ON THE CONNECTION BETWEEN EINSTEIN SURFACE EQUATIONS, RAYCHAUDHURI EQUATION AND THE INTRINSIC GEOMETRY OF NULL HYPERSURFACES

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If teleparallelism is chosen as an affine connection on a null hypersurface $\overset{*}{V}_3$, then expansion θ and distortion $|\sigma|$ may be represented on it in terms of intrinsic geometry of the $\overset{*}{V}_3$. The Einstein surface equations together with the Raychaudhuri equation for θ yield an equation which restricts the intrinsic geometry of that null hypersurface.

In the theory of gravitational radiation three optical parameters are often used: expansion θ, rotation ω and distortion σ. These parameters are determined by a space-time filling congruence of null geodesics with tangent vector field k^α. Evolution of the optical parameters along the null geodesics is described by the Sachs propagation equations [(1]).

Consider all null geodesics lying on one null hypersurface $\overset{*}{V}_3$, which is given by the equation $\varphi(x^\lambda) = 0$ and being a gradient field of this hypersurface: $k_\alpha = \partial_\alpha \varphi$. In this case the rotation ω vanishes on the $\overset{*}{V}_3$. A problem arises: Is it possible to express the parameters θ and $|\sigma|$ in terms of quantities of the intrinsic geometry of the $\overset{*}{V}_3$ only? The response is affirmative.

In order to relate the optical parameters with the intrinsic geometry of a null hypersurface, an affine structure must be given on it. Intrinsically, the $\overset{*}{V}_3$ is characterized only by a metric tensor $\bar{g}_{ik}$ which is degenerate: det $(\bar{g}_{ik}) = 0$; then the Christoffel symbols of second kind do not exist on it. In those conditions an arbitrary number of affinities may be introduced on the $\overset{*}{V}_3$ and one has to decide which of them to use. This problem has been studied by a number of investigators ([2–8]). The Christoffel connection in a Riemann space is both metric and symmetric, but it is not the case in null hypersurfaces. It may be proved that for a singular metric these two properties are not compatible — a symmetric connection cannot be metric and vice versa. Most of the authors (the only ex-

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ception known to me is paper [8]) have chosen torsionless connections; their works have been based, as a rule, on the proposition [9] that a symmetric connection of a space A_n uniquely determines a symmetric connection in a subspace X_{n-1} provided that X_{n-1} is rigged in the A_n. Dautcourt ([3]) has discussed a class of symmetric connections and one of them has been obtained by projecting the Christoffel symbols of the imbedding space-time into a null hypersurface.

It seems to me, however, that of these two properties of an affine connection, the property of being metric is much more important than that of being symmetric. When a metric connection is used, the length of any vector remains unchanged under parallel transport. Torsion complicates some formulas as compared with a symmetric connection, but it has no effect on a geodesic equation. The condition that the connection should be symmetric does not determine it uniquely, further requirements are needed. The curvature tensor for an affine connection does not play so important geometrical role as the Riemann-Christoffel tensor in the Riemann spaces does. Therefore it would be very convenient to put the curvature tensor equal to zero everywhere; in fact it turns out that only asymmetric connections can be so chosen as to satisfy this requirement.

The approach with the use of symmetric connections seems to me to be unsatisfactory and I make use of a connection called teleparallelism, introduced for the first time by Einstein ([10]). With the aid of this connection some new results have been obtained; in this paper I shall present one of them, concerning the geometrical interpretation of optical parameters on a null hypersurface.

The metric tensor $\bar{g}_{ik}$ determines (up to 4-parameter transformation group) orthonormal triads of vectors $\{\underset{a}{w}^i\}$ and $\{\overset{a}{v}_i\}$, $a, i = 1, 2, 3$, where $\underset{1}{w}^i$ is an eigenvector of the metric

$$\bar{g}_{ik}\underset{1}{w}^k = 0 \quad \text{and} \quad \bar{g}_{ik} = -\sum_{A=2}^{3} \overset{A}{v}_i \overset{A}{v}_k .$$

The teleparallelism defined as

$$\Gamma^i_{kl} \equiv \underset{a}{w}^i \partial_k \overset{a}{v}_l$$

is a metric connection

$$\nabla_i \overset{a}{v}_k = \nabla_i \underset{a}{w}^k = 0$$

and is flat

$$R_{ikl}{}^m = 2\partial_{[i}\Gamma^m_{k]l} + 2\Gamma^m_{[i|n|}\Gamma^n_{k]l} = 0,$$

however, torsion $S_{ik}{}^l \equiv \Gamma^l_{[ik]}$ does not vanish in general. We introduce in a space-time a coordinate system in which the $\overset{*}{V}_3$ has an equation $x^0 \overset{*}{=} 0$ and a space-time metric has components: $g_{01} \overset{*}{=} 1$; $g_{1i} \overset{*}{=} 0$. In this case the space components of the tensor $g_{\alpha\beta}$ are equal to the $\overset{*}{V}_3$ metric $g_{ik} \overset{*}{=} \bar{g}_{ik}$, moreover $k^\alpha = \delta^\alpha_1$.

For every tensor $T_{\alpha\beta}$ determined at points of the $\overset{*}{V}_3$ its components T_{00}, T_{0i} and T_{ik} behave as a scalar, vector and tensor respectively, with respect to coordinate transformations in the $\overset{*}{V}_3$. Now we replace spatial partial derivatives with covariant derivatives, e. g.

$$\partial_i T_{0k} \equiv \nabla_i T_{0k} + \Gamma^n_{ik} T_{0n}.$$

This procedure allows us to express all the Christoffel symbols at points of the $\overset{*}{V}_3$ in terms of $g_{\alpha\beta}$, $\partial_0 g_{\alpha\beta}$ and Γ^i_{kl}, e. g.

$$\begin{Bmatrix} 0 \\ ik \end{Bmatrix} \overset{*}{=} 2 g_{n(i} S_{k)1}{}^n;$$

formulas for the remaining Christoffel symbols are more complicated. Having all $\begin{Bmatrix} \alpha \\ \lambda\mu \end{Bmatrix}$ expressed in this manner we are able to calculate the expansion and the distortion on the $\overset{*}{V}_3$. We have:

$$\theta \equiv \tfrac{1}{2} \nabla_\alpha k^\alpha \overset{*}{=} \tfrac{1}{2} \begin{Bmatrix} \alpha \\ \alpha 1 \end{Bmatrix} = \underset{1}{w}^i S_{ik}{}^k. \tag{1}$$

Similarly

$$|\sigma|^2 = \underset{1}{w}^i \underset{1}{w}^k S_{km}{}^n [S_{in}{}^m - \bar{g}_{np} S_{il}{}^p (\underset{2}{w}^l \underset{2}{w}^m + \underset{3}{w}^l \underset{3}{w}^m)] - \theta^2. \tag{2}$$

At every point of the $\overset{*}{V}_3$ the vectors $\{\underset{a}{w}^i\}$ may be subjected to transformations of a 4-parameter group of null rotations without changing the metric $\bar{g}_{ik}$. This transformation group is obtained from a null rotation group of a null tetrad $\{k^\alpha, m^\alpha, t^\alpha, \bar{t}^\alpha\}$, which was introduced by Jordan, Ehlers and Sachs [11]. In fact, there is a close connection between vectors $\{k^\alpha, m^\alpha, t^\alpha, \bar{t}^\alpha\}$ and triads $\{\underset{a}{w}^i\}$ and $\{\overset{a}{v}_i\}$. With the aid of the null rotations it is always possible to choose the vectors $\underset{a}{w}^i$ as to satisfy a "gauge" condition

$$\bar{g}_{n[i} S_{kl]}{}^n = 0, \tag{3}$$

which considerably simplifies all calculations.

Taking into account (3) we get

$$|\sigma|^2 = 2 \underset{1}{w}^i \underset{1}{w}^k S_{in}{}^l S_{kl}{}^n - \theta^2. \tag{4}$$

We see that the θ and $|\sigma|$ are completely determined by the intrinsic geometry of the null surface and do not depend on its imbedding in the space V_4.

The Raychaudhuri equation ([1])

$$\frac{d\theta}{dv} + \theta^2 + |\sigma|^2 = \tfrac{1}{2} R_{\alpha\beta} k^\alpha k^\beta$$

$\left(\text{here } v \text{ is an affine parameter along the null geodesics defined by } k^\alpha = \frac{\partial x^\alpha}{\partial v}\right)$ takes on the $\overset{*}{V}_3$ the form:

$$\underset{1}{w}^i \underset{1}{w}^k (\nabla_i S_{kl}{}^l + 2S_{in}{}^l S_{kl}{}^n) = \tfrac{1}{2} R_{\alpha\beta} k^\alpha k^\beta. \tag{5}$$

In our coordinates $G^0{}_i = R^0{}_i \overset{*}{=} R_{1i}$ and $R_{\alpha\beta} k^\alpha k^\beta \overset{*}{=} R_{11}$. Thus the Einstein surface equations $G^0{}_i = 0$ result in

$$\underset{1}{w}^i \underset{1}{w}^k (\nabla_i S_{kl}{}^l + 2S_{in}{}^l S_{kl}{}^n) = 0. \tag{6}$$

This equation (representing the equation $G^0{}_1 \overset{*}{=} 0$) gives restrictions on a possible choice of the metric of any null hypersurface.

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