say, $b_1 = b$, $b_2 = b_3 = 0$. Using the notation $\theta'^i := dx^i$ and

$$\lambda := -c_2 - c_3, \quad \alpha := \tfrac{1}{2}|c_2 - c_3|, \quad \Delta := |b^2 - \alpha^2|^{\frac{1}{2}}, \tag{3.14}$$

we can write the line element and the constant μ as follows:

$$|b| < \alpha \Rightarrow \begin{cases} \mu = -\frac{3}{4}\lambda^2 - \Delta^2, \\ ds^2 = dt^2 - R^2 \exp(2\lambda F)(\theta'^1)^2 \\ -\Delta^{-1} R^2 \exp(-\lambda F)[\alpha \exp(2\Delta F)(\theta'^2)^2 + \alpha \exp(-2\Delta F)(\theta'^3)^2 + 2b\theta'^2\theta'^3]; \end{cases} \tag{3.15a}$$

$$|b| < \alpha \Rightarrow \begin{cases} \mu = -\frac{3}{4}\lambda^2 + \Delta^2, \\ ds^2 = dt^2 - R^2 \exp(2\lambda F)(\theta'^1)^2 \\ -\Delta^{-1} R^2 \exp(-\lambda F)\}[|b| + \alpha \cdot \sin(2\Delta F)](\theta'^2)^2 + [|b| - \alpha \sin(2\Delta F)(\theta'^3)^2 \\ \quad + 2(\operatorname{sgn} b)\, \alpha \cdot \cos(2\Delta F)\theta'^2\theta'^3\}; \end{cases} \tag{3.15b}$$

$$|b| = \alpha \Rightarrow \begin{cases} \mu = -\frac{3}{4}\lambda^2, \\ ds^2 = dt^2 - R^2 \exp(2\lambda F)(\theta'^1)^2 - R^2 \exp(-\lambda F)[(\theta'^2)^2 \\ \quad + (1 + 4b^2F^2)(\theta'^3)^2 - 4bF\theta'^2\theta'^3]. \end{cases} \tag{31.5cd}$$

In all these cases the components of spin are

$$S_{23} = 2bR^{-1} \exp(-\lambda F), \quad S_{12} = S_{13} = 0. \tag{3.16}$$

These solutions reduce to those recently found by Kopczyński [3] and the author [6] if we put $\alpha = 0$ or $\lambda = 0$, respectively. (3.15*b*) and (3.16) with $\alpha = \lambda = 0$ represent the general solution of type I with the Robertson-Walker line element.

4. *Type II*

Applying a certain rotation in the e_2^*/e_3^* plane one can obtain $\theta_{23} = 0$. Eqs (I.4) imply that $\theta_{12} = \theta_{13} = 0$ and $S_1 = 0$. From Eqs (I.5) and the Jacobi identities (I.3) it results

$$(\theta_{11} - \theta_{22})S_3 = (\theta_{11} - \theta_{33})S_2 = 0,$$

which under the assumption $S \neq 0$ requires $\theta_{11} = \theta_{22}$ or $\theta_{11} = \theta_{33}$, but neither of these equalities is admitted by Eqs (I.6), (I.7).

There are no type II solutions with non-vanishing spin.

5. *Type VII$_0$*

Appendix II shows that the assumptions $n_1 \neq n_2$, $S \neq 0$ are incompatible with each other. In this section the case $n_1 = n_2 =: n$ is considered. We choose the frame (θ^{*i}) such that $\Omega_3 = 0$. From Eqs (I.3) and (I.4) it results that the triad (e_i^*) is Fermi propagated,

$$\Omega_i = 0, \tag{5.1}$$

and

$$\theta_{12} = 0,\ \theta_{13} = -\tfrac{1}{2}S_2,\ \theta_{23} = \tfrac{1}{2}S_1. \tag{5.2}$$

Using Eqs (I.9) we find

$$S_i = 2b_i R^{-3}, \tag{5.3}$$

where b_i are constants. Eq. (I.7) and the Jacobi identities (I.2) yield the relations (written with suitable recaling of R)

$$\theta_{11} = \theta_{22} = -R^{-1}\dot{R}+\frac{c}{2}R^{-3},$$

$$\theta_{33} = -R^{-1}\dot{R}-cR^{-3}, \tag{5.4}$$

$$n = R^{-1}\exp(-cF), \tag{5.5}$$

where c is a constant. Applying constant rotation in the e_1^*/e_2^* plane we can come at $S_1 = 0$. From this it follows that we may assume $b_1 = 0$ without loss of generality. Then the exterior derivative of the basis takes the form:

$$d[\theta^{*i}] = dt \wedge (R^{-1}\dot{R}+M)[\theta^{*i}]+n\theta^{*3} \wedge N[\theta^{*i}], \tag{5.6}$$

where

$$M = \begin{bmatrix} -\frac{c}{2} & -b_3 & 2b_2 \\ b_3 & -\frac{c}{2} & 0 \\ 0 & 0 & c \end{bmatrix}, \quad N = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.7}$$

The field equations reduce to the Friedmann Eqs (2.11), (2.12) with $R^* = 0$ and $\mu = b_3^2-\frac{3}{4}c^2$. They are the same as for type I.

If $b_2 = 0$ then the rotation about the axis e_1^* by the angle φ, satisfying $d\varphi = -n\theta^{*3}$, leads to the elimination of n without a change of other quantities. It is easy to show that *the type VII_0 solution with non-vanishing spin is the case of type I if, and only if, the spin is an eigenvector of n^{ij}, i.e. $b_2 = 0$ in (5.7).*

We are interested in the solution written in the basis (θ'^i) invariant under the group of type VII_0. (θ'^i) may be chosen in such a way that

$$d\theta'^1 = \theta'^3 \wedge \theta'^2,$$

$$d\theta'^2 = -\theta'^3 \wedge \theta'^1, \tag{5.8}$$

$$d\theta'^3 = 0.$$

The structure constants do not change under the transformations described by the constant matrices of the form

$$\begin{pmatrix} A & B & C \\ -kB & kA & 0 \\ 0 & 0 & k \end{pmatrix}, \quad k = \pm 1. \tag{5.9}$$

The basis (θ^{*i}) may be connected with (θ'^i) by unknown matrix $X(t)$ with components X^i_j,

$$\theta^{*i} = RX^i{}_j\theta'^j. \tag{5.10}$$

Using (5.6)–(5.9) the exterior differentiation of (5.10) leads to the equation

$$\dot{X} = R^{-3}MX \tag{5.11}$$

with the constraints

$$X^3{}_3 = \exp(cF), \quad X^3{}_1 = X^3{}_2 = 0, \quad X^1{}_1 = X^2{}_2, \quad X^2{}_1 = -X^1{}_2. \tag{5.12}$$

Depending on whether $b_3^2+c^2$ is zero or not, we obtain two classes of solutions. With suitable choice of basis (θ'^i) the metric and the spin tensor may be written as follows

$$b_3^2+c^2 \neq 0 \Rightarrow \begin{cases} \mu = b_3^2 - \frac{3}{4}c^2, \\ ds^2 = dt^2 - R^2\exp(-cF)\,[(\theta'^1)^2+(\theta'^2)^2] - (1+a^2)R^2\exp(2cF)\,(\theta'^3)^2 \\ +2aR^2\exp\left(\frac{c}{2}F\right)[\sin(b_3F)\theta'^1+\cos(b_3F)\theta'^2], \\ [S_{ij}] = R^{-3}([g_{ij}]\Lambda - \Lambda^{\mathrm{T}}[g_{ij}]), \quad \Lambda = M(b_2=0); \end{cases} \tag{5.13a}$$

$$b_3 = c = 0 \Rightarrow \begin{cases} \mu = 0, \\ ds^2 = dt^2 - R^2[(\theta'^1)^2+(\theta'^2)^2+(1+a^2F^2)(\theta'^3)^2 - 2aF\theta'^2\theta'^3]. \\ S_{23} = aR^{-1}, \ S_{12} = S_{13} = 0. \end{cases} \tag{5.13b}$$

The constant a, depend on b_i and c in the following way

$$a = 2b_2(\tfrac{9}{4}c^2+b_3^2)^{-1/2}, \tag{5.14a}$$

$$a = 2b_2. \tag{5.14b}$$

The metric and the spin given by (5.13), where θ'^i satisfy Eqs (5.8), R satisfies the Friedmann Eqs (3.4), (3.5), and a, b, c are arbitrary constants, is the general type VII_0 solution with non-vanishing spin.

g_{ij} and S_{ij} in Eq. (5.13a) are of the same kind as the transformed tensors for the case (b) of type I, and for (5.13b) are of the same kind as for the case (c) of type I. Putting $a = c = 0$ in (5.13a) we obtain the solution with the isotropic metric, which is also invariant under the group of type I.

To describe the solution in coordinates we may use the following representation of (θ'^i):

$$\begin{aligned} \theta'^1 &= \cos x^3\,dx^1 + \sin x^3\,dx^2, \\ \theta'^2 &= -\sin x^3\,dx^1 + \cos x^3\,dx^2, \\ \theta'^3 &= dx^3. \end{aligned} \tag{5.15}$$

6. Type VI_0

There are two families of the type VI_0 spacetimes. Appendix II deals with solutions for which n_1+n_2 is not zero. It is shown that *if the group is type VI_0 and $(n_1+n_2)S \neq 0$ then the equation of state is determined up to a constant.*

In this section we consider the case $n_1 = -n_2 = :n$. Eqs (I.3) and (I.4) imply

$$\theta_{12} = 0,\ \theta_{13} = \tfrac{1}{2}S_2,\ \theta_{23} = -\tfrac{1}{2}S_1, \tag{6.1}$$

$$\Omega_3 = \tfrac{1}{2}S_3,\ \Omega_2 = S_2,\ \Omega_1 = S_1. \tag{6.2}$$

Substituting these relations and the equality $\theta_1 = \theta_2$, derived from (I.2), in (I.6) and (12) we obtain $S_1 = S_2 = 0$. From (I.8) it results in $S_3 = 2bR^{-3}$, where b is a constant. Eq. (I.7) yields a differential link between the functions R and n

$$\partial_0[R^3(R^{-1}\dot{R}+n^{-1}\dot{n})]+4n^2R^3 = 0. \tag{6.3}$$

In the case $n_1 = -n_2 = :n$ of type VI_0 the field equations reduce to Eq. (6.3), *the Friedmann Eqs* (*2.11*), (*2.12*), *and the relations*

$$\theta_{ij} = 0 \quad (i \neq j), \quad \theta_{11} = \theta_{22} = -\tfrac{3}{2}R^{-1}\dot{R}-\tfrac{1}{2}n^{-1}\dot{n}, \quad \theta_{33} = n^{-1}\dot{n}, \tag{6.4}$$

$$\Omega_1 = \Omega_2 = 0,\ \Omega_3 = bR^{-3}, \tag{6.5}$$

$$S_1 = S_2 = 0,\ S_3 = 2bR^{-3}. \tag{6.6}$$

7. Types IX, VIII

First we consider the case when some eigenvalues n_i are equal. Under the assumption $S \neq 0$ the equality $n_1 = n_2 = n_3$ is ruled out by Eq (I.4). This means that the expansion tensor is anisotropic and the metric cannot be that of the Robertson-Walker.

If only two eigenvalues n_i are the same, then we may assume $n_1 = n_2 =: n \neq n_3$ without the loss of generality. Eqs (I.4) and the Jacobi identities (I.3) imply

$$\theta_{12} = 0,\ \theta_{13} = -\tfrac{1}{2}n(n-n_3)^{-1}S_2,\ \theta_{23} = \tfrac{1}{2}n(n-n_3)^{-1}S_1, \tag{7.1}$$

$$\Omega_{1,2} = \tfrac{1}{2}n_3(n_3-3n)(n-n_3)^{-2}S_{1,2},\ S_3 = 0.$$

Substituting (7.1) and the equality $\theta_{11} = \theta_{22}$, resulting from (I.2), in Eqs (I.6) and (12) we find $n_3 = 3n$ under the assumption $S \neq 0$. Then (I.2) shows that $\theta_{11} = \theta_{22} = \theta_{33}$, what is inconsistent with (1.7).

If all n_i are different, Eqs (I.2)–(I.4) are equivalent to the following relations:

$$\theta_{ii} = \tfrac{1}{2}n_j^{-1}\dot{n}_j+\tfrac{1}{2}n_k^{-1}\dot{n}_k, \tag{7.2}$$

$$\theta_{ij} = \tfrac{1}{2}(n_i-n_j)^{-1}n_kS_k, \tag{7.3}$$

$$\Omega_i = \tfrac{1}{2}[1-n_i(n_j+n_k)(n_j-n_k)^{-2}]S_i, \tag{7.4}$$

where (ijk) is any even permutation of (123). From (7.2) it follows that $R^{-3} = \text{const.} n_1 n_2 n_3$. The comparison of (I.5) with (I.8) by using (7.2)–(7.4) leads to the conditions of consistency

$$\partial_0(n_i^{-1}n_j^{-1}R^*)S_k - n_i^{-1}n_j^{-1}R^*(n_i+n_j-n_k)(n_i-n_j)(n_i-n_k)^{-1}(n_j-n_k)^{-1}S_iS_j = 0, \quad (7.5)$$

$$S_iS_j = 0 \text{ if } n_i+n_j-n_k = 0. \quad (7.6)$$

Summing up Eqs (7.5) multiplied by

$$n_in_j(n_i-n_j+n_k)(-n_i+n_j+n_k)(n_i-n_k)(n_j-n_k)(n_i-n_j)S_iS_j,$$

and using the identity

$$\sum_{(ijk)} \partial_0(n_i^{-1}n_j^{-1}R^*)n_in_j(n_i-n_j+n_k)(-n_i+n_j+n_k) = 0, \quad (7.7)$$

we obtain

$$R^*\Big[\prod_{(ijk)}(n_i+n_j-n_k)\Big]\cdot\Big[\sum_{ij}(n_i-n_j)^2S_i^2S_j^2\Big] = 0. \quad (7.8)$$

Below we consider all the cases, for which (7.8) is satisfied.

(a) $R^* = 0$

This is the case of type IX with $n_3 = (\sqrt{n_1} \pm \sqrt{n_2})^2$. From (7.2) it results in

$$\theta_{33}-\theta_{11} = (1+h)^{-1}(\theta_{22}-\theta_{11}), \quad (7.9)$$

where $h := \pm\sqrt{n_1n_2^{-1}}$. The comparison of (I.6) with (I.7) by using (7.9) leads to the inconsistency

$$(1+h)^{-2}h^{-2}h^2+(2+h)^{-2}S_1^2+(1+2h)^{-2}S_2^2+(1-h)^{-2}S_3^2+4(h^2+h+1)n_2^2 = 0.$$

Since we have not used the assumption $S \neq 0$ we see that *there are no type IX solutions with $R^* = 0$.*

(b) $n_i+n_j = n_k$ for fixed (ijk), and $S_i^2+S_j^2 \neq 0$ for any different i, j.

We may assume $n_1 = n_2+n_3$ without the loss of generality. Considering the equality $S_1S_2 = 0$, resulting from (7.6), and Eqs (7.5), which take the form

$$S_2\partial_0 \ln|1+n_3n_2^{-1}|+2n_2(n_2-n_3)^{-1}S_1S_3 = 0,$$

$$S_3\partial_0 \ln|1+n_2n_3^{-1}|+2n_3(n_2-n_3)^{-1}S_1S_2 = 0, \quad (7.10)$$

we obtain $n_3 = \text{const}\cdot n_2$ and $S_1 = 0$. The latter equality is inconsistent with (I.8) for $i = 1$.

(c) $S_1S_2 = S_1S_3 = S_2S_3 = 0$

This means that S^i is an eigenvector of both the expansion tensor and the tensor n_{ij}. Giving up the assumption concerning signs of n_i when the group is of type VIII, we may assume $S_1 = S_2 = 0$, $S_3 \neq 0$ without the loss of generality. From (I.8) it results in

$$S_3 = bn_1n_2n_3, \qquad b = \text{const.} \quad (7.11)$$

Eqs (7.5) imply $R^* = \text{const.}\, n_1 n_2$ which shows

$$n_3 = n_1 + n_2 + 2c\sqrt{|n_1 n_2|}, \qquad c = \text{const.} \tag{7.12}$$

All quantities θ_{ij}, Ω_i, S_i can be represented as explicit functions of $n_1, n_2, \dot{n}_1, \dot{n}_2$. (I.6) and (I.7) reduce to the complete first order system of differential equations with respect to n_1, n_2, which is compatible under some assumptions concerning c. The Friedmann equations yield the expressions for ε and p in terms of n_1, n_2.

We conclude that

Only the type IX and VIII solutions with non-vanishing spin are those described in the case (c). If matter expands then the equation of state is determined up to a constant.

For these types there exist such solutions that ε, p are constant. Neither of them satisfies the condition $-\frac{1}{3}\varepsilon \leqslant p \leqslant \varepsilon$.

8. *Type V*

We use the freedom of rotation in the e_2^*/e_3^* plane to make $\theta_{23} = 0$. From the Jacobi identities and Eqs (I.4) it results

$$\theta_{12} = -\tfrac{1}{6} S_3, \qquad \theta_{13} = \tfrac{1}{6} S_2,$$

$$\Omega_3 = \tfrac{1}{3} S_3, \qquad \Omega_2 = \tfrac{1}{3} S_2, \tag{8.1}$$

$$a^{-1}\dot{a} = -\theta_1, \qquad -2\theta_{11} + \theta_{22} + \theta_{33} = 0, \tag{8.2}$$

With these restrictions the sum of Eqs (I.6) and (I.7) shows that $S_2 = S_3 = 0$; therefore S^i is a shear eigenvector. Eq. (23) implies that $\Omega_1 = 0$, or, if $\theta_{22} = \theta_{33}$ we may choose such Ω_1 which does not violate this equation. From Eqs (I.6) and (8.2) it follows that $\theta_i = R^{-1}\dot{R} + c_i R^{-3}$, $a = R^{-1}$, where $c_1 = 0$, $c_3 = -c_2$ and c_2 is a constant. Considering the equality $S_1 = 2bR^{-3}$(b = const.) resulting from (I.8) we may write the exterior derivative of the basis forms in the following way

$$d[\theta^{*i}] = dt \wedge (R^{-1}\dot{R} + R^{-3}M)[\theta^{*i}] - R^{-1}\theta^{*1} \wedge [\theta^{*i}], \tag{8.3}$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c_2 & -b \\ 0 & b & -c_2 \end{pmatrix}. \tag{8.4}$$

The field equations reduce to the Friedmann equations

$$3R^{-2}\dot{R}^2 - 3R^{-2} + \mu R^{-6} = \varepsilon, \tag{8.5}$$

$$-2R^{-1}\ddot{R} - R^{-2}\dot{R}^2 + R^{-2} + \mu R^{-6} = p,$$

where

$$\mu = -\tfrac{1}{2}\operatorname{Tr} M^2 = b^2 - c_2^2.$$

For type V there exists a frame (θ'^i) invariant under the group and such that

$$d[\theta'^i] = -\theta'^1 \wedge [\theta'^i]. \tag{8.6}$$

The structure constants do not change under the trnasformations represented by constant matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ A & B & C \\ D & E & G \end{pmatrix}. \tag{8.7}$$

If we define the matrix $X(t)$ with components $X^i{}_j$ by the relations $\theta^{*i} = RX^i{}_j\theta'^j$ then (8.3) together with (8.6) give

$$\dot{X} = R^{-3}MX, \tag{8.8}$$

with the conditions

$$X^1{}_1 = 1, \ X^1{}_2 = X^1{}_3 = 0. \tag{8.9}$$

Eq. (8.8) may be solved similarly as the corresponding equation for the case $b_2 = b_3 = 0$ of type I. The constraints (8.9) imply that the constants of integration can be eliminated by using a transformation of the form (8.7).

The metric and the spin given by (3.15), (3.16), where θ'^i satisfy (8.6), R satisfies the Friedmann Eqs (8.5), and $c_3 = -c_2$, is the general solution of type V.

This solution has already been found [6] but without the proof of generality. Putting $b = 0$ we obtain all solutions without spin. (3.15b) and (3.16) with $c_2 = c_3 = 0$ represent the general solution of type V with the Robertson-Walker line element given in the frame invariant under the group.

To express the solution in coordinates we may use the following representation of θ'^i:

$$\theta'^1 = -dx^1, \quad \theta'^2 = \exp x^1 dx^2, \quad \theta'^3 = \exp x^1 dx^3. \tag{8.10}$$

9. Types VII_h, VI_h, IV.

First let us discuss the special case of type VII_h when $n_2 = n_3 = : n$. Then we may assume $\Omega_1 = 0$ without the loss of generality. From Eqs (I.2) and (01) it follows that $\theta_{ii} = -R^{-1}\dot{R}$, $n = d \cdot R^{-1}$ (d = const.), and from the third Eq. of (I.3) $\theta_{23} = 0$. Substituting these relations in (23), (I.6) and (I.7) we obtain either $\Omega_2 = \Omega_3 = 0$ or $\theta_{12} = \theta_{13} = 0$. Eqs (I.3), (02), (03) imply that all $\Omega_2, \Omega_3, \theta_{12}, \theta_{13}, S_2, S_3$ are required to vanish. Eqs (I.8) yield $S_1 = 2bR^{-3}$ where b is a constant, and (I.1) yield $a = R^{-1}$ with suitable rescaling of R. The exterior derivative of the basis (θ^{*i}) may be written in the following way:

$$d[\theta^{*i}] = dt \wedge (R^{-1}\dot{R}+R^{-3}M)[\theta^{*i}]+R^{-1}\theta^{*1} \wedge N[\theta^{*i}], \tag{9.1}$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -b \\ 0 & b & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & d \\ 0 & -d & 1 \end{pmatrix}. \tag{9.2}$$

The field equations reduce to the Friedmann equations (2.11), (2.12) with $R^* = 6R^{-2}$ and $\mu = -b^2$.

The coefficient n can be eliminated by means of rotation about the axis e_1^* by an angle φ satisfying the condition $d\varphi = -n\theta^{*1}$ (existence of φ is secured by (9.1) and (9.2)). The other quantities do not change under this transformation. We see that *the type VII_h solutions with $n_2 = n_3$ constitute the subset of the type V solutions with the isotropic metric.*

If the group is of type VII_h with $n_2 \neq n_3$, or type VI_h, or IV then the conditions of consistency of the field equations are very complex. One can show that solutions with $S_2 = S_3 = 0$, $S_1 \neq 0$ exist and are singular. In this case the equation of state is determined up to a constant and satisfies the energy dominant conditions $-\frac{1}{3}\varepsilon \leqslant p \leqslant \varepsilon$. We are not able to state whether other solutions of these types exist.

10. Some properties of the solutions of types I, VII_0, V

In Sections 3 to 9 we have shown that for all types except type II there exist solutions with non-vanishing spin. For types I, VII_0, V, we have found representations of solutions in terms of R and F. Under the assumption $S \neq 0$ these types contain all solutions with the Robertson-Walker line element.

Behaviour of type I, VII_0, or V solutions is determined by function R. They are non-singular if, and only if, $R(t)$ is regular and positive for any t. Let us assume the equation of state of the form $p = p(\varepsilon)$ [4]. Development of R is determined by the Friedmann equations which read

$$3R^{-2}\dot{R}^2 - 3kR^{-2} + \mu R^{-6} = \varepsilon \tag{10.1}$$

$$-2R^{-1}\ddot{R} - R^{-2}\dot{R} + kR^{-2} + \mu R^{-6} = p, \tag{10.2}$$

where $k = 0$ for types I, VII_0, and $k = 1$ for type V. If $\dot{R} \neq 0$ then Eq. (10.2) may be replaced by the conservation law of enthropy

$$\dot{\varepsilon} + 3R^{-1}\dot{R}(\varepsilon + p) = 0, \tag{10.3}$$

which implies $\varepsilon = \varepsilon(R)$. (10.1) takes the form of the "energy integral",

$$\dot{R}^2 + V(R) = 0, \tag{10.4}$$

where

$$V(R) = \tfrac{1}{3}\mu R^{-4} - \tfrac{1}{3}R^2\varepsilon(R) - k. \tag{10.5}$$

We look for physically reasonable conditions which secure the non-singularity of R. Assuming that

$$p \leqslant \gamma\,\varepsilon, \qquad \gamma = \text{const.} < 1, \tag{10.6}$$

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[4] One can consider $p = p(\varepsilon, S)$, since from the conservation law of spin it results $S = \text{const.}\ R^{-3}$, so further conclusions are the same as for $p = p(\varepsilon)$.

one can show, by the integral form of (10.3), that $\varepsilon \leqslant \text{const.}\ R^{\alpha}$, $\alpha > -6$ for $R \to 0$, therefore $V \xrightarrow[R\to 0]{} \infty$ if the density of spin exceeds the doubled scalar of shear, i.e.

$$\mu > 0. \tag{10.7}$$

Conditions (10.6), (10.7) prevent the solution from ever approaching zero. If moreover

$$|p| \leqslant \varepsilon \tag{10.8}$$

then $R(t)$ is regular for any t and $R \to \infty$, $F \to$ const. $\pm$ for $t \to \pm\infty$.

The types I, VII_0, V models, satisfying conditions (10.6), (10.7), (10.8), are non--singular and tend asymptotically to the solutions with the isotropic metric.

This latter results from the representations of metric and spin in terms of R and F, and the freedom of transformations conserving the structure constants. The asymptotic behaviours of solutions do not depend on the assumption (10.7).

Estimations of the minimum average length R_m and other quantities for the non--singular type I models with the vanishing pressure and shear may be found in [4]. One can show that these models give the best estimations amongst the types I, VII_0, V models satisfying $0 \leqslant p \leqslant \varepsilon$, $\varepsilon(t_0) = \varepsilon_0$, $S(t_0) = S_0$ where t_0, ε_0, S_0 are fixed. With a suitable choice of R, such that $R(t_0)$ is fixed, we obtain the following inequalities:

$$R_m(V, p, \sigma) < R_m(I, p, \sigma) \leqslant \left\{ \begin{matrix} R_m(I, 0, \sigma) \\ R_m(I, p, 0) \end{matrix} \right\} \leqslant R_m(I, 0, 0). \tag{10.9}$$

The expressions in brackets indicate the type of the group, the equation of state, and the value of shear[5]. The energy and spin densities are maximum for $R = R_m$ and they obey the opposite inequalities to R_m. We are not able to show the occurrence of the non-singular solutions of other types. If they exist and $R^* > 0$ then

$$R_m(R^* > 0, p, \sigma) < R_m(I, p, 0). \tag{10.10}$$

Let v^a denote the normalized vector ($v^a v_a = 1$), which is tangent to time geodetic. In the frame (θ^{*a}) the geodetic equation for the type I models reads

$$\dot{v}^i + (\theta^i{}_j + \tfrac{1}{2} S^i{}_j) v^j = 0 \tag{10.11}$$

and gives

$$[v^i] = R^{-1} A E_A^{-1} [\alpha^i] \tag{10.12}$$

where α^i are arbitrary constants.

It seems that the motion of a particle with spin is described by an equation more complex than (10.11), while particles without spin move along Riemannian geodetics [10]. In the latter case we must replace (10.11) by the equation

$$\dot{v}^i + (\theta^i{}_j - \tfrac{1}{2} S^i{}_j) v^j = 0 \tag{10.13}$$

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[5] The estimations for type I and VII_0 are the same.

which yields

$$[v^i] = R^{-1}A^{-\mathrm{T}}E_A^{-\mathrm{T}}[\alpha^i]. \tag{10.14}$$

Eqs (10.12) and (10.14) together with (3.12) and Table II imply that geodetics may twist in the non-rotating rest frame of matter defined by $\tilde{\Omega}^i = 0$. There exist such universes where the global change of direction of v^i may exceed a multiple of 2π. Such behaviour is impossible when the spin vanishes.

Properties of geodetics for type VII_0 and V are similar to those for type I, however, their physical interpretation is not clear.

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APPENDIX I

With respect to the frame (e_a^*) the Jacobi identities read as follows:

$$\begin{aligned} &\dot{a} - \theta_{11}a = 0, \\ &a(\theta_{12} + \tfrac{1}{2}S_3 - \Omega_3) = 0, \\ &a(\theta_{13} - \tfrac{1}{2}S_2 + \Omega_2) = 0, \end{aligned} \tag{I.1}$$

$$\begin{aligned} &\dot{n}_1 + (\theta_{11} - \theta_{22} - \theta_{33})n_1 = 0, \\ &\dot{n}_2 + (-\theta_{11} + \theta_{22} - \theta_{33})n_2 = 0, \\ &\dot{n}_3 + (-\theta_{11} - \theta_{22} + \theta_{33})n_3 = 0, \end{aligned} \tag{I.2}$$

$$\begin{aligned} &(n_1 - n_2)(\tfrac{1}{2}S_3 - \Omega_3) - (n_1 + n_2)\theta_{12} = 0, \\ &(n_3 - n_1)(\tfrac{1}{2}S_2 - \Omega_2) - (n_1 + n_3)\theta_{13} = 0, \\ &(n_2 - n_3)(\tfrac{1}{2}S_1 - \Omega_1) - (n_2 + n_3)\theta_{23} = 0, \end{aligned} \tag{I.3}$$

and the field equations except the Friedmann equations are

$$\begin{aligned} &a(2\theta_{11} - \theta_{22} - \theta_{33}) + (n_2 - n_3)\theta_{23} - \tfrac{1}{2}n_1S_1 = 0, && (01) \\ &3a\theta_{12} + (n_3 - n_1)\theta_{13} + \tfrac{1}{2}aS_3 - -\tfrac{1}{2}n_2S_3 = 0, && (02) \\ &3a\theta_{13} + (n_1 - n_2)\theta_{12} - \tfrac{1}{2}aS_2 - \tfrac{1}{2}n_3S_3 = 0, && (03) \end{aligned} \tag{I.4}$$

$$\begin{aligned} &-\dot{\theta}_{12} - \theta\theta_{12} + (\theta_{22} - \theta_{11})\Omega_3 + \theta_{13}\Omega_1 - \theta_{23}\Omega_2 = 0, && (12) \\ &-\dot{\theta}_{13} - \theta\theta_{13} + (\theta_{11} - \theta_{33})\Omega_2 - \theta_{12}\Omega_1 + \theta_{23}\Omega_3 = 0, && (13) \end{aligned} \tag{I.5}$$

$$-\dot{\theta}_{23}-\theta\theta_{23}+(\theta_{33}-\theta_{22})\Omega_1+\theta_{12}\Omega_2-\theta_{13}\Omega_3+a(n_2-n_3) = 0, \qquad (23)$$

$$\dot{\theta}_{11}-\dot{\theta}_{22}+\theta(\theta_{11}-\theta_{22})-4\theta_{12}\Omega_3+2\theta_{13}\Omega_2+2\theta_{23}\Omega_1+(n_2-n_1)(n_1+n_2-n_3) = 0, \qquad (I.6)$$

$$\dot{\theta}_{11}-\dot{\theta}_{33}+\theta(\theta_{11}-\theta_{33})-2\theta_{12}\Omega_3+4\theta_{13}\Omega_2-2\theta_{23}\Omega_1+(n_3-n_1)(n_1-n_2+n_3) = 0, \qquad (I.7)$$

$$\dot{S}_i+\theta S_i+\varepsilon_{ijk}\Omega^j S^k = 0. \qquad (I.8)$$

APPENDIX II

In the type VII_0 or VI_0 case the Jacobi identities (I.2) yield the principal expansions in the form

$$\theta_{11} = -\tfrac{3}{2} R^{-1}\dot{R}-\tfrac{1}{2} n_1^{-1}\cdot\dot{n}_1, \quad \theta_{22} = -\tfrac{3}{2} R^{-1}\dot{R}-\tfrac{1}{2} n_2^{-1}\dot{n}_2,$$

$$\theta_{33} = \tfrac{1}{2} n_1^{-1}\dot{n}_1+\tfrac{1}{2} n_2^{-1}\dot{n}_2. \qquad (II.1)$$

Eqs (I.3) and (I.4) allow us to express $\theta_{ij}(i \neq j)$ and Ω_i in terms of n_i and S_i,

$$\theta_{12} = 0, \quad \theta_{13} = -\tfrac{1}{2} n_2 n_1^{-1} S_2, \quad \theta_{23} = \tfrac{1}{2} n_1 n_2^{-1} S_1,$$

$$\Omega_1 = -\tfrac{1}{2}(n_1-n_2)n_2^{-1}S_1, \quad \Omega_2 = \tfrac{1}{2}(n_1-n_2)n_1^{-1}S_2, \quad \Omega_3 = \tfrac{1}{2} S_3 \quad (\text{if } n_1 \neq n_2) \qquad (II.2)$$

We assume $n_1 \neq |n_2|$ on an open neighbourhood, since the cases $n_1 = \pm n_2$ are considered in Sections 5, 6. Eq. (12) gives the first integral

$$2(\theta_{11}-\theta_{22})S_3+(n_1-n_2)^2 n_1^{-1} n_2^{-1} S_1 S_2 = 0. \qquad (II.3)$$

The comparison of (I.5) with (I.9) by using (II.1), (II.2) leads to other first integrals:

$$n_2(-\dot{f}+2\theta_{22}-2\theta_{11})S_2+n_1 S_1 S_3 = 0,$$

$$n_1\dot{f}S_1+n_2 S_2 S_3 = 0, \qquad (II.4)$$

where

$$f := \ln\,[n_1(n_1-n_2)^2 R^3]. \qquad (II.5)$$

Eqs (II.3) and (II.4) show that $S_1 S_2 = 0$ since if not, they yield the inconsistency

$$n_2^2 S_1^{-2} S_3^2+n_1^2 S_2^{-2} S_3^2+(n_1-n_2)^2 = 0.$$

We may assume $S_2 = 0$ without the loss of generality. Eqs (II.3), (II.4) have two kinds of solutions with non-vanishing spin, but one of them: $S_1 = S_2 = 0$, $\theta_{11} = \theta_{22}$, is inconsistent with Eq. (I.6). So the proper solution is

$$S_2 = S_3 = 0, \qquad (II.6)$$

$$R^3 = \text{const.}\ n_1^{-1}(n_1-n_2)^{-1}. \qquad (II.7)$$

Eqs (I.8) imply that

$$S_1 = 2bR^{-3}, \ b = \text{const.} \qquad (II.8)$$

Thus all quantities θ_{ij}, Ω_i, S_i can be expressed analytically by the variables n_i, $\dot{n}_i$.

The substitution of (II.7) in (II.1) gives

$$\theta_{3\bar{3}}-\theta_{11} = -(n_1+n_2)(n_1-n_2)^{-1}(\theta_{22}-\theta_{11}). \tag{II.9}$$

Comparison of (I.6) with (I.7) by using (II.9) leads to the integral

$$4(\theta_{22}-\theta_{11})^2(n_1-n_2)^{-2}+S_1^2n_2^{-2}+2n_1n_2^{-1}+n_2n_1^{-1}+1 = 0. \tag{II.10}$$

If $n_1, n_2 > 0$ then the left-hand side of the integral is positive, so there are no type VII models with $n_1 \neq n_2$. Eq. (II.10) is the first order differential equation with respect to n_1, n_2. The second independent equation of this type is obtained by substitution of $\theta_{22}-\theta_{11}$, calculated from (II.10), in Eq. (I.6).

Among the type VI_0 models with $n_1 \neq -n_2$ there exist such solutions where p, ε, S are constant. They satisfy the equation of state $p = -3\varepsilon$.

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