WAVE EQUATIONS FOR UNSTABLE PARTICLES AND RESONANCES: GENERAL CONSIDERATIONS AND SOLUBLE MODELS

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We propose a quantum-mechanical description of an unstable object, which is characterized by an equation of motion with complex energy (mass) parameter. In order to satisfy conventional axioms of QM, we introduce the source term which is proportional to the root of the imaginary part of the complex energy, and in the stable limit disappears. The decay properties are therefore characterized by a complex energy (mass) parameter and the source term which describes the deviation from "nonunitary" space-time development. We consider models with space variables, nonrelativistic and relativistic ones. The explicit formulas for simple models of wave functions are given. Finally, we present within our framework the description of an unstable V-particle in $N\Theta$-sector of the Lee model.

1. *Introduction*

Wigner's success in describing the states of stable particles as vectors in irreducible unitary representation of the Poincaré group led several authors to efforts relating the description of unstable particles with nonunitary representations of the Poincaré group [1–5] or the Poincaré semigroup [6]. The idea of having a wave function of an unstable object, characterized only by complex mass, is very appealing because the decay is described by beautifully exact exponential decay law. Besides, the parametrization is very simple, and unifies the description of stable and unstable particles.

One introduces such nonunitary wave functions by looking for the solutions of the Schrödinger equation [7]

$$i\frac{\partial}{\partial t}|M;t\rangle_0 = \hat{H}_0^M|M;t\rangle_0, \tag{1.1}$$

where $M = m_0 - i\gamma$ denotes a complex mass, and the nonhermitean Hamiltonian $\hat{H}_0^M$

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is obtained from the free Hamiltonian $\hat{H}_0^m$ for stable particle by analytic continuation $m \to M$.

Unfortunately, if $\gamma \neq 0$ the introduction of states $|M; t\rangle_0$ leads to a violation of the conventional axiom of QM, which states that the energy spectrum should be real and positive. Besides, in relativistic QM, the space-time propagation of nonunitary states violates causality [8–9]. One has the following two choices:

a) To enlarge the space of states by considering also complex eigenvalues of energy. In this way we have, in theory, unstable elementary states, but we are facing many difficulties such as asymptotic expansions of the wave functions, troubles with unitarity, analyticity and causality [10].

b) To preserve the conventional energy spectrum, and all remaining axioms [11]. It follows from such an approach that the nonunitary state be only an approximation of the state describing physical unstable particles [18].

In this paper we shall follow second possibility. We shall assume that the Hilbert space $\mathscr{H}$ of states is spanned by the eigenvectors of a total Hamiltonian $\hat{H}$. In further discussion it is sufficient to consider only the continuous part of the spectrum of $\hat{H}$, i.e. we shall assume the following spectral decomposition

$$\hat{H} = \int_0^\infty E d\hat{P}_E, \tag{1.2}$$

and the following definition of $\mathscr{H}$

$$\mathscr{H} : \{|f\rangle \in \mathscr{H} : f(E) = \langle E|f\rangle \in L_2\}. \tag{1.3}$$

The space $\mathscr{H}$ can be decomposed as follows

$$\mathscr{H} = \mathscr{H}_U \oplus \mathscr{H}_D, \tag{1.4}$$

where $\mathscr{H}_U$ describes unstable states, and $\mathscr{H}_D$ its decay products.

Our aim is to introduce the unstable state *which is characterized by complex mass, but does not lead to the violation of the spectral condition (1.2)*. We propose the following modification of equation (1.1)

$$\left(i\frac{\partial}{\partial t} - \hat{H}_0^M\right)|M; t\rangle = \left(\frac{\gamma}{\pi}\right)^{1/2} |S; t\rangle, \tag{1.5}$$

where the state vector $|S; t\rangle$ describes *the deviation from the nonunitary time development* and can be treated as *the source term in the equation for unstable object*. Equation (1.5) due to the presence factor $\gamma^{1/2}$ on the right-hand side, gives in the limit $\gamma \to 0$ the conventional equation for the wave function of the stable object

$$i\frac{\partial}{\partial t}|m; t\rangle_0 = \hat{H}_0^m|m; t\rangle_0. \tag{1.6}$$

The source term in (1.5) is responsible for the introduction of threshold and background effects. It is interesting to mention that if we derive Eq. (1.5) from dynamical equations (e.g. the Lee model), it follows that the parameter γ is proportional to the coupling constant. We see, therefore, that equation (1.5) incorporates the basic property of unstable particles — *it appears only in the presence of interactions.*

In this paper we discuss only the following wave functions

$$\Psi_M(t) = \langle M|M;t\rangle \tag{1.6a}$$

which describes the projection of $|M;t\rangle$ on the space $\mathscr{H}_U$ (see (1.4)).

Sometimes the function (1.6a) is called the nondecay probability amplitude [24]. Such time development leads to the decay property

$$P_M(t) = |\Psi_M(t)|^2 \leqslant 1.$$

Introducing the complete set of states $|D\rangle_i$ in $\mathscr{H}_D$, and the projections

$${}_i\langle D|M;t\rangle = \chi_M^i(t),$$

one can write

$$P_M(t) = 1 - |\chi_M^i(t)|^2.$$

It should be also mentioned, that in the case of unstable object it is possible to define the internal time variable τ, and the state $|\tau\rangle$ is defined as follows:

$$|\tau\rangle = \frac{1}{2\pi}\int_0^\infty e^{-i\tau E}|E\rangle dE.$$

One can characterize the state $|U\rangle$ by means of the wave function $\Phi_m(\tau) = \langle\tau|U\rangle$ defining the "Schrödinger representation". The function $\Phi_m(\tau)$ defines the "time shape" of the unstable process. The discussion of such a Schrödinger representation for the state vector, satisfying the unhomogeneous equation (1.5) leads to the formalism with two time variables: the internal time τ and "historical time" t, unitarily implemented in the Hilbert space (1.4) by the operator [25]

$$\exp\left[-i\hat{H}t\right]$$

The plan of our paper is as follows:
In Sect. 2 we consider the simplest one-dimensional case of QM with only one time variable. Firstly we discuss model (1.5) without specification of the source term. Further, we consider an example describing a wide class of unstable wave functions. In Sect. 3 we present similar considerations for three-dimensional nonrelativistic QM. The relativistic case is considered in Sect. 4. In accordance with the spirit of Eq. (1.5), we calculate the space-time development of the wave function of an unstable particle satisfying the following equation

$$\left((\Box - m_0^2)^2 + \frac{\gamma^2}{4}\right)\Psi_M(x) = \gamma\delta_+(x), \tag{1.7}$$

where

$$\delta_+(x) = \int_{V_+} d^4p e^{ipx}. \tag{1.7a}$$

In Sect. 2–4 the source term is introduced a priori, without giving any dynamical justification. In Sect. 5 we discuss the physical meaning of the source term by considering an $N\Theta$ sector of the Lee model. It appears that in dynamical models the real and imaginary parts of the mass occuring in (1.5) become opertors, i.e. we have

$$M \to \tilde{M}(E) = \tilde{M}_0(E) - i\tilde{\Gamma}(E). \tag{1.8a}$$

The choice of constant complex mass M corresponds to a generalized mass renormalization procedure, which is defined by means of one complex equation (i.e. two real equations):

$$M = \tilde{M}(M) - i\tilde{\Gamma}(M). \tag{1.8b}$$

It can be shown that the solution of (1.8b) describes the position of a complex pole of the resolvent $(z-\hat{H})^{-1}$, continued analytically on the second sheet.

In Sect. 1–5 we discuss only the one-channel Hamiltonian $\hat{H}$ with its eigenstates describing elastic scattering. In such a framework we are able to describe only a *"free" unstable particle* [19] which has been formed and subsequently decayed in the course of the resonant elastic scattering process. In order to discuss interacting unstable particles, one should introduce the Hamiltonian describing at least two coupled channels. The definition of the wave function for an unstable particle in the presence of other stable or unstable particles will be given in the forth-coming paper by one of the present authors.

2. *One-dimensional QM*

a) General considerations

We shall assume that the geometric group of motion is represented only by one--parameter group of time translations. In such a case the free Hamiltonian is reduced to a number ($\hat{H}_0^m = m$; mass is equal to energy (and equation (1.6)) has a form:

$$\left(i\frac{d}{dt} - m\right)|m; t\rangle_0 = 0, \tag{2.1}$$

or ($|m; 0\rangle_0 \equiv |m\rangle_0$)

$$|m; t\rangle_0 = e^{-imt}|m\rangle_0. \tag{2.2}$$

In order to discuss the state describing unstable particles let us introduce the source vector $|S; t\rangle$ which for $t = 0$ can be decomposed in the continuous energy basis of the total Hamiltonian as follows:

$$|S; 0\rangle \equiv |S\rangle = \int_0^\infty dE S(E)|E\rangle. \tag{2.3}$$

Using the formulae

$$|M; t\rangle = e^{-i\hat{H}t}|M\rangle = \int_0^\infty dE e^{-iEt} M(E)|E\rangle, \tag{2.4}$$

we obtain from (1.5) that

$$M(E) = \left(\frac{\gamma}{\pi}\right)^{1/2} \frac{S(E)}{E-M}. \tag{2.5}$$

From (2.5) we can calculate the wave function for an unstable particle

$$\Psi_M(t) = \langle M|e^{-i\hat{H}t}|M\rangle$$

$$= \frac{\gamma}{\pi}\int_0^\infty \frac{|S(E)|^2 \exp[-iEt]dE}{(E-M)(E-M^*)} = \frac{\gamma}{\pi}\int_0^\infty \frac{|S(E)|^2 \exp(-iEt)}{(E-m_0)^2+\gamma^2} dE. \tag{2.6}$$

The wave function (2.6) satisfies the following equation

$$\left\{\left(i\frac{d}{dt}-m_0\right)^2+\gamma^2\right\}\Psi_M(t) = \left(\frac{\gamma}{\pi}\right)^{1/2}\tilde{S}(t), \tag{2.7}$$

where

$$\tilde{S}(t) = \langle S|e^{-i\hat{H}t}|S\rangle = \int_0^\infty |S(E)|^2 e^{-iEt}dE \tag{2.8}$$

characterizes the time development of the source.

Equation (2.7) is determined if we know the *source function* $\tilde{S}(t)$. In connection with this problem we would like to make the following remarks:

Remark 1. The source function $\tilde{S}(t)$ can not vanish for any finite time interval. This result follows from the fact that $\tilde{S}(t)$ is a boundary value of an analytic function holomorphic in a lower complex t halfplane.

Remark 2. The Breit-Wigner formula for the description of wave function of an unstable particle is obtained if we assume that $S^{BW}(t) = \delta(t)$. In such a case the spectral condition for the unstable system is violated. The wave function has the form

$$\Psi_M^{BW}(t) = e^{-iMt}\Theta(t)+e^{-iM^*t}\Theta(-t) \tag{2.9}$$

which differs from the nonunitary wave functions only at $t = 0$.

Remark 3. The role of $\tilde{S}(t)$ is to introduce the threshold and proper asymptotic behaviour of the energy spectrum if $E \to \infty$. A resonable choice of the source function should not complicate, however, the analytic structure of the spectral function. We shall assume that *$S(E)$ is holomorphic for Re $E > 0$*. In such a way the Laplace transform $G_M(z)$ of the propagation function

$$G_M(z) = \langle M|\frac{1}{H-z}|M\rangle = \frac{\gamma}{\pi}\int_0^\infty \frac{|S(E)|^2 dE}{[(E-m_0)^2+\gamma^2](E-z)} \tag{2.10}$$

analytically continued to the second sheet by means of the formula (Im $z < 0$)

$$G_M^{II}(z) = G_M(z)-2\pi i\frac{\gamma}{\pi}\frac{|S(z)|^2}{[(z-m_0)^2+\gamma^2]}, \tag{2.11}$$

has in the whole quadrant ($\mathrm{Re}\, z > 0$, $\mathrm{Im}\, z < 0$) of the second Riemann sheet only one pole at $E = M = m_0 - i\gamma$.

Remark 4. We shall use the normalization condition

$$\Psi_M(0) = \langle M|M\rangle = \frac{\gamma}{\pi}\int_0^\infty \frac{|S(E)|^2 dE}{(E-m_0)^2+\gamma^2}. \tag{2.12}$$

Another normalization condition

$$\langle M|\hat{H}|M\rangle = m_0 = \frac{\gamma}{\pi}\int_0^\infty \frac{|S(E)|^2 E dE}{(E-m_0)^2+\gamma^2} \tag{2.13}$$

is a self-consistency equation, which is not easily solvable.

Remark 5. In general case, due to interaction, we obtain instead of (2.6)

$$\Psi(t) = \frac{1}{\pi}\int_0^\infty \frac{|S(E)|^2 \tilde{\Gamma}(E) e^{-iEt} dE}{[E-\tilde{M}(E)]\,[E-\tilde{M}^*(E)]}, \tag{2.14}$$

or instead of (1.5)

$$\left[i\frac{d}{dt} - \tilde{M}\left(\frac{1}{i}\frac{\partial}{\partial t}\right)\right]|M; t\rangle = \left[\frac{\tilde{\Gamma}\left(\frac{1}{i}\frac{\partial}{\partial t}\right)}{\pi}\right]^{1/2} |S; t\rangle. \tag{2.15}$$

The choice of our unstable particle state corresponds to the replacement of the mass-operator $\tilde{M}$ by a complex constant, describing positions of poles of the integrand in (2.14). The replacement of $\Psi(t)$ by $\Psi_M(t)$ is therefore a simplification of the formula (2.14) satisfying the following conditions:

a) The residual contributions from the points $E = M$, $E = M^*$ are the same.

b) The threshold ($E \approx 0$) and asymptotic ($E \to \infty$) behaviours of spectral function are preserved.

It should be mentioned that the wave function $\Psi_M(t)$ is not equal to the contribution, obtained by deforming of the integration contour in (2.14) through the pole at $E = M$. It is easy to see that such a contribution describes purely exponential nonunitary wave function with renormalized value of complex mass parameter.

b) Model

Let us assume that

$$|S_{\nu;\beta}(E)|^2 = \Theta(E) E^\nu e^{-\beta E} \frac{1}{N^{1/2}}. \tag{2.16}$$

It should be mentioned that the power ν determines the threshold behaviour, and β leads to the exponential damping at infinite energies.

The wave function of our unstable object is given by the formula

$$\Psi_M^{(\nu,\beta)}(t) = \frac{1}{N}\int_0^\infty \frac{E^\nu e^{-iEt-\beta E}}{(E-m_0)^2+\gamma^2}\,dE, \tag{2.17}$$

where N is chosen in order to satisfy the condition (2.12).

b1) General case ($\nu > -1, \quad \beta > 0$)

The integral (2.17) (and value of N) can be calculated if we use the following formula (see for instance [21])

$$\int_0^\infty \frac{E^\nu e^{-pE}}{E+p}\,dE = \Gamma(\nu)p^{-\nu}\Phi(1, 1-\nu; bp)+\Gamma(\nu+1)\Gamma(-\nu)b^\nu e^{bp},$$

where $\Phi(\alpha, \beta; z)$ denotes the hypergeometric series [22]. We obtain finally

$$\Psi_M^{(\nu,\beta)}(t) = \Psi_M^{(\nu,0)}(t-i\beta) = \frac{1}{Ni\gamma}\left\{\frac{\Gamma(\nu)}{(\beta+it)^\nu}\left[\Phi(1, 1-\nu; -M^*(\beta+it))\right.\right.$$

$$\left.\left. -\Phi(1, 1-\nu; -M(\beta+it))\right]+(-1)^\nu \frac{\pi}{\sin\pi\nu}\left[M^\nu e^{-M(\beta+it)}-M^{*\nu}e^{-M^*(\beta+it)}\right]\right\}, \tag{2.18}$$

and the normalization factor $N = \psi_M^{(\nu,\beta)}(0)$. The asymptotic behaviour at $t \approx 0$ is as follows

$$\Psi_M^\nu(t-i\beta) = C_\nu^1(\beta+it)^{-\nu}\sum_{k=1}^\infty \alpha_M^{(\nu,k)}(\beta^k+itk\beta^{k-1})+C_\nu^2(1-it)+O(t^2), \tag{2.19}$$

where

$$C_\nu^1 = \frac{\Gamma(\nu)\Gamma(1-\nu)}{Ni\gamma},$$

$$C_\nu^2 = (-1)^\nu \frac{\pi}{\sin\pi\nu}(M^\nu e^{-M\beta}-M^{*\nu}e^{-M^*\beta}),$$

$$\alpha_M^{(\nu,k)} = (-1)^k\frac{M^{*k}-M^k}{\Gamma(1-\nu tk)}.$$

The asymptotic behaviour for $t \to \infty$ is the following

$$\Psi_M^\nu(t-i\beta) \sim \frac{1}{N|M|^2}\frac{\Gamma(1+\nu)}{(\beta+it)^{\nu+1}}. \tag{2.20}$$

We are able to calculate the limit $\nu \to n$, $n = 0, 1, 2, \ldots$ in formula (2.18), and then we obtain the following form of the wave function

$$\Psi_M^n(t-i\beta) = \lim_{\nu\to n}\Psi_M^\nu(t-i\beta) = \frac{1}{i\gamma}\{F_{M^*}^n(t-i\beta)-F_M^n(t-i\beta)\}, \tag{2.21}$$

where

$$F^n_M(t-i\beta) = (-1)^{n-1}(-M)^n \exp[-iM(t-i\beta)]$$

$$\times Ei[iM(t-i\beta)] + \sum_{k=1}^{n} (k-1)!M^{n-k}(\beta+it)^k. \tag{2.21a}$$

b2) The limit $\beta \to 0_+$

Let us assume that there is no damping in our formulas (i.e. $\beta = 0$), The wave function of an unstable object, in this case, has the form

$$\Psi^\nu_M(t-i0_+) = \frac{1}{Ni\gamma}\left\{\frac{\Gamma(\nu)}{(it)^\nu}[\Phi(1, 1-\nu; -iM^*t)\right.$$

$$\left.-\Phi(1, 1-\nu; -iMt)] + (-1)^\nu \frac{\pi}{\sin \pi\nu}[M^\nu e^{-iMt} - M^{*\nu} e^{-iM_* t}]\right\}, \tag{2.22}$$

and

$$iN\gamma = (-1)^\nu \frac{\pi}{\sin \pi\nu}[M^\nu - M^{*\nu}], \tag{2.23}$$

where $\nu \neq 1, 2, \ldots$

The asymptotic behaviours are correspondingly:

1) Small times ($t \approx 0$)

$$\Psi^\nu_M(t \approx 0) \sim \frac{1}{Ni\gamma}\left\{\frac{\gamma}{i}\,\frac{\Gamma(\nu)\Gamma(1-\nu)}{\Gamma(2-\nu)}(it)^{1-\nu} + (-1)^{\nu+1}(1-it)[M^{*\nu} - M^\nu]\right. . \tag{2.24}$$

2) The wave function for $t \to \infty$ is given by formula (2.20).

b3) Special cases

1. ($\beta = 0$, $\nu = 0$)

The normalization factor has the following form in this case

$$N = -\frac{i}{\gamma}\log\frac{M}{M^*}, \tag{2.25}$$

and the wave function is as follows

$$\Psi^0_M(t) = \frac{1}{Ni\gamma}[e^{-iMt}Ei(iMt) - e^{-iM^*t}Ei(iM^*t)], \tag{2.26}$$

where the function Ei is defined as follows

$$Ei(in) = \log n + C + \sum_{n=1}^{\infty}\frac{(-1)n^{2n}}{2n(2n)!} + \sum_{n=0}^{\infty}\frac{(-1)^n n^{2n+1}}{(2n+1)(2n+1)!}.$$

The asymptotic behaviour for $t \sim 0$ is given by

$$\Psi_M^0(t \sim 0) \sim 1 - im_0 t - \frac{i}{N} t[\log|t| - i\pi + C - 1], \tag{2.27}$$

and for a large t

$$\Psi_M^0(t) \xrightarrow[t\to\infty]{} \frac{1}{N\gamma}\left\{\frac{1}{Mt} - \frac{1}{M^* t}\right\}. \tag{2.28}$$

2) ($\beta = 0$, $\nu = 1/2$)

The wave function of the unstable objects has the following asymptotic behaviours: ($i\gamma N = \pi i(\sqrt{M} - \sqrt{M^*})$)

a) Small time

$$\Psi_M^{1/2}(t) \sim \frac{1}{\pi} - \frac{2\sqrt{\pi i}}{N} t^{1/2} - \frac{i}{\sqrt{\pi}} t. \tag{2.29}$$

b) Large time

$$\Psi_M^{1/2}(t) \xrightarrow[t\to\infty]{} -\frac{\sqrt{\pi i}}{2N|M|^2} t^{-3/2}. \tag{2.30}$$

It should be mentioned that the value $\nu = 1/2$ corresponds to the physical S-wave threshold behaviour (see also Sect. 3).

3. *Nonrelativistic QM (three space dimensions)*

a) General considerations

In ordinary QM, Eq. (1.6) is as follows

$$\left(i\frac{\partial}{\partial t} - \frac{\hat{p}^2}{2m}\right)|m; t\rangle = 0. \tag{3.1}$$

Expanding the solution of 3.1 in the complete set of three-momentum eigenvectors

$$|m\rangle = \int d^3p\,\xi(\bar{p})|\bar{p}; m\rangle, \tag{3.2}$$

we get the following time development

$$|m, t\rangle = \int d\bar{p}\,\xi(\bar{p})e^{-i\frac{\hat{p}^2}{2m}t}|\bar{p}; m\rangle. \tag{3.3}$$

The usual formula for the wave function is as follows:

$$\Psi_m(\bar{x}, t) = \langle m|\exp[i\bar{p}\bar{x} - i\hat{H}_0^m t]|m\rangle$$

$$= \int d\bar{p}|\xi(\bar{p})|^2 e^{i\bar{p}\bar{x} - i\frac{\bar{p}^2}{2m}t}. \tag{3.4}$$

For an unstable state $|M; t\rangle$ we introduce the following equation

$$\left(i\frac{\partial}{\partial t}-\frac{\hat{p}^2}{2M}\right)|M; t\rangle = \left(\frac{\gamma}{\pi}\right)^{1/2}\frac{|\hat{p}|}{M}|S; t\rangle, \tag{3.5}$$

where

$$|S\rangle = \int_0^\infty dE \int d\bar{p} S(E, \bar{p})\, |E, \bar{p}\rangle, \tag{3.6}$$

and $|E, \bar{p}\rangle$ describes scattering states with a continuous energy spectrum,

$$\hat{H}|E, \bar{p}\rangle = E|E, \bar{p}\rangle,\ E \geqslant 0 \tag{3.7}$$

is normalized as follows:

$$\langle E, \bar{p}|E', \bar{p}'\rangle = \delta(E-E')\delta(\bar{p}-\bar{p}'). \tag{3.7a}$$

Thus we have

$$|S; t\rangle = e^{-i\hat{H}t}|S\rangle = \int_0^\infty dE \int d\bar{p} e^{-iEt} S(E, \bar{p})\, |E, \bar{p}\rangle. \tag{3.8}$$

Introducing the spectral representation

$$|M, t\rangle = \int_0^\infty dE \int d\bar{p} e^{-iEt} M(E, \bar{p})\, |E, \bar{p}\rangle \tag{3.9}$$

one obtains (we recall $M = m_0 - i\gamma$)

$$M(E, \bar{p}) = \left(\frac{\gamma}{2\pi}\right)^{1/2} \frac{S(E, \bar{p})}{E-\dfrac{\bar{p}^2}{2M}}\frac{|\bar{p}|}{M}. \tag{3.10}$$

The wave function for our unstable object is given by the formula:

$$\Psi_M(\bar{x}, t) = \langle M|e^{i\bar{p}\bar{x}}e^{-i\hat{H}t}|M\rangle$$

$$= \frac{\gamma}{2\pi|M|^2}\int_0^\infty dE \int d\bar{p} e^{-iEt} e^{i\bar{p}\bar{x}} \frac{|\bar{p}|S(E, \bar{p})}{\left(E-\dfrac{\bar{p}^2}{2M}\right)\left(E-\dfrac{\bar{p}^2}{2M^*}\right)}. \tag{3.11}$$

Function (3.11) satisfies the following equation:

$$\left(i\frac{\partial}{\partial t}-\frac{\Delta}{2M}\right)\left(i\frac{\partial}{\partial t}-\frac{\Delta}{2M^*}\right)\Psi_M(\bar{x}, t) = \frac{\gamma}{2\pi}\frac{\Delta}{|M|^2}\tilde{S}(\bar{x}, t), \tag{3.12}$$

where

$$\tilde{S}(\bar{x}, t) = \int_0^\infty dE e^{-iEt} \int d\bar{p} e^{i\bar{p}\bar{x}} |S(E, \bar{p})|^2 \tag{3.15}$$

describes the four-dimensional source function.

Formulae (3.11) and (3.12) describe the general wave function of unstable particle in QM. The factor, occuring in the front of the source function, is chosen in such a way because

$$\lim_{\gamma\to 0}\frac{1}{2\pi}\,\frac{\gamma\bar{p}^2/|M|^2}{\left(E-\dfrac{\bar{p}^2}{2M}\right)\left(E-\dfrac{\bar{p}^2}{2M^*}\right)}=\delta\left(E-\frac{\bar{p}^2}{2m_0}\right), \tag{3.14}$$

i.e. we obtain

$$\lim_{\gamma\to 0}\Psi_M(\bar{x},t)=\Psi_{m_0}(\bar{x},t), \tag{3.15}$$

where Ψ_{m_0} is defined by (3.4) and

$$\xi(\bar{p})=S\left(\frac{\bar{p}^2}{2m_0},\,\bar{p}\right). \tag{3.15a}$$

In nonrelativistic QM, one can localize the source function in space, i.e. one can assume that

$$\tilde{S}(\bar{x},t)=\tilde{S}(t)\delta^3(\bar{x}) \tag{3.16}$$

which corresponds to the assumption in (3.13) that

$$|S(E,\bar{p})|^2=|S(E)|^2.$$

Remark 1. The integration in (3.11) over dE can be replaced by the integration over dm in accordance with the formulae:

$$E=\frac{p^2}{2m},\qquad dE=-\frac{p^2}{2m^2}\,dm. \tag{3.17}$$

One obtains (compare with (3.11))

$$\Psi(\bar{x},t)=\frac{\gamma}{\pi}\int_0^\infty\frac{dm}{(m-M)(m-M^*)}\int d\bar{p}\left|S\left(\frac{\bar{p}^2}{2m},\,\bar{p}\right)\right|^2\exp\left[i\overline{px}-i\frac{\bar{p}^2}{2m}\,t\right]. \tag{3.18}$$

We see, therefore, that the normalization of the source function becomes the same as in one dimensional model. The "mass representation" (3.19) leads in a transparent way to the result of (3.15).

Remark 2. If we assume (3.16), the wave function is rotationally invariant.

b) Model

In this model we are using the same form of damping as at Sect. 2. It is given by the factor $\exp(-\beta E)$, $\beta > 0$. We shall discuss the space-time behaviour of the following function

$$\Psi_M^\nu(\bar{x}, t+i\beta) = \frac{2|M|^2}{N} \int_0^\infty dE E^\nu e^{-\alpha E} \int d\bar{p} \frac{e^{-i\bar{p}\bar{x}}}{(\bar{p}^2 - a)(\bar{p}^2 - a^*)}, \tag{3.19}$$

where

$$\alpha = \beta + it, \qquad a = 2ME.$$

b1) General case ($\beta \neq 0$, $\nu > 0$)

We can easily calculate the value of the integrals

$$\int d\bar{p} \frac{\exp(-i\bar{p}\bar{x})}{p^2 - a} = \frac{2\pi^2}{r^2} \exp(-i\sqrt{2ME}\, r),$$

$$\int d\bar{p} \frac{\exp(-i\bar{p}\bar{x})}{\bar{p}^2 - a^*} = \frac{2\pi^2}{r^2} \exp(i\sqrt{2M^*E}\, r). \tag{3.19a}$$

Putting $k^2 = E$ we obtain from (3.19) and (3.19a) the following formula [21] ($r \equiv |\bar{x}|$)

$$\Psi_M^\nu(r, t+i\beta) = \frac{8\pi^2|M|^2}{Ni\gamma r} \Gamma(\nu) 2^{-\nu/2} (\beta+it)^{-\nu/2}$$

$$\times \left\{ \exp\left(-\frac{M^* r^2}{4(\beta+it)}\right) D_{-\nu}\left(\frac{-ir\sqrt{M^*}}{\sqrt{\beta+it}}\right) - \exp\left[-\frac{Mr^2}{4(\beta+it)}\right] D_{-\nu}\left(\frac{ir\sqrt{M}}{\sqrt{\beta+it}}\right)\right\}. \tag{3.20}$$

Let us write (3.20) for $t = 0$

$$\Psi_M^\nu(r) = \frac{8\pi^2|M|^2}{i\gamma r} \Gamma(\nu) (2\beta)^{-\nu/2} \left\{ \exp\left[-\frac{M^* r^2}{4\beta}\right] D_{-\nu}\left(-\frac{ir\sqrt{M^*}}{\sqrt{\beta}}\right) \right.$$

$$\left. - \exp\left[-\frac{Mr^2}{4\beta}\right] D_{-\nu}\left(\frac{ir\sqrt{M}}{\sqrt{\beta}}\right)\right\}. \tag{3.21}$$

If $\beta > 0$ we can choose the constant N in the following way

$$\int d^3r |\Psi_M^\nu(r)|^2 = N^2, \tag{3.21a}$$

which is consistent with the probability interpretation of function (3.19). The asymptotic behaviour at $r^2/\beta+it| \sim 0$ is as follows:

$$\Psi_M^\nu(r, t+i\beta) \sim C_\nu^1 (\beta+it)^{-(\nu+1)/2} + C_\nu^2 \frac{r}{(\beta+it)^{\frac{\nu+2}{2}}} + C_\nu^3 \frac{r^2}{(\beta+it)^{\frac{\nu+3}{2}}},$$

$$C_\nu^1 = \frac{8\pi^2|M|^2\Gamma(\nu)2^{-\nu}}{N\gamma} \frac{\sqrt{2\pi}}{\Gamma\left(\frac{\nu}{2}\right)} (\sqrt{M} + \sqrt{M^*}),$$

$$C_\nu^2 = -\frac{8\pi^2|M|^2\Gamma(\nu)2^{-\nu}}{N}\frac{\sqrt{\pi}}{\Gamma\left(\frac{\nu+1}{2}\right)}\frac{\nu}{2},$$

$$C_\nu^3 = \frac{8\pi^2|M|^2\Gamma(\nu)2^{-\nu}}{N\gamma}\frac{(1-\nu)\sqrt{2\pi}}{6\cdot\Gamma\left(\frac{\nu}{2}\right)}(M\sqrt{M}+M^*\sqrt{M^*}). \tag{3.22}$$

The asymptotic behaviour for $\frac{r^2}{|\beta+it|}\to\infty$:

$$\Psi_M^\nu(r, t+i\beta) \sim C_\nu r^{-\nu-1},$$

$$C_\nu = \frac{8\pi^2|M|^2}{Ni\gamma}\Gamma(\nu)2^{-\nu/2}\left[\left(\frac{i}{\sqrt{M^*}}\right)^\nu-\left(\frac{1}{i\sqrt{M}}\right)^\nu\right]. \tag{3.23}$$

b2) $\beta\to 0_+$

It is not difficult to get a respective formulae for the wave function and its asymptotic behaviour from formulae (3.21–3.23). The limit $\beta\to 0_+$ implies, however, difficulties with the normalization constant. If one assumes $\beta = t = 0$ and calculates the wave function at $t = 0$, one obtains

$$\Psi^\nu(r) = \Lambda\frac{\Gamma(\nu)}{r^{\nu+1}}, \tag{3.24a}$$

where

$$\Lambda = \frac{8\pi^2|M|^2}{\gamma}\left\{\frac{1}{\sqrt{2M}}-\frac{1}{\sqrt{2M^*}}\right\}.$$

It is easy to see that N is divergent. It appears that only if $\nu = \frac{1}{2}$ one can regain the physical normalization of the wave function by the following procedure:
Let us introduce the modified function $\tilde{\Psi}_\nu(r)$ as follows

$$\tilde{\Psi}_{1/2-\xi}(r) = \sqrt{\frac{1}{\Gamma(\xi)}}\Psi_{1/2-\xi}(r) \tag{3.24b}$$

and let us introduce the modified norm

$$\tilde{N}(\xi) = \frac{1}{\Lambda^2\Gamma^2(\nu)}\int|\tilde{\Psi}_{1/2-\xi}(r)|^2 d\bar{r} = \frac{\Gamma^2(\nu-\xi)}{\Gamma^2(\nu)}\int\frac{r^{2\xi-3}}{\Gamma(\xi)}d\bar{r}. \tag{3.25}$$

It is easy to see, using the formula [23]

$$\lim_{\lambda\to -n}\frac{r^\lambda}{\Gamma\left(\frac{\lambda+n}{2}\right)} = \delta^n(r)$$

and putting in (3.25)

$$\xi = \frac{\lambda + n}{2}, \qquad n = 3,$$

that the integral exists if $\xi \to 0$, and

$$\lim_{\xi \to 0} \tilde{N}(\xi) = 1.$$

4. *Four-dimensional relativistic QM*

a) General considerations

In relativistic theory the wave packet of stable particles is described by the equation

$$\left(i \frac{\partial}{\partial t} - \sqrt{\hat{p}^2 - M_0^2} \right) |M_0; t\rangle = 0, \tag{4.1}$$

or by the positive energy solutions of the equation

$$\left[\frac{\partial^2}{\partial t^2} - (\hat{p}^2 + M_0^2) \right] |M_0; t\rangle = 0. \tag{4.1a}$$

Assuming that

$$M_0\rangle = \int d\bar{p} \, \frac{f(\bar{p})}{(2\omega)^{1/2}} \, |\bar{p}; M_0\rangle \tag{4.2}$$

we obtain the following formula for the wave function

$$\Psi_{M_0}(x) = \langle M_0 | e^{iP_\mu x^\mu} | M_0 \rangle = \int \frac{d\bar{p}}{2\omega_{po}} \, |f(\bar{p})| \, e^{i(\bar{p}\bar{x} - \omega_{po} x_0)}$$
$$= \int d\bar{x}' d\bar{x}'' \tilde{f}^*(\bar{x} - \bar{x}') \Delta^{(+)}(\bar{x}' - \bar{x}''; x_0; M_0^2) \tilde{f}(\bar{x}''), \tag{4.3}$$

where the two-point Wightman function $\Delta^{(+)}(x, M_0^2)$ describes the space-time propagation of the excitations obtained by putting $\tilde{f}(\bar{x}) = \delta^3(\bar{x})$. The unstable relativistic particle is obtained by means of the formula:

$$\left[\frac{\partial^2}{\partial t^2} - (\hat{p}^2 + M^2) \right] |M; t\rangle = \left(\frac{\gamma}{\pi} \right)^{1/2} |S; t\rangle. \tag{4.4}$$

We assume that

$$|S; t\rangle = \int_0^\infty d\kappa^2 \int d\bar{p}/(2\omega)^{1/2} S(\bar{p}; \kappa^2) \, |\kappa^2, \bar{p}\rangle, \tag{4.5}$$

where the scattering states $|p^2, \bar{p}\rangle$ are the eigenfunctions of the total four-momentum operator $\hat{P}_\mu$

$$\hat{\bar{P}}|p^2, \bar{p}\rangle = \bar{p}|p^2, \bar{p}\rangle,$$

$$\hat{P}_0|p^2, \bar{p}\rangle = (\bar{p}^2+p^2)^{1/2}|p^2, \bar{p}\rangle \tag{4.6a}$$

and are normalized as follows

$$\langle p^2, \bar{p}|p'^2, \bar{p}'\rangle = \delta(p^2-p'^2)\delta^3(\bar{p}-\bar{p}'). \tag{4.6b}$$

Using for $|M, t\rangle$ analogous formula to (4.5) one obtains

$$M(\bar{p}, \kappa^2) = \left(\frac{\gamma}{\pi}\right)^{1/2} \frac{S(\bar{p}; \kappa^2)}{\kappa^2 - M^2}. \tag{4.7}$$

Introducing the wave function for an unstable object (4.4), one obtains

$$\Psi_M(x) = \langle M|e^{iP_\mu x^\mu}|M\rangle = \frac{\gamma}{\pi}\int_0^\infty d\kappa^2 \int \frac{dp}{2\omega} \frac{|S(\bar{p}, \kappa^2)|^2 e^{ipx}}{(\kappa^2-M^2)(\kappa^2-M^{*2})} \tag{4.8}$$

which satisfies the field equation

$$(\Box - M^2)(\Box - M^*)\Psi_M(x) = [(\Box - m_0^2)^2+\gamma^2]\Psi_M(x) = \frac{\gamma}{\pi} S(x), \tag{4.9}$$

where $M^2 = m_0^2 - i\gamma$. Assuming that $M = M_0 - i\Gamma$, we have the following relation

$$m_0{}^2 = M_0{}^2 - \Gamma^2, \ \gamma = 2M_0\Gamma.$$

If one writes

$$|S(\bar{p}; \kappa^2)|^2 = |f(\bar{p})|^2 |S(\kappa^2)|^2, \tag{4.10}$$

the wave function (4.8) is determined by formula (4.3) with $\Delta^{(+)}(x; M_0^2)$ replaced by

$$\Delta_M^{(+)}(x) = \frac{\gamma}{\pi}\int_0^\infty dx^2 \frac{|S(\kappa^2)|^2 \Delta^+(x; M_0^2)}{(\kappa^2-M^2)(\kappa^2-M^{*2})}. \tag{4.11}$$

In particular we see that

$$\lim_{\gamma\to 0} \Delta_M^{(+)}(x) = \Delta^{(+)}(x; M_0^2), \tag{4.12}$$

provided that $|S(M_0^2)|^2 = 1$.

The space-time propagation of relativistic unstable objects is therefore determined by the class of the Wightman functions (4.11).

b) Model

We shall calculate the wave function (4.8) with $|S(\bar{p}, \kappa^2)| = 1$ which satisfies the following equation

$$(\Box - M^2)(\Box - M^{*2})\Psi_M(x) = \int_{V_+} d^4p e^{ipx}. \tag{4.13}$$

At first we shall solve the following equation

$$(\Box - M^2) f_M(x) = \int_{V_+} d^4 p e^{ipx}. \tag{4.14}$$

One can continue analytically this equation in a complex positive plane

$$x_\mu \to z_\mu = x_\mu + i y_\mu, \quad x^2 \to z^2 = z_\mu z^\mu.$$

We shall assume this solution in a covariant form $f_M(z^2)$. Then, the solution of equation (4.14) is a boundary value of $f_M(z^2)$ when $y_\mu \to 0_+$. It is easy to check this in the course of analytic continuation.

$$\Box_x f(x^2) \to \Box_z f(z^2) = -4z^2 \left(\frac{d}{dz^2}\right)^2 f(z^2) - 8 \frac{d}{dz^2} f(z^2)$$

and Eq. (4.14) takes a simple form

$$u^2 \frac{d^2}{du^2} Z(u) + u \frac{d}{du} Z(u) + (u^2 - 1) Z(u) = u^{-1}, \tag{4.15}$$

where

$$u = Mz, \quad f_M(z^2) = \frac{M}{\pi^2} z^{-1} Z(Mz), \quad z = (z^2)^{1/2}.$$

The solution of Inhomogeneous Bessel Equation (4.15) are Lommel's functions [22] $(S_{-2,1}(u))$:

$$Z(u) = S_{-2,-1}(u) = \lim_{\nu \to -1} S_{\nu-1,\nu}(u) = \lim_{\nu \to -1} \Bigg\{ 2^{\nu-2} \Gamma(\nu) \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2} u)^{2m+\nu}}{m! \Gamma(\nu+m+1)}$$
$$\times [2 \log(\tfrac{1}{2} u) - \Psi(\nu+m+1) - \Psi(m+1)] - \pi 2^{\nu-2} \Gamma(\nu) Y_\nu(u) \Bigg\},$$

where $\Gamma(z)$ is Euler's function, $\Psi(z)$ is the logarithmic derivative of the function $\Gamma(z)$, i.e.

$$\Psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)};$$

and $J_\nu(z), Y_\nu(z)$ are Bessel's functions.

We obtain finally

$$S_{-2,-1}(u) = 2^{-3} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{u}{2}\right)^{2m+1}}{m! \Gamma(m+2)} \Bigg[2 \log^2 \left(\frac{u}{2}\right)$$
$$-3 \log\left(\frac{u}{2}\right) \Psi(m+2) - \pi^2 + \Psi^2(m+2) + \Psi(m+2)\Psi(m+1) - \Psi'(m+2) \Bigg]$$
$$+ \log\left(\frac{u}{2}\right) \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{u}{2}\right)^{2m-1}}{m! \Gamma(m)} \Psi(m) + \Psi(1) \left(\frac{u}{2}\right)^{-1}. \tag{4.16}$$

The asymptotic behaviour of the wave function $f_M(z^2)$ for large $|z|(\arg z| < \pi)$ is given by the formula

$$f_M(z^2) \sim \frac{1}{M^2\pi^2 z^4} \tag{4.17}$$

and for small z ($|z| \sim 0$),

$$f_M(z^2) \cong \frac{1}{8\pi^2}\left\{-\frac{2\log\left(\frac{Mz}{2}\right)}{z^2} + 2\frac{\Psi(1)}{z^2} + \tfrac{3}{2}M^2\log\left(\frac{Mz}{2}\right)\right.$$

$$\left.+[\Psi^2(m+2)+\Psi(m+2)\Psi(m+1)-\Psi'(m+2)-\pi]\frac{M^2}{2} + M^2\log^2\left(\frac{Mz}{2}\right)\right\}. \tag{4.17a}$$

The solution of (4.13) is given by the following formula

$$\Psi_M(z^2) = \frac{1}{i\gamma}\{f_{M^*}(z^2) - f_M(z^2)\}. \tag{4.18}$$

The asymptotic behaviours for total wave function are given correspondingly:
1. Large z ($|z| \to \infty$)

$$\Psi_M(z^2) \sim -\frac{1}{\pi^2|M|^4 z^4}. \tag{4.18a}$$

2. Small z ($|z| \sim 0$)

$$\Psi_M(z^2) \sim a_1\frac{1}{z^2} + a_2\log z + a_3\log^2 z + a_4, \tag{4.18b}$$

where

$$a_1 = \frac{1}{8\pi^2}\frac{2}{i\gamma}\log\frac{M}{M^*}, \qquad a_3 = \frac{1}{8\pi^2},$$

$$a_2 = \frac{1}{4\pi^2 i\gamma}\left\{\tfrac{3}{2}i\gamma + M^{*2}\log(M^*/2) - M^2\log\left(\frac{M}{2}\right)\right\},$$

$$a_4 = \frac{1}{8\pi^2 i\gamma}\left\{\tfrac{3}{2}\left(M^{*2}\log\left(\frac{M^*}{2}\right) - M^2\log\left(\frac{M}{z}\right)\right) + M^{*2}\log^2\left(\frac{M^*}{2}\right)\right.$$

$$\left.-M^2\log^2\left(\frac{M}{2}\right) + [\Psi^2(2)+\Psi(2)\Psi(1)-\Psi'(2)-\pi^2]\tfrac{1}{2}c\gamma\right\}.$$

The integral on the right hand side of (4.13) can be also written in the form the Kallen--Lehman representation

$$S^{(+)}(z) = \int_{V_+} d^4p e^{ipz} = \int_0^\infty d\kappa^2 \Delta_+(z;\kappa^2), \tag{4.19}$$

where $(H_1^{(1)}(u)$ is the first Hankel Function)

$$\Delta_+(z); \kappa^2) = \frac{\kappa^2}{8\pi i} \frac{H_1^{(1)}(\kappa z)}{\kappa z}.$$

One can generalize (4.19) by assuming that

$$S_\nu^{(+)}(z) = \int_0^\infty d\kappa^2 (\kappa^2)^{\nu/2} \Delta_+(z; \kappa^2),$$

which corresponds to the power-like threshold behaviour $\varrho(E) \sim E^\nu$ in Sect. 2 and Sect. 3.

If one takes $\nu = 0$ one obtains equation (4. 13). In generalized case (see [23] for instance)

$$S_\nu^{(+)}(z) = \frac{-i}{\pi^2} 2^\nu \Gamma\left(1+\frac{\nu}{2}\right) \Gamma\left(2+\frac{\nu}{2}\right) \frac{1}{(z^2)^{2+\nu/2}}. \tag{4.19a}$$

Now (4.15) turns into the following

$$u^2 \frac{d^2 Z'(u)}{du^2} + u \frac{dZ'(u)}{du} + (u^2-1)Z'(u) = u^{-\nu-1}. \tag{4.20}$$

The solution of (4.20) can be expressed by means of the function $S_{-2,\,-1}(z)$. The function $f_M^\nu(z^2)$ analogous to the $f_M(z^2)$ has the form

$$f_M^\nu(z^2) = -\frac{i}{\pi^2} 2^\nu \Gamma\left(1+\frac{\nu}{2}\right) \Gamma\left(2+\frac{\nu}{2}\right) M^{\nu+1} z^{-1} Z'(Mz). \tag{4.21}$$

In this case the total wave function, which is the solution of Eq. (4.13) with the right-hand side replaced by (4.19) (or (4.19a)) is given by means of the formula analogous to (4.18), i.e.

$$\Psi_M^\nu(z^2) = \frac{1}{i\gamma} \{f_{M^*}^\nu(z^2) - f_M^\nu(z^2)\}. \tag{4.22}$$

Remark. It should be mentioned that the solutions of equation (4.13) are given by the covariant formula (4.21) and (4.22) and describe *improper wave functions* of the unstable system, obtained by the assumption that $|f(\bar{p})|^2 = \text{const.}$, in (4.10). In order to obtain *normalizable wave functions* one has to break the relativistic covariance by assuming that in our model

$$S(\bar{p}; \kappa^2) = 1 \rightarrow S(\bar{p}; \kappa^2) = f(\bar{p}),$$

where $f(\bar{p})$ is chosen in such a way that the wave function (4.8) is normalizable at $t = 0$. Such physical wave functions can be obtained only if we assume some noncovariant source term in equation (4.9).

5. *Dynamic example: NΘ-resonant scattering in the Lee model*

In order to calculate an unstable V-particle state *one should express the eigenstate of the free Hamiltonian in terms of the states NΘ E,* which are the S-wave physical $N\Theta$ scattering states. Using the result of [20] one can write

$$|V\rangle_0 = \frac{1}{(2\pi)^{3/2}} \int d^3k\beta(k)|N\Theta;\bar{k}\rangle, \tag{5.1}$$

where ($\omega = (k^2+\mu^2)^{1/2}$),

$$\beta(k) = \frac{g_0 f(\omega)}{\sqrt{2\omega}} \cdot \frac{1}{(m_V - m_N - \omega - \Phi(\omega + i0))}, \tag{5.1a}$$

and ($\mu \equiv m_\theta$)

$$\Phi(z) = \frac{g_0^2}{4\pi^2} \int_\mu^\infty \frac{f^2(\omega')kd\omega'}{\omega' - z}. \tag{5.1b}$$

The scattering states $|N\Theta;\bar{k}\rangle$ are normalized as follows

$$\langle N\Theta;\bar{k}|N\Theta;\bar{k}'\rangle = \delta^3(\bar{k}-\bar{k}'). \tag{5.1c}$$

Introducing the total energy variable $E = m_N + (k^2+\mu^2)^{1/2}$ and using $\omega dE = kdk$, $d^3k = k^2 d\Omega$, one obtains the S-wave state

$$|N\Theta;E\rangle = \frac{k^{1/2}(E)\omega^{1/2}(E)}{(2\pi)^{1/2}} \int d\Omega|N\Theta;\bar{k}\rangle \tag{5.2}$$

normalized to $\delta(E-E')$. One can write (5.1) as follows

$$|V\rangle_0 = \frac{1}{2\pi} \int dE\tilde{\beta}(E)|N\Theta;E\rangle, \tag{5.3}$$

where

$$\tilde{\beta}(E) = g_0 \frac{f(E-m_V)}{[2\omega(E)]^{1/2}} \cdot \frac{k^2(E)}{m_V - E - \Phi(E - m_V + iO)}. \tag{5.3a}$$

The mass operator is given by the formula

$$\tilde{M}(E) = m_V - \Phi(E - m_V + iO). \tag{5.4}$$

We obtain the following equation for the complex mass renormalization term $\delta m_V = M_V - m_V$

$$\delta m_V = -\Phi^{\text{II}}(\delta m_V), \tag{5.5}$$

where the analytic function $\Phi^{\text{II}}(z)$ is defined for $\text{Re}\, z \geqslant \mu$ as follows

$$\Phi^{\text{II}}(z) = \Phi(z) - \frac{ig_0^2}{2\pi} k(z) f^2(z). \tag{5.6}$$

Formula (5.3) describes *exact* unstable V-particle state.

One can, however, introduce a model of a wave function, preserving the basic properties of (5.3), i.e. having the same position of complex pole, the same residuum, and an identical threshold ($E = m_V + \mu$) as well as asymptotic behaviours ($E \to \infty$). We modify (5.3) as follows

$$\tilde{\beta}_M(E) = g_0 \frac{S(E)}{E - M_V}, \tag{5.7}$$

where $S(E)$ satisfies the condition

$$S(M_V) = \frac{f(\delta m_V)}{[2\omega(M_V)]^{1/2}} \cdot \frac{k^2(M_V)}{1 + \left.\dfrac{d\Phi^{\text{II}}(z)}{dz}\right|_{z=\delta m_V}}, \tag{5.7a}$$

and it is chosen in such a way that

$$\frac{\tilde{\beta}_M(E)}{\tilde{\beta}(E)} \to 1 \quad \text{if} \quad \begin{matrix} E \to m_V + \mu \\ E \to \infty. \end{matrix} \tag{5.7b}$$

The main difference between (5.3) and (5.7) consists in the replacement of the mass operator (5.4) appropriately chosen constant renormalized complex mass.

We see that (5.7) describes an example of the wave function (2.5), where γ is proportional to g_0^2.

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