HIGH ENERGY BEHAVIOUR OF NON-PLANAR AND PLANAR DUAL MULTILOOP AMPLITUDES

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Multi-loop four-point amplitudes constructed from iteration of non-planar or planar orientable self-energy operators are studied in the asymptotic limit of large s. We find expected factorization properties to sum up the leading contributions of multi-loop graphs of arbitrary order. This leads to the definition of renormalized Pomeron and Regge trajectories.

1. Introduction

A large progress in the attempt of implanting unitarity perturbatively into the dual resonance model has been achieved by the construction of multiloop amplitudes in terms of the Abelian integrals on closed Riemann surfaces [1]. Another useful and equivalent representation can be obtained by iteration of primitive loop operators [2, 3]. Although explicitly known, the practical handling of multiloop amplitudes is a widely unsolved problem. This concerns, in particular, the high energy behaviour. Internal consistency requires, e. g. for the four-point N-loop function with a $(N+1)$-fold iterated Regge trajectory $\alpha(t)$ in the t-channel an asymptotic behaviour like $s^{\alpha(t)}(\ln s)^N (\Sigma(t))^N$, where $\Sigma(t)$ is some trajectory correction function. For $N = 1$ the asymptotic investigations had been carried out some time ago [4–5]. Using a domain variational technique Karpf and Liehl [6] proved recently that planar multiloop amplitudes have the necessary factorization properties to obtain the expected asymptotic behaviour. Then the sum of all the leading contributions defines a renormalized Regge trajectory.

The aim of the work presented here is mainly to study the related question of the asymptotic behaviour of non-planar orientable multiloop diagrams which can be con-

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sidered as iterations of the Pomeron-loop investigated in Ref. [5]. Some work in this direction has been already done by two of us [7] by computing the first correction to the asymptotic behaviour $\sim s^{\alpha_P(t)}$ of the bare Pomeron. There we used the formulation of the two-loop non-planar orientable amplitude in terms of the Abelian integrals. We obtained the asymptotic behaviour by expansion of the Jacobi-transformed functions near the parabolic point being connected with the vanishing of the handles of the associated Riemann surface. A generalization of this approach to arbitrary N loops seems to be rather involved as it is difficult to find the right parametrization of the Riemann surface suited for the expansion around the adequate parabolic point. Instead, we found that it is convenient to start from an expression of the multiloop amplitude generated iteratively by one-loop self-energy operators [2–3].

As our main result we have proved the necessary factorization properties of the non--planar N-loop amplitude, and obtained the renormalized Pomeron singularity $\alpha_P^r(t) = \alpha_P(t) + g^2\Pi(t, \alpha_P^r(t))$. Moreover, we have shown factorization with respect to the ordinary Regge trajectory so that the same class of non-planar diagrams leads to a renormalized trajectory $\alpha^r(t) = \alpha(t) + g^2\tilde{\Sigma}(t, \alpha^r(t))$ for a Reggeon coupling directly to the Pomeron. For completeness we have investigated closely parallel the corresponding iterations of the planar loop reproducing very quickly the results of Ref. [6]. All the calculations have been performed for simplicity in the conventional dual model, though we believe that the extension to more recent ghost-free dual models [8] should not be difficult.

This paper is organized as follows. In Chapter 2 we present the iterated N-loop non--planar and planar self-energy operators and the four-point functions with these chains of self-energy diagrams exchanged in the t-channel. Chapter 3 contains the derivation of the Pomeron dominated s-asymptotic behaviour. In Chapter 4 we give the results for the asymptotic behaviour due to Reggeon exchange calculated from both the non--planar and planar orientable diagrams.

2. The N-loop dual amplitudes

The four-point function with a chain of N non-planar orientable or planar self-energy loops exchanged in the t-channel (see Fig. 1a–c) is obtained easily if one constructs first the corresponding N-loop self-energy operators $\Sigma_N^T(a^\dagger, b; p)$ or $\Sigma_N(a^\dagger, b; p)$, respectively. (The symbols $a^\dagger$, b stand for two independent sets of oscillator operators, and we define $p = p_1 + p_2 = -(p_3 + p_4)$, $t = p^2$, $s = (p_2 + p_3)^2$. Conventions and notations are the same as in Ref. [2].) To do this we connect the well-known one-loop operators Σ_1^T [2] or Σ_1 [9] by projected twisted propagators D_T [2], treating exponentials of quadratic operators by complex Gaussian functional integration ([1, 3], compare Ref. [9]).

The N-loop operators represented by the diagrams shown in Fig. 2 have the following structure

$$\Sigma_N^{(T)}(a^\dagger, b; p) = \int d\mu_N \langle 0_b| \exp\{\langle p|F_N^R|b\rangle - \langle p|F_N^L|a^\dagger\rangle + \tfrac{1}{2}\langle a^\dagger|E_N^L|a^\dagger\rangle + \tfrac{1}{2}\langle b|E_N^R|b\rangle + \langle a^\dagger|X_N|b\rangle\}|0_a\rangle. \tag{2.1}$$

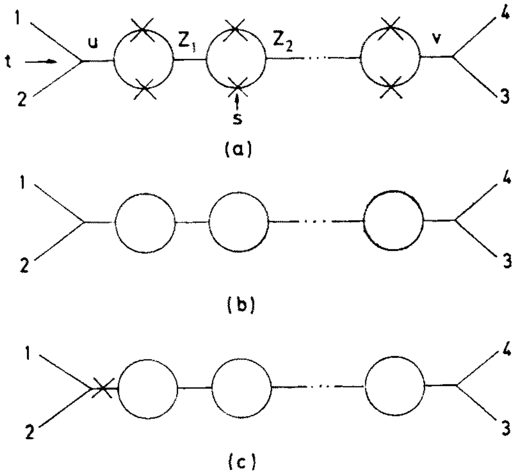

Fig. 1. Non-planar and planar multiloop four-point amplitudes considered in the text

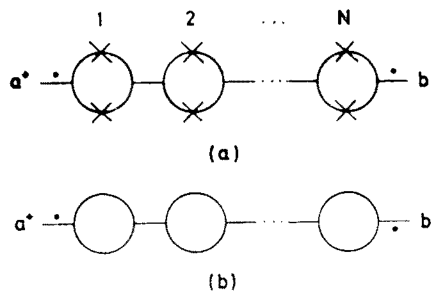

Fig. 2. Non-planar and planar N-loop self-energy operators

The matrices in the exponent are given by the following iteration scheme $\left(M_z \equiv M^t_-\left(\frac{z}{z-1}\right)M_-\right)$

$$E^{\mathrm{L}}_{N+1} = E^{\mathrm{L}}_N + X_N M_{z_N} E^{\mathrm{L}}_1 \Phi_N M_{z_N} X^t_N,$$

$$E^{\mathrm{R}}_{N+1} = E^{\mathrm{R}}_1 + X^t_1 M_{z_N} E^{\mathrm{R}}_N M_{z_N} \Phi^t_N X_1,$$

$$X_{N+1} = X_N M_{z_N} \Phi^t_N X_1,$$

$$F^{\mathrm{L}}_{N+1} = F^{\mathrm{L}}_N - (z_N) X_N + [F^{\mathrm{L}}_1 + \{(z_N) - (F^{\mathrm{R}}_N + (z_N) E^{\mathrm{R}}_N) M_{z_N}\} E^{\mathrm{L}}_1] \Phi_N M_{z_N} X_N,$$

$$F^{\mathrm{R}}_{N+1} = F^{\mathrm{R}}_1 - (z_N) X_1 + [F^{\mathrm{R}}_N + \{(z_N) - (F^{\mathrm{L}}_1 + (z_N) E^{\mathrm{L}}_1) M_{z_N}\} E^{\mathrm{R}}_N] M_{z_N} \Phi^t_N X_1, \tag{2.2}$$

$$\Phi_N = (1 - M_{z_N} E^{\mathrm{R}}_N M_{z_N} E^{\mathrm{L}}_1)^{-1}. \tag{2.3}$$

From the non-planar self-energy operator $\Sigma^T_1[2]$ we know the one-loop matrices.

$$E^{\mathrm{L}}_1 = E^{\mathrm{R}}_1 = 2(1-\omega)\,[E(\omega)]\,(1-\omega),$$

$$X_1 = (1-\omega)\,[E_T(x,\omega)]\,(1-\omega),$$

$$F^{\mathrm{L}}_1 = F^{\mathrm{R}}_1 = [F_T(x,\omega)]\,(1-\omega), \tag{2.4}$$

while in the corresponding planar case one has [9]

$$
\begin{aligned}
E_1^{\mathrm{L}} &= 2(1-x)\,[E(\omega)]\,(1-x),\\
E_1^{\mathrm{R}} &= 2\left(1-\frac{\omega}{x}\right)[E(\omega)]\left(1-\frac{\omega}{x}\right),\\
X_1 &= (1-x)\left[G\left(\frac{\omega}{x},\, x\right)\right]\left(1-\frac{\omega}{x}\right),\\
F_1^{\mathrm{L}} &= \left[F\left(x,\, \frac{\omega}{x}\right)\right](1-x),\\
F_1^{\mathrm{R}} &= \left[F\left(\frac{\omega}{x},\, x\right)\right]\left(1-\frac{\omega}{x}\right).
\end{aligned} \tag{2.5}
$$

For explicit representations of the matrices $E(\omega)$, $E_T(x, \omega)$, $F_T(x, \omega)$ and $F(x, y)$, $G(x, y)$ we refer the reader again to the works [2] and [9].

The integration measure is given by the iteration formula ($\alpha(t) = \alpha_0+\frac{1}{2}t$)

$$
\begin{aligned}
d\mu_{N+1} = {}& d\mu_N d\omega_{N+1} dx_{N+1} dz_N\; 4\pi^2 g^2 \frac{f^{-4}(\omega_{N+1})}{\ln^2 \omega_{N+1}}\, \omega_{N+1}^{-\alpha_0-1}\, \frac{1}{x_{N+1}}\\
&\times h(x_{N+1}, \omega_{N+1}; t) z_N^{-1-\alpha(t)}(1-z_N)^{2\alpha_0-1-\alpha(t)}[\det(\Phi_N^{-1})]^{-2}\\
&\times \exp\{t\langle 1|\, [\{F_N^{\mathrm{R}}+\tfrac{1}{2}(z_N)E_N^{\mathrm{R}}+F_1^{\mathrm{L}}+\tfrac{1}{2}(z_N)E_1^{\mathrm{L}}\}(z_N)+\tfrac{1}{2}\{F_N^{\mathrm{R}}+(z_N)E_N^{\mathrm{R}}\}M_{z_N}E_1^{\mathrm{L}}\Phi_N\\
&\times M_{z_N}\{(F_N^{\mathrm{R}})^t+E_N^{\mathrm{R}}(z_N)\}+\tfrac{1}{2}\{F_1^{\mathrm{L}}+(z_N)E_1^{\mathrm{L}}\}\Phi_N M_{z_N} E_N^{\mathrm{R}} M_{z_N}\{(F_1^{\mathrm{L}})^t+E_1^{\mathrm{L}}(z_N)\}\\
&-\{F_1^{\mathrm{L}}+(z_N)E_1^{\mathrm{L}}\}\Phi_N M_{z_N}\{(F_N^{\mathrm{R}})^t+E_N^{\mathrm{R}}(z_N)\}]\,|1\rangle\},
\end{aligned} \tag{2.6}
$$

where

$$
d\mu_1 = 4\pi^2 g^2 \frac{d\omega_1}{\ln^2 \omega_1} f^{-4}(\omega_1)\omega_1^{-\alpha_0-1} \frac{dx_1}{x_1}[h(x_1, \omega_1)]. \tag{2.7}
$$

The functions h in the measure are different for the non-planar and planar cases ([2, 9])

$$
\begin{aligned}
h^T(x, \omega; t) &= \left[\frac{\psi_T(x, \omega)}{1-\omega}\right]^t.\\
h(x, \omega; t) &= \left[\frac{\psi(x, \omega)^2}{(1-x)\left(1-\frac{\omega}{x}\right)}\right]^{\frac{t}{2}} \frac{1-\omega}{(1-x)\left(1-\frac{\omega}{x}\right)}.
\end{aligned} \tag{2.8}
$$

The functions ψ, ψ_T are the elliptic functions as defined in Ref. [2].

In Eqs (2.2), (2.3) and (2.6) all the one-loop matrices have to be taken with the arguments $x = x_{N+1}$ and $\omega = \omega_{N+1}$. As preliminary integration region we take $0 \leqslant \omega_i \leqslant 1$, $\omega_i \leqslant x_i \leqslant 1$ and $0 \leqslant z_j \leqslant 1$. Later on we shall see that there is a further limitation in the z-integrations for the non-planar case. Questions of necessary regularizations will be touched on later.

The four-point function can now easily be written down by attaching vertices and propagators to $\Sigma_N^{(T)}(a^\dagger, b; p)$, and by taking the vacuum expectation value corresponding to four external (scalar) ground state particles. We obtain

$$T_N(s, t) = g^2 \int_0^1 du \int_0^1 dv(uv)^{-\alpha(t)-1}[(1-u)(1-v)]^{\alpha_0-1} \int d\mu_N(x_i, \omega_i, z_j; t)$$

$$\times \exp\left\{\frac{s}{2}\langle 1|(u)X_N(v)|1\rangle - \frac{t}{2}[\langle 1|F_N^{\mathrm{L}}(u)|1\rangle + \langle 1|F_N^{\mathrm{R}}(v)|1\rangle]\right.$$

$$\left. + 2\alpha_0[\langle 1|(u)X_N(v)|1\rangle - \tfrac{1}{2}\langle 1|(u)E_N^{\mathrm{L}}(u)|1\rangle - \tfrac{1}{2}\langle 1|(v)E_N^{\mathrm{R}}(v)|1\rangle]\right\}. \tag{2.9}$$

The sewing procedure performed here, connecting the one-loop non-planar or planar self-energy operators by twisted projected propagators, originally leads to a diagram with twisted intermediate lines in the non-planar case and with one twisted intermediate line (u-t-term) in the planar case (Fig. 1c). Due to the fact that the spurious states of the model are projected out, operatorial duality works for any part of a diagram [10]. One easily sees that the non-planar four-point amplitude is unchanged by a twist operation on any of the internal lines corresponding to u, v- or z_j-integrations. In the planar case the diagrams of Fig. 1b and 1c have to be added by hand to get signaturized trajectories.

3. *Contribution of the Pomeron singularity to the asymptotic behaviour of the orientable non-planar N-loop amplitude*

For the further asymptotic computations we introduce Jacobi transformed variables q_i, σ_i instead of ω_i, x_i

$$q_i = e^{\frac{2\pi^2}{\ln \omega_i}}, \qquad \sigma_i = \frac{2\pi \ln x_i}{\ln \omega_i}. \tag{3.1}$$

We have to investigate now the behaviour of the integrand in Eq. (2.9) near the critical points of the function $\langle 1|(u) X_N(v)|1\rangle$ which multiplies the asymptotic variable s in the exponent[1]. For simplicity we restrict ourselves to the most leading Pomeron contribution connected with the critical point $q_i = 0$ $(i = 1 \ldots N)$. This corresponds to the simultaneous factorization of all the N Pomeron singularities contained in the diagram under consideration. Then, we have to expand the integrand around this point and to perform the integrations over all q_i.

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[1] The limit $|s| \to \infty$ will be performed parallel to the imaginary axis. Questions of analytic continuation into the whole s-plane and of possible saddle points inside the region of integration have not been considered here (see Ref. [5]).

The relevant one-loop matrices (Eq. (2.4)) behave for $q \to 0$ as follows [11]

$$\{\langle 1|F_1\}_n = \left\{\langle 1|\left[-\sum_{k\neq 0}\left(\frac{1}{k}\right)\right]\right\}_n + O(q) \equiv \{\langle 1|\tilde{F}\}_n + O(q),$$

$$(E_1)_{mn} = \left\{\sum_{k\neq 0}\left(\frac{1}{k}\right)M_+\left(-\frac{1}{k}\right)\right\}_{mn} + O(q^2) \equiv \tilde{E}_{mn} + O(q^2),$$

$$(X_1)_{mn} = q\,\frac{\sqrt{mn}}{m!n!}\left[e^{i\sigma}(2\pi i)^{m+n} + e^{-i\sigma}(-2\pi i)^{m+n}\right] + O(q^2) \equiv q\tilde{X}_{mn}(\sigma) + O(q^2). \qquad (3.2)$$

Using the iteration equations (2.2) one can easily determine the respective lowest powers in the q_i-variables of the N-loop matrices ($0(q_i)$ here means a quantity vanishing at least as one of the q_i's)

$$F_N^{\mathrm{R}} \sim F_N^{\mathrm{L}} = \tilde{F} + O(q_i),$$

$$E_N^{\mathrm{R}} \sim E_N^{\mathrm{L}} = \tilde{E} + O(q_i^2), \qquad (3.3)$$

$$X_N \sim (q_1 q_2 \ldots q_N)\left(\prod_{i=1}^{N-1}\tilde{X}(\sigma_i)M_{z_i}(1-(\tilde{E}M_{z_i})^2)^{-1}\right)\tilde{X}(\sigma_N) \equiv (q_1 q_2 \ldots q_N)\tilde{X}_N. \qquad (3.4)$$

The measure $d\mu_N$ in Eq. (2.9) factorizes as we take the lowest powers in the q_i only (σ_i — integrations run from 0 to 2π; $\alpha_{\mathrm{P}}(t) = \frac{1}{3} + t/4$)

$$d\mu_N \sim \left(\prod_{i=1}^{N}\frac{dq_i}{\ln^3 q_i}\,q_i^{-\alpha_{\mathrm{P}}(t)-1}\right)(2\pi)^{-t}(-2\pi^3 g^2)$$

$$\times\left(\prod_{j=1}^{N-1}(-2\pi^3 g^2)\,\frac{dz_j[z_j(1-z_j)]^{-\alpha(t)-1}(1-z_j)^{2\alpha_0}}{[\det(1-(M_{z_j}\tilde{E})^2)]^2}\,[F(z_j)]^t d\sigma_j\right)d\sigma_N. \qquad (3.5)$$

The function $F(z)$ originates from the exponent in Eq. (2.6) and reads

$$F(z) = \frac{1}{2\pi}\left(\frac{\sin\pi z}{\pi z}\right)^4 \exp\left\{-\langle 1|\,(\tilde{F}+(z)\tilde{E})\,\frac{1}{1+M_z\tilde{E}}\,M_z(\tilde{F}+\tilde{E}(z))\,|1\rangle\right\}. \qquad (3.6)$$

We have used here the relations

$$\langle 1|\tilde{F}(z)\,|1\rangle = \tfrac{1}{2}\langle 1|\,(z)\tilde{E}(z)\,|1\rangle = \ln\frac{\sin\pi z}{\pi z}. \qquad (3.7)$$

The logarithmic factors $\ln^{-3} q_i$ in Eq. (3.5) make the Pomeron a unitarity violating cut in the model considered here. Since they are absent in the more satisfactory model with a ghost-free spectrum ($\alpha_0 = 1$), and with space-time dimensions $D = 26$, where one obtains $\alpha_{\mathrm{P}}(t) = 2 + t/4$ [12], we do suppress them in the following. (If we take it for granted that the projection upon physical, i. e. "transverse" intermediate states only deminishes the power of the total multiloop partition function from D to D–2, as it is suggested by the string formalism and one-loop results [18], then our way of calculation is directly transferable to that version of the dual model.)

Taking into account Eqs (2.9), (3.3–3.5) we now perform the q-integrations

$$\int_0^1 dq_1 \ldots \int_0^1 dq_N e^{-(q_1 q_2 \ldots q_N)\frac{s}{2}(-\langle 1|(u)\tilde{X}_N(v)|1\rangle)}(q_1 q_2 \ldots q_N)^{-\alpha_P(t)-1}$$

$$\underset{|s|\to\infty}{\sim} \frac{1}{(N-1)!}\left(\frac{\partial}{\partial v}\right)^{N-1}\left[\left(\frac{s}{2}\right)^{v}\Gamma(-v)\left(-\langle 1|(u)\tilde{X}_N(v)|1\rangle\right)^{v}\right]_{|v=\alpha_P(t)} \tag{3.8}$$

The essential problem to be solved now is to show how to factorize the function $\langle 1|(u)\tilde{X}_N(v)|1\rangle$. Let us first rewrite the one-loop matrix $\tilde{X}(\sigma)$ given by Eq. (3.2) as a dyadic product

$$\tilde{X}(\sigma) = 2\,\mathrm{Re}\,(|k\rangle e^{i\sigma}\langle k|) \equiv \{|k\rangle e^{i\sigma}\langle k|\}, \tag{3.9}$$

where

$$(\langle k|)_n = (|k\rangle)_n = \frac{\sqrt{n}}{n!}(2\pi i)^n \tag{3.10}$$

and $\frac{1}{2}\{M\}$ is a shorthand notation for the real part of a matrix M. Using the abbreviation

$$H_i = M_{z_i}(1-(\tilde{E}M_{z_i})^2)^{-1} \tag{3.11}$$

we obtain with Eq. (3.4) the following form for the quantity in question.

$$\langle 1|(u)\tilde{X}_N(v)|1\rangle = \{\langle 1|(u)|k\rangle e^{i\sigma_1}\langle k|\}\left(\prod_{j=1}^{N-2} H_j\{|k\rangle e^{i\sigma_{j+1}}\langle k|\}\right)$$

$$\times H_{N-1}\{|k\rangle e^{i\sigma_N}\langle k|(v)|1\rangle\}. \tag{3.12}$$

Due to the identity

$$\langle 1|(u)|k\rangle = 2\sin \pi u e^{i\pi(u+1/2)} \tag{3.13}$$

and the redefinitions

$$\pi(u+\tfrac{1}{2})+\sigma_1 \to \sigma_1, \qquad \pi(v+\tfrac{1}{2})+\sigma_N \to \sigma_N, \tag{3.14}$$

we see that we have to deal with a u- and v- independent quantity R:

$$\langle 1|(u)\tilde{X}_N(v)|1\rangle = 4\sin \pi u \sin \pi v R,$$

$$R = \{e^{i\sigma_1}\langle k|\}\left(\prod_{j=1}^{N-2} H_j\{|k\rangle e^{i\sigma_{j+1}}\langle k|\}\right)H_{N-1}\{|k\rangle e^{i\sigma_N}\}. \tag{3.15}$$

We factorize R successively, using the following identity valid for real matrices H_1

$$\{e^{i\sigma_1}\langle k|\}H_1\{|k\rangle e^{i\sigma_2}\langle k|\} = \{[e^{i\sigma_1}\langle k|H_1|k\rangle + e^{-i\sigma_1}\langle \bar{k}|H_1|k\rangle]e^{i\sigma_2}\langle k|\}. \tag{3.16}$$

Defining

$$\varphi_i(\sigma, z_i) = \arg(e^{i\sigma}\langle k|H_i|k\rangle + e^{-i\sigma}\langle\bar{k}|H_i|k\rangle),$$
$$C(\sigma, z_i) = |e^{i\sigma}\langle k|H_i|k\rangle + e^{-i\sigma}\langle\bar{k}|H_i|k\rangle|, \tag{3.17}$$

we obtain for the right hand side of Eq. (3.16)

$$C(\sigma_1, z_1)\{e^{i(\sigma_2+\varphi_1(\sigma_1,z_1))}\langle k|\}. \tag{3.18}$$

Continuation of this procedure rightwards leads to

$$R = 2\,(\prod_{j=1}^{N-1} C(\varrho_j, z_j))\cos(\varrho_N), \tag{3.19}$$

where ($\varphi_0 = 0$)

$$\varrho_j = \sigma_j + \varphi_{j-1}(\varrho_{j-1}, z_{j-1}). \tag{3.20}$$

The Jacobian of the variable transformation $\sigma_j \to \varrho_j$ is one, and due to the periodicity of the integrand we can integrate over ϱ_j again between 0 and 2π The integration over ϱ_N leads to the Pomeron signature factor times a B_4-function

$$\int_0^{2\pi} d\varrho_N(-\cos\varrho_N)^{\nu}_{|\nu=\alpha_P(t)} = (1+e^{i\pi\nu})B_4\left(\frac{\nu+1}{2}, \tfrac{1}{2}\right)_{|\nu=\alpha_P(t)}. \tag{3.21}$$

Collecting all the u-dependent factors in Eq. (2.9) and taking into account Eqs (3.3), (3.7), (3.8) and (3.15) we find as the result of the u-integration the usual Pomeron form--factor (ν corresponds to $\alpha_P(t)$, see Eq. (3.8)) [7]

$$f_P(t, \nu) = 2^{\nu-\alpha(t)}(2\pi)^{2\alpha_0-1}\int_0^1 du[u(1-u)]^{\alpha_0-1}(\sin\pi u)^{\nu-\alpha(t)-\alpha_0}. \tag{3.22}$$

The v-integral provides the same expression once more.

Gathering up all the results, the asymptotic behaviour for $s \to \infty$ produced by the most leading Pomeron singularity of the N-loop amplitude (Eq. (2.9) $N \geqslant 1$) is given by

$$T_N^{(P)}(s, t) \underset{|s|\to\infty}{\sim} \text{const}\, g^2 \frac{1}{(N-1)!}\left(\frac{\partial}{\partial\nu}\right)^{N-1}$$
$$\times \left\{\left[\left(-\frac{s}{2}\right)^{\nu} + \left(\frac{s}{2}\right)^{\nu}\right] g^2 P(\nu) f_P^2(t, \nu)\,[g^2 \Pi(t, \nu)]^{N-1}\right\}_{|\nu=\alpha_P(t)}. \tag{3.23}$$

The function $\Pi(t, \nu)$ is defined by a two-fold integral:

$$\Pi(t, \nu) = -2\pi^3 \int_0^{2\pi} d\varrho \int dz \frac{[z(1-z)]^{-\alpha(t)-1}(1-z)^{2\alpha_0}}{[\det(1-(\tilde{E}M_z)^2)]^2}[F(z)]^{\nu}[C(\varrho, z)]^{\nu}, \tag{3.24}$$

where the functions $F(z)$ and $C(\varrho, z)$ are defined by Eqs (3.6) and (3.17). The z-integration region will be specified later. In Eq. (3.23) we introduced

$$P(\nu) = 2^{\nu} B_4\left(\frac{\nu+1}{2}, \frac{1}{2}\right)\Gamma(-\nu). \tag{3.25}$$

Eq. (3.23) exhibits the factorization properties conjectured in Ref. [7]. In that work we started from the two-loop amplitude formulated in terms of automorphic functions on closed Riemann surfaces as given in Ref. [1]. The asymptotic behaviour was obtained there after doing the Jacobi transformation of the *whole* two-loop expression. The function $\Pi(t, \nu)$ presented there is analogous in structure to the right hand side of Eq. (3.24), though a direct identification of the constituents of the integrands in both representations is difficult due to the different variables used.

As discussed in Ref. [7] the factorization property allows for summing up the Mellin transforms of the asymptotic contributions of all the N-loop diagrams ($N \geqslant 1$). We obtain

$$\frac{g^2 P(l) g^2 f_{\mathrm{P}}^2(t, l) + O(g^6)}{l - (\alpha_{\mathrm{P}}(t) + g^2 \Pi(t, l) + O(g^4))} + \text{regular terms}. \tag{3.26}$$

This defines a renormalized Pomeron trajectory

$$\alpha_{\mathrm{P}}^{\mathrm{r}}(t) = \alpha_{\mathrm{P}}(t) + g^2 \Pi(t, \alpha_{\mathrm{P}}^{\mathrm{r}}(t)) + O(g^4) \tag{3.27}$$

by the vanishing of the denominator. Accordingly, $g^2\Pi(t, \alpha_{\mathrm{P}}(t))$ is the first order correction in g^2 to the Pomeron trajectory. If we want to compute higher order corrections, too, we have to iterate, e. g. the exchange of two-loop aggregates as shown in Fig. 3 with only one of the variables ω_i near the parabolic point 1, but with the "marginal" variables ω_1 and ω_N near 1 in any case. (ω_1 or $\omega_N \not\approx 1$ contribute to form-factor renormalizations not considered here. The factorization of these iterated higher order contributions could then be performed along the same lines as shown here.

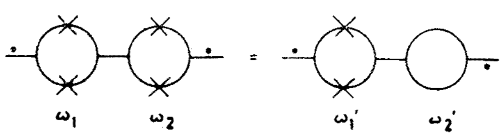

Fig. 3. Operator identity for non-planar two-loop graphs

Finally, let us say a few words about the singularities in the integrand of $\Pi(t, \nu)$ that make it a divergent quantity. Closely related with this is also the decision about the correct z-integration interval. Due to Ref. [3] we can write the determinant in Eq. (3.24) as an infinite product of common partition functions $[f(\omega)]^{-4}$

$$[\det(1-(\tilde{E}M_z)^2)]^{-2} = \prod_{P^{\alpha}{}_{12}} [f(K_\alpha)]^{-4},$$

$$f(\omega) = \prod_{n=1}^{\infty}(1-\omega^n), \tag{3.28}$$

where the product runs over all multiplicative combinations P_{12}^{α} of Möbius transformation generators $(P_1)^n$ and $(P_2)^m$ ($|n|, |m| \geqslant 1$) with a different multiplier K_α. The two generators P_1, P_2 are given in our case by ($\omega_1 = \omega_2 = 1$)

$$P_1 = \Gamma P \Gamma \quad \text{and} \quad P_2 = M_z^{-1} P M_z \tag{3.29}$$

with $\Gamma = M_+ M_-$ and $P = M_-(-1)$. To avoid multiple counting one has to limit the integration in z in such a way that $K_\alpha \leqslant 1$ for all P_{12}^α in Eq. (3.28). Obviously, we have a divergence of the two-loop partition function, if a parabolic point $K_\alpha = 1$ is reached somewhere. As stated in Ref. [13] it is sufficient to consider $P_1 \cdot P_2$ and $P_1 \cdot P_2^{-1}$ and check whether the multiplier can surpass one or not. In our case only $P_1 \cdot P_2$ leads to restrictions and one finds for the corresponding multiplier the following solutions

$$K^{(1)}(P_1 \cdot P_2) = \left(\frac{z}{1-z}\right)^2, \quad K^{(2)}(P_1 \cdot P_2) = \left(\frac{1-z}{z}\right)^2. \tag{3.30}$$

Thus there is a divergence of the integrand at $z = \frac{1}{2}$ ("third divergence phenomenon"; see Ref. [14], where also some regularization schemes are given), and we are led to a restriction of the z-integration region, either $0 \leqslant z \leqslant \frac{1}{2}$ or $\frac{1}{2} \leqslant z \leqslant 1$, depending on which solution was chosen in Eq. (3.30). Analogously to a very similar problem studied by Cremmer [3] we choose the first solution ($K^{(1)}$) and have to integrate z in Eq. (3.24) between 0 and $\frac{1}{2}$. We remark that in the case of the planar (unprojected) propagator beeing used for sewing single Pomeron loops, we have generally to omit all the matrices containing the diagonal matrix (z), and have to replace M_z by (z) there-after. The upper limit of the z-integration comes out then as $\frac{1}{4}$.

4. *Reggeon asymptotic behaviour of non-planar and planar multiloop diagrams*

Besides the critical point $q_i = 0$ contributing to the leading Pomeron behaviour $s^{\alpha_P(t)} \ln^{N-1} s$ considered above, one can easily see that the function $\langle 1|(u) X_N(v)|1\rangle$ multiplying s in the exponent of Eq. (2.9) becomes zero, too, if one of the variables (u, v, z_j) $(1 \leqslant j \leqslant N-1)$ vanishes or if u or v is equal to one. (The points $z_j = 1$ are excluded by the argument given at the end of the last chapter, though the upper limit of the z_j-integrations depend now on the "neighbouring" variables ω_j and ω_{j+1}.) We shall see that these critical points are connected with Reggeon asymptotic behaviour. Similarly to the Pomeron case we limit ourselves to the leading s-behaviour for every N-loop amplitude. This is equivalent to a restriction to the contribution of the four corners given by $(u, v, z_j) = (0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(1, 0, 0)$. Expanding the integrand in Eq. (2.9) around the first corner we obtain

$$\begin{aligned} T_N^{(\mathrm{R})(1)} \underset{s\to\infty}{\sim} g^2 & \left(\prod_{i=1}^{N} 4\pi^2 g^2 \int_0^1 \frac{d\omega_i}{\ln^2 \omega_i} f^{-4}(\omega_i)\omega_i^{-\alpha_0 - 1} \int_{\omega i}^1 \frac{dx_i}{x_i} \left(\frac{\psi_T(x_i)}{1-\omega_i}\right)^t\right) \\ & \times \int_0^1 du \int_0^1 dv \left(\prod_{j=1}^{N-1} \int dz_j\right) (uvz_1 z_2 \ldots z_{N-1})^{-\alpha(t)-1} \\ & \times \exp\left\{-\frac{s}{2}(uvz_1z_2 \ldots z_{N-1})\left(\prod_{i=1}^{N} -(X_1(\omega_i, x_i))^{11}\right)\right\}. \end{aligned} \tag{4.1}$$

Obviously, the product of variables $(u \cdot v \prod_j z_j)$ technically plays here an analogous role as the product $(\prod_i q_i)$ for the Pomeron case (Eq. (3.8)) ($(X_1)^{11}$ is the (11)-matrix element of the matrix X_1 introduced in Eq. (2.4)).

The point (1, 1, 0) gives the identical contribution. Let us now consider the corner (0, 1, 0). We find

$$\langle 1|\,(u)X_N(v)\,|1\rangle \sim -u\,\frac{1-v}{\omega_N}\left(\prod_{j=1}^{N-1}(-z_j)\,(X_1(\omega_j, x_j))^{11}\right)(X_1(\omega_N, x_N))^{11}, \tag{4.2}$$

$$\langle 1|F_N^{\mathrm{R}}(v)\,|1\rangle \sim \langle 1|\,[F_1(x_N, \omega_N)]\,(v)\,|1\rangle \sim \ln\left(\frac{1-v}{\omega_N}\right),$$

$$\tfrac{1}{2}\langle 1|\,(v)E_N^{\mathrm{R}}(v)\,|1\rangle \sim \tfrac{1}{2}\langle 1|\,(v)\,[E_1(\omega_N)]\,(v)\,|1\rangle \sim \ln\left(\frac{1-v}{\sqrt{\omega_N}}\right). \tag{4.3}$$

This leads to a similiar integral as that in Eq. (4.1), where v has to be replaced by $(1-v)/\omega_N$ and we have $-s/2$ instead of $s/2$ due to the minus sign in front of the right-hand side of Eq. (4.2). The fourth corner gives the same contribution so that we obtain altogether

$$T_N^{(\mathrm{R})}(s, t) \underset{|s|\to\infty}{\sim} 2g^2/N!\left(\frac{\partial}{\partial v}\right)^N \left\{\left[\left(-\frac{s}{2}\right)^v + \left(\frac{s}{2}\right)^v\right]\Gamma(-v)\,[g^2\tilde{\Sigma}(t, v)]^N\right\}_{|v\to\alpha(t)}. \tag{4.4}$$

where

$$\tilde{\Sigma}(t, v) = 4\pi^2 \int_0^1 \frac{d\omega}{\ln^2\omega} f^{-4}(\omega)\omega^{-\alpha_0-1} \int_\omega^1 \frac{dx}{x}\left(\frac{\psi_T(x)}{1-\omega}\right)^t |(X_1)^{11}|^v \tag{4.5}$$

and

$$(X_1)^{11} = -(1-\omega)^2\,\frac{\partial^2}{\partial\lambda\partial\mu}\,[\ln\,\psi_T(x\lambda\mu)]_{|\lambda=\mu=1}. \tag{4.6}$$

For $N = 1$ Eq. (4.4) contains the one-loop result of Ref. [5]. By taking the Mellin transform of the sum of the multiloop contributions, we find the expected renormalized Regge trajectory

$$\alpha^{\mathrm{r}}(t) = \alpha(t) + g^2\,\tilde{\Sigma}(t, \alpha^{\mathrm{r}}(t)) + \mathrm{O}(g^4). \tag{4.7}$$

In particular, we see that we have a renormalization of the leading trajectory of positive signature. Obviously, the renormalization correction $\tilde{\Sigma}$ contributes only to trajectories with vacuum quantum numbers coupling to the Pomeron directly (e. g. f-f'). Similiar to the argument following Eq. (3.27) we could calculate along the same lines higher order corrections to Eq. (4.7) by iterating only partially "contracted" ($z_j \approx 0$) multiloop subdiagrams.

The corresponding corrections to the Regge-trajectory originating from the exchange of a chain of planar loops (Fig. 1c) is obtained analogously. The critical point of the

s-multiplying function of the planar multiloop $\langle 1|(u)X_N(v)|1\rangle$ is now $(u, v, z_j) = (0,0,0)$ alone, as can easily be checked by Eqs (2.2) and (2.5). Around this corner of the integration region we have again a factorized expression

$$\langle 1|(u)X_N(v)|1\rangle \sim -uv\left(\prod_{j=1}^{N-1} z_j\right)\left(\prod_{i=1}^{N} -(X_1^i)^{11}\right), \tag{4.8}$$

where according to Eq. (2.5) the (11)-matrix element is given by

$$(X_1)^{11} = (1-x)(1-y)\left(\frac{\pi}{\ln\omega}\right)^2\left[-\sin^{-2}\left(\frac{\sigma}{2}\right) + 8\sum_{m=1}^{\infty}\frac{mq^{2m}}{1-q^{2m}}\cos m\sigma\right], \tag{4.9}$$

$$((X_1)^{11} \leqslant 0).$$

We obtain for the leading asymptotical behaviour of the planar amplitude shown in Fig. 1c an integral of the type of the right-hand sides of Eqs (3.8) and (4.1). Therefore the leading contribution is again given by the $(N)^{\text{th}}$ derivative of the Regge factor times the $(N)^{\text{th}}$ power of a planar self-energy correction $\Sigma(t, v)$[2]

$$T_N(s,t) \sim g^2\left(\prod_{i=1}^{N} 4\pi^2 g^2 \int_0^1 \frac{d\omega_i}{\ln^2\omega_i} f^{-4}(\omega_i)\omega_i^{-\alpha_0-1}\int_{\omega_i}^1 \frac{dx_i}{x_i}\left[\frac{\psi(x_i)^2}{(1-x_i)\left(1-\frac{\omega_i}{x_i}\right)}\right]^{\frac{t}{2}}\right.$$

$$\left.\times\frac{1-\omega_i}{(1-x_i)\left(1-\frac{\omega_i}{x_i}\right)}\right)\int_0^1 du\int_0^1 dv\left(\prod_{j=1}^{N-1}\int_0^1 dz_j\right)(uvz_1z_2\ldots z_{N-1})^{-\alpha(t)-1}$$

$$\times\exp\left[-\frac{s}{2}(uvz_1z_2\ldots z_{N-1})\left(\prod_{i=1}^{N} -(X_1(\omega_i, x_i))^{11}\right)\right]$$

$$\underset{\substack{|s|\to\infty \\ \operatorname{Re}(s)>0}}{\sim} \frac{g^2}{N!}\left(\frac{\partial}{\partial v}\right)^N\left\{\left(\frac{s}{2}\right)^v \Gamma(-v)\,[g^2\Sigma(t,v)]^N\right\}_{|v=\alpha(t)}. \tag{4.10}$$

The planar self-energy correction $\Sigma(t, v)$, which has been already computed by Neveu and Scherk [4] on the basis of the investigation of the one-loop diagram, reads

$$\Sigma(t,v) = 4\pi^2\int_0^1 \frac{d\omega}{\ln^2\omega} f^{-4}(\omega)\omega^{-\alpha_0-1}\int_\omega^1 \frac{dx}{x}\left[\frac{\psi(x)^2}{(1-x)\left(1-\frac{\omega}{x}\right)}\right]^{\frac{t}{2}}$$

$$\times\frac{(1-\omega)}{(1-x)\left[1-\frac{\omega}{x}\right]}|X_1^{11}(\omega, x)|^v. \tag{4.11}$$

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[2] Due to the configuration of dots of the self-energy operators used, we have to do with a u–t-diagram effectively. Hence, the integral is convergent in the right half-plane of s.

Summing up, we obtain again a renormalized Regge trajectory

$$\alpha^{r}(t) = \alpha(t)+g^2\,\Sigma(t,\alpha^{r}(t))+O(g^4). \tag{4.12}$$

This is just equivalent to the result of Karpf and Liehl [6].

A slight difference to the renormalization corrections originating in non-planar (Pomeron) loops concerns the emergency of the signature factor. It does not arise here automatically but has to be produced by adding u–t and s–t–diagrams (Fig. 1b, c).

As it is well known [4], the divergence of the partition-function $f^{-4}(\omega)$ for $\omega \to 1$ requires the introduction of a regularizing counterterm in any planar amplitude. This regularization procedure is not unique in general [15]. In the final result for the asymptotic behaviour of multiloop planar diagrams only the *one-loop* self-energy correction (Eq. (4.11) appears. Thus, any choice for the one-loop counterterm, e. g. the one given by Neveu and Scherk, solves the divergence problem of the multiloop asymptotic behaviour, too.

As was stated by Scherk [15], the counterterm can be generated by a corresponding operator. Thus, the regularization can be already performed on the one-loop operator level. We could start from the very beginning with regularized one-loop planar operators and end up with a finite unitarity correction in Eq. (4.12). Of course, such a procedure would not succeed in making the planar amplitudes T_N finite, as the multiloop divergences [16] connected with the infinities of higher partition functions (Eq. (3.28)) are not extinguished.

5. *Conclusion and discussion*

In this paper we have shown, on the basis of the operator formalism of the dual model, the factorization of the asymptotically leading contributions of both iterated Pomeron exchange and Reggeon exchange in orientable non-planar and planar diagrams. The results have the expected structures of manifold Pomeron or ordinary Regge poles in the complex angular momentum plane. Here we have dealt only with trajectory renormalization, and have not discussed the renormalization of residua. The latter could be done relatively easy using the same methods for the investigation of other "critical" corners of the integration region.

As stated above, a direct generalization of the iteration scheme for multiloop subdiagrams would allow for the calculation of higher renormalization corrections to the trajectories $\alpha_P(t)$ and $\alpha(t)$. But of course, a satisfactory solution of the regularization problem, for all the relevant divergencies apparent in the partition functions, should be found ealier. Especially we can expect "mixed" corrections to the Regge trajectory $\alpha(t)$ of the type $g^4\,\tilde{\Sigma}(t)\,\Sigma(t)$, corresponding to the right-hand configuration, in Fig. 3. Such a term could be obtained by sewing planar and non-planar self-energy operators or by using the operator identity illustrated in Fig. 3, and then finding out the respective duality transformation for the integration variables in the 4-point function. One is led then to the contributions of additional critical points to the Regge-asymptotic behaviour of the non-planar N-loop amplitudes (Fig. 1a) not considered in our paper.

As can be seen, by inclusion of isospin factors à la Chan-Paton, the non-planar renormalization correction $\tilde{\Sigma}$ contributes only to $I = 0$ trajectories (i. e. the f-trajectory) coupling to the Pomeron directly, while planar (regularized) corrections Σ are present for all trajectories. Thus $\tilde{\Sigma}$ leads to a breaking of the $f\varrho\omega A_2$-degeneracy to order g^2 [5].

Similiar factorization properties, as those discussed here for the conventional dual model, should be expected also for the more recent types of dual models with a ghost--killing Virasoro algebra and additional spin-mode creation and annihilation operators. Working with ghost-free models, a better understanding of the reflection mechanism of gauges extending recent one-loop results [17] in multiloop amplitudes is desirable.

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