CONNECTION OF THE PAPAPETROU PSEUDOTENSOR AND THE BEL-ROBINSON SUPERENERGY TENSOR

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The expansion of the Papapetrou pseudotensor in the Riemann normal coordinates is obtained in the zeroth, first, and second orders in the coordinates. Pure gravitational terms appear only in the second order and the corresponding coefficient of the expansion is a tensor of the fourth rank. Being symmetrized, this tensor exactly coincides with the Bel-Robinson superenergy tensor which can be hence interpreted as the relative energy-momentum density of the gravitational field.

It is characteristic for the gravitational field described in a general covariant form, irrelative to frames of reference, to act purely geometrically upon test masses, without involving any 4-vector forces. However, the analysis of the geodesic deviation equation shows that the curvature tensor plays the role of the relative field intensity (see e.g. [1]), and it makes it natural to take the Bel-Robinson superenergy tensor quadratic in curvature for the relative energy-momentum density of the gravitational field. This approach is confirmed by the result obtained by Garecki [3], who showed that some components of the Bel-Robinson superenergy tensor essentially enter the expansion of the Einstein pseudotensor.

We are going to examine the Papapetrou energy-momentum symmetric pseudotensor which was introduced already in 1948 with the help of the well known Belinfante-Rosenfeld method. The symmetry of the pseudotensor suggests its closer connection with the Bel--Robinson tensor, symmetric in its all four indices. On the other hand, it is possible to derive the Papapetrou pseudotensor not only by the ordinary way (see e.g. [1], here the Einsteinian Lagrangian can be used as well as the density of the scalar curvature), but also as Burlankov showed in 1963 [5], by a new one based on the ideas of the bimetric formalism [6] (see also [1]), considered by Papapetrou, too. The method reveals the physical sense of the Papapetrou pseudotensor as the energy-momentum density, and links its

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symmetry with that of the second metric. According to the idea of Rosen, Burlankov used the covariant derivative relative to the second metric (denoted by $A_{\mu|\nu}$), defined with the help of the second connection

$$\gamma^{\lambda}_{\mu\nu} := \tfrac{1}{2} e^{\kappa\lambda}(e_{\kappa\nu,\mu} + e_{\mu\kappa,\nu} - e_{\mu\nu,\kappa}) \tag{1}$$

($e_{\mu\nu}$ is the second metric associated with a certain background flat space-time). Then, according to the Noether theorem, the tensor density

$$\pi^{\sigma\tau} \equiv \pi^{\tau\sigma} := -2 \frac{\partial \mathscr{L}}{\delta e_{\sigma\tau}} \tag{2}$$

satisfies the weak conservation law

$$\pi^{\sigma\tau}_{|\tau} = 0, \tag{3}$$

written in Cartesian second metric coordinates, with the help of partial divergence. In these coordinates the very pseudotensor (genuine tensor from the viewpoint of the bimetric formalism) is equal to

$$\pi^{\sigma\tau} = -\frac{1}{2\kappa}(\mathfrak{g}^{\sigma\tau}_{,\mu,\nu}\delta^{\mu\nu} + \mathfrak{g}^{\mu\nu}_{,\mu,\nu}\delta^{\sigma\tau} - 2\mathfrak{g}^{\mu(\sigma}_{,\mu,\nu}\delta^{\tau)\nu}) \tag{4}$$

(Gothic letters are used for densities: $\mathfrak{g}^{\sigma\tau} := \sqrt{-g} g^{\sigma\tau}$) which is equivalent to the expression for the Einstein equations when the wave part (right-hand side) and the sources, including nonlinear part of the gravitational field (left-hand side), are separated. The instructive analysis of the Einstein equations structure, carried out by Gupta and Halpern on the base of the Papapetrou pseudotensor [7, 8] is worth mentioning. Note also that it is possible to rewrite identically the Einstein equations to obtain the relation

$$\begin{aligned}\mathfrak{g}^{\sigma\tau}_{,\mu,\nu}g^{\mu\nu} + \mathfrak{g}^{\mu\nu}_{,\mu,\nu}g^{\sigma\tau} - 2\mathfrak{g}^{\mu(\sigma}_{,\mu,\nu}g^{\tau)\nu} = 2\kappa\mathscr{I}^{\sigma\tau} + \sqrt{-g}\,[&g^{\sigma\mu}_{,\nu}g^{\tau\nu}_{,\mu} - g^{\sigma\tau}_{,\mu}g^{\mu\nu}_{,\nu} + g^{\sigma\mu}_{,\lambda}g^{\tau\nu}_{,\varrho}g^{\lambda\varrho}g_{\mu\nu} \\ &- g^{\sigma\mu}_{,\lambda}g^{\lambda\varrho}_{,\nu}g^{\tau\nu}g_{\mu\varrho} - g^{\sigma\nu}g^{\tau\mu}_{,\lambda}g^{\lambda\varrho}_{,\nu}g_{\varrho\mu} + \tfrac{1}{2}\, g^{\mu\nu}_{,\lambda}g^{\lambda\varrho}_{,\nu}g_{\mu\varrho}g^{\sigma\tau} \\ &- \tfrac{1}{2}\,(g^{\varepsilon\omega}_{,\nu}g_{\varepsilon\omega}g^{\lambda\varrho}_{,\mu}g_{\lambda\varrho} + g^{\varepsilon\omega}_{,\mu}g_{\varepsilon\omega,\nu})\,(g^{\sigma\mu}g^{\tau\nu} - \tfrac{1}{2}\, g^{\sigma\tau}g^{\mu\nu}) \\ &- \tfrac{1}{2}\, g^{\varepsilon\omega}_{,\mu}g_{\varepsilon\omega}(g^{\sigma\tau}_{,\nu}g^{\mu\nu} - 2g^{\nu(\sigma}_{,\nu}g^{\tau)\mu})],\end{aligned} \tag{5}$$

which is useful in the applications of the Papapetrou pseudotensor (e.g. in calculations of the energy flux density of weak gravitational waves).

In order to analyse the Papapetrou pseudotensor in the Riemann normal coordinates, we need the expansion of the metric in the coordinates up to the 4-th order (see [9], p. 45 and [10]):

$$\begin{aligned}g_{\varepsilon\omega}(y) = \overset{0}{g}_{\varepsilon\omega} - \tfrac{1}{3}\,\overset{0}{R}_{\varepsilon\mu\omega\nu}y^{\mu}y^{\nu} - \tfrac{1}{3}\,\overset{0}{R}_{\varepsilon\lambda\omega\nu;\mu}y^{\mu}y^{\nu}y^{\lambda} \\ + \frac{1}{5!}(-6\overset{0}{R}_{\varepsilon\varrho\omega\lambda;\nu;\mu} + \tfrac{16}{3}\,\overset{0}{R}_{\mu\omega\nu.}{}^{\theta}\overset{0}{R}_{\lambda\varepsilon\varrho\theta})y^{\mu}y^{\nu}y^{\lambda}y^{\varrho} + \dots .\end{aligned} \tag{6}$$

Since we are interested in the derivatives of the covariant components of the metric density, we take its Taylor series

$$(\overset{0}{g}_{\varepsilon\omega} = \overset{0}{g}{}^{\varepsilon\omega} = \delta_{\varepsilon\omega} = \mathrm{diag}\,(1,\,-1,\,-1,\,-1)),$$

$$\mathfrak{g}^{\sigma\tau}(y) = \delta^{\sigma\tau} + \frac{1}{2!}\,\overset{0}{\mathfrak{g}}{}^{\sigma\tau}_{,\alpha,\beta}y^{\alpha}y^{\beta} + \frac{1}{3!}\,\overset{0}{\mathfrak{g}}{}^{\sigma\tau}_{,\alpha,\beta,\lambda}y^{\alpha}y^{\beta}y^{\lambda} + \frac{1}{4!}\,\overset{0}{\mathfrak{g}}{}^{\sigma\tau}_{,\alpha,\beta,\lambda,\varrho}y^{\alpha}y^{\beta}y^{\lambda}y^{\varrho} \tag{7}$$

and correspondingly

$$\mathfrak{g}^{\sigma\tau}_{,\mu,\nu}(y) = \overset{0}{\mathfrak{g}}{}^{\sigma\tau}_{,\mu,\nu} + \overset{0}{\mathfrak{g}}{}^{\sigma\tau}_{,\mu,\nu,\lambda}y^{\lambda} + \tfrac{1}{2}\,\overset{0}{\mathfrak{g}}{}^{\sigma\tau}_{,\mu,\nu,\lambda,\varrho}y^{\lambda}y^{\varrho} \tag{8}$$

(the symbol "0" over a quantity indicates that it is taken in the Riemann normal coordinates at the initial point, and the number in round brackets signifies the order of the approximation). It is not difficult to compute that the following equalities are identically satisfied in the Riemann normal coordinates:

$$\overset{0}{\mathfrak{g}}{}^{\sigma\tau}_{,\mu,\nu} = (\tfrac{1}{2}\,\delta^{\varepsilon\omega}\delta^{\sigma\tau} - \delta^{\varepsilon\sigma}\delta^{\omega\tau})\overset{0}{g}_{\varepsilon\omega,\mu,\nu}, \tag{9}$$

$$\overset{0}{\mathfrak{g}}{}^{\sigma\tau}_{,\mu,\nu,\lambda} = (\tfrac{1}{2}\,\delta^{\varepsilon\omega}\delta^{\sigma\tau} - \delta^{\varepsilon\sigma}\delta^{\omega\tau})\overset{0}{g}_{\varepsilon\omega,\mu,\nu,\lambda}, \tag{10}$$

$$\begin{aligned}\overset{0}{\mathfrak{g}}{}^{\sigma\tau}_{,\mu,\nu,\lambda,\varrho} = {} & [(\tfrac{1}{4}\,\delta^{\sigma\tau}\delta^{\varepsilon\omega}\delta^{\kappa\eta} - \tfrac{1}{2}\,\delta^{\sigma\tau}\delta^{\varepsilon\kappa}\delta^{\omega\eta} - \tfrac{1}{2}\,\delta^{\kappa\eta}\delta^{\varepsilon\sigma}\delta^{\omega\tau} - \tfrac{1}{2}\,\delta^{\varepsilon\omega}\delta^{\sigma\kappa}\delta^{\tau\eta} + \delta^{\varepsilon\kappa}\delta^{\sigma\eta}\delta^{\omega\tau} \\ & + \delta^{\varepsilon\sigma}\delta^{\omega\kappa}\delta^{\tau\eta})\,(\overset{0}{g}_{\kappa\eta,\nu,\varrho}\overset{0}{g}_{\varepsilon\omega,\mu,\lambda} + \overset{0}{g}_{\kappa\eta,\nu,\lambda}\overset{0}{g}_{\varepsilon\omega,\mu,\varrho} + \overset{0}{g}_{\varepsilon\omega,\mu,\nu}\overset{0}{g}_{\kappa\eta,\lambda,\varrho}) + (\tfrac{1}{2}\,\delta^{\sigma\tau}\delta^{\varepsilon\omega} - \delta^{\varepsilon\sigma}\delta^{\omega\tau})\overset{0}{g}_{\varepsilon\omega,\mu,\nu,\lambda,\varrho}].\end{aligned} \tag{11}$$

On the other hand it is known that

$$\overset{0}{g}_{\varepsilon\omega,\mu,\nu} = \tfrac{2}{3}\,\overset{0}{R}_{\varepsilon(\mu\nu)\omega}, \tag{12}$$

$$\overset{0}{g}_{\varepsilon\omega,\mu,\nu,\lambda} = \overset{0}{R}_{\varepsilon(\underline{\lambda}\underline{\nu}\omega;\underline{\mu})}, \tag{13}$$

$$\overset{0}{g}_{\varepsilon\omega,\mu,\nu,\lambda,\varrho} = \tfrac{1}{5}\,[-6\overset{0}{R}_{\varepsilon(\underline{\varrho}\omega\underline{\lambda};\underline{\nu};\underline{\mu})} + \tfrac{16}{3}\,\overset{0}{R}_{(\underline{\mu}\omega\underline{\nu}.}{}^{\theta}\overset{0}{R}_{\underline{\lambda}\varepsilon\underline{\varrho})\theta}] \tag{14}$$

(all quantities should be expressed through the Cartesian metric and the curvature tensor). Round brackets mean symmetrization over all underlined indices involved.

From the expansion (8) follows that (9) corresponds to the zero approximation of the second partial derivative of the metric density, (10) corresponds to the first and (11) to the second approximation. We construct for each approximation the expression for $\pi^{\sigma\tau}$ with the help of (8) and express the coefficients of the expansion from (12)–(14). We present here the detailed computation only for the comparatively simple zeroth approximation:

$$-2\kappa\overset{(0)}{\pi}{}^{\sigma\tau} = \overset{0}{\mathfrak{g}}{}^{\sigma\tau}_{,\mu,\nu}\delta^{\mu\nu} + \overset{0}{\mathfrak{g}}{}^{\mu\nu}_{,\mu,\nu}\delta^{\sigma\tau} - 2\overset{0}{\mathfrak{g}}{}^{\mu(\sigma}_{,\mu,\nu}\delta^{\tau)\nu}$$

$$= [(\tfrac{1}{2}\,\delta^{\varepsilon\omega}\delta^{\sigma\tau} - \delta^{\varepsilon\sigma}\delta^{\omega\tau})\delta^{\mu\nu}\overset{0}{g}_{\varepsilon\omega,\mu,\nu} + (\tfrac{1}{2}\,\delta^{\varepsilon\omega}\delta^{\mu\nu} - \delta^{\varepsilon\mu}\delta^{\omega\nu})\delta^{\sigma\tau}\overset{0}{g}_{\varepsilon\omega,\mu,\nu}$$

$$-(\tfrac{1}{2}\,\delta^{\varepsilon\omega}\delta^{\sigma\mu}-\delta^{\varepsilon\sigma}\delta^{\omega\mu})\delta^{\tau\nu}\overset{0}{g}_{\varepsilon\omega,\mu,\nu}-(\tfrac{1}{2}\,\delta^{\varepsilon\omega}\delta^{\tau\mu}-\delta^{\varepsilon\tau}\delta^{\omega\mu})\delta^{\sigma\nu}\overset{0}{g}_{\varepsilon\omega,\mu,\nu}]$$

$$= (\delta^{\varepsilon\omega}\delta^{\sigma\tau}\delta^{\mu\nu}-\delta^{\varepsilon\sigma}\delta^{\omega\tau}\delta^{\mu\nu}-\delta^{\varepsilon\mu}\delta^{\omega\nu}\delta^{\sigma\tau}-\delta^{\varepsilon\omega}\delta^{\sigma\mu}\delta^{\tau\nu}+\delta^{\varepsilon\sigma}\delta^{\omega\mu}\delta^{\tau\nu}$$

$$+\delta^{\varepsilon\tau}\delta^{\omega\mu}\delta^{\sigma\nu})\cdot\tfrac{2}{3}\,\overset{0}{R}_{\varepsilon(\mu\nu)\omega} = -2(\overset{0}{R}{}^{\sigma\tau}-\tfrac{1}{2}\,\delta^{\sigma\tau}\overset{0}{R}). \tag{15}$$

Thus we have

$$\overset{(0)}{\pi}{}^{\sigma\tau} = -\overset{0}{\mathscr{I}}{}^{\sigma\tau}. \tag{16}$$

During the calculation of the first approximation, as it is clear from (13), the covariant derivatives of the Riemann-Christoffel tensor appear in the expression for $\overset{(1)}{\pi}{}^{\sigma\tau}$. With the help of the Bianchi identities they are reduced to the derivatives of the Ricci tensor and we have

$$-2\kappa\overset{(1)}{\pi}{}^{\sigma\tau} = 2(\overset{0}{R}{}^{\sigma\tau}_{;\lambda}-\tfrac{1}{2}\,\delta^{\sigma\tau}\overset{0}{R}_{;\lambda})y^{\lambda}, \tag{17}$$

or

$$\overset{(1)}{\pi}{}^{\sigma\tau} = -\overset{0}{\mathscr{I}}{}^{\sigma\tau}_{;\lambda}y^{\lambda}. \tag{18}$$

The computation of the second approximation turns out to be particularly cumbersome. In order to simplify it we considered here empty space, i.e. put $R_{\mu\nu} = 0$ (precisely for this case the Bel-Robinson tensor is defined with all its properties). In this approximation the second covariant derivatives of the curvature tensor appear as the expansion coefficients. Here also they are reduced with the help of the Bianchi identities (with the preceding alternation of the covariant derivatives which brings additional terms, quadratic in the curvature tensor) to the derivatives of the Ricci tensor. Here, in computations we used the identities of the Lanczos type [11] in empty space (see [1]):

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}g_{\lambda\varrho} = 4R_{\alpha\beta\gamma\lambda}R^{\alpha\beta\gamma}{}_{\cdot\cdot\cdot\varrho}, \tag{19}$$

$$R^{\mu\tau\sigma\theta}R_{\mu\varrho\lambda\theta}-R^{\mu\tau\cdot\theta}_{\cdot\cdot\varrho\cdot}R_{\mu\cdot\lambda\theta}^{\cdot\sigma\cdot\cdot}-R^{\mu\sigma\cdot\theta}_{\cdot\cdot\varrho\cdot}R_{\mu\cdot\lambda\theta}^{\cdot\tau\cdot\cdot}$$

$$+R^{\mu\sigma\cdot\theta}_{\cdot\cdot\varrho\cdot}R_{\mu\lambda\cdot\theta}^{\cdot\cdot\tau\cdot} = \tfrac{1}{8}\,R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}(g^{\tau}_{\varrho}g^{\sigma}_{\lambda}-g^{\sigma\tau}g_{\lambda\varrho}). \tag{20}$$

We omit here all the cumbersome detailes and give the final result for the second approximation of the Papapetrou pseudotensor density in empty space in the Riemann normal coordinates

$$\overset{(2)}{\pi}{}^{\sigma\tau} = \tfrac{1}{3}\,[\overset{0}{R}{}_{\mu\cdot\cdot;\nu}^{\cdot\tau\varrho\cdot}\overset{0}{R}{}^{\mu\lambda\sigma\nu}+\overset{0}{R}{}_{\mu\cdot\cdot;\nu}^{\cdot\tau\lambda\cdot}\overset{0}{R}{}^{\mu\varrho\sigma\nu}-\tfrac{1}{8}\,\overset{0}{R}{}^{\alpha\beta\gamma\delta}\overset{0}{R}_{\alpha\beta\gamma\delta}\delta^{(\sigma\tau}\delta^{\lambda\varrho)}]y_{\lambda}y_{\varrho}. \tag{21}$$

So we see here that only in this approximation the tensor construction emerges

$$\mathscr{P}^{\sigma\tau\lambda\varrho}: = R_{\mu\cdot\cdot;\nu}^{\cdot\tau\varrho\cdot}R^{\mu\lambda\sigma\nu}+R_{\mu\cdot\cdot;\nu}^{\cdot\tau\lambda\cdot}R^{\mu\varrho\sigma\nu}-\tfrac{1}{8}\,R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}g^{(\sigma\tau}g^{\lambda\varrho)}, \tag{22}$$

and it is included in $\overset{(2)}{\pi}{}^{\varrho\tau}$ as

$$\overset{(2)}{\pi}{}^{\sigma\tau} = \frac{\sqrt{-g}}{3}\,\overset{0}{\mathscr{P}}{}^{\sigma\tau\lambda\varrho}y_{\lambda}y_{\varrho} \tag{23}$$

(the indices of the Riemann coordinates y_{μ} are lowered by the pseudo-Cartesian metric $\delta_{\mu\nu}$).

The definition of the tensor $\mathcal{P}^{\sigma\tau\lambda\varrho}$ shows that the symmetry

$$\mathcal{P}^{\sigma\tau\lambda\varrho} = \mathcal{P}^{\tau\sigma\lambda\varrho} = \mathcal{P}^{\lambda\varrho\sigma\tau} = \mathcal{P}^{\lambda\varrho\tau\sigma} \tag{24}$$

exists. Also it is not difficult to verify that $\mathcal{P}^{\sigma\tau\lambda\varrho}$, when contracted in any pair of indices, vanishes in vacuum. However, the divergence of the tensor, in general, differs from zero. Being symmetrized in all its four indices the tensor $\mathcal{P}^{\sigma\tau\lambda\varrho}$ becomes

$$\mathcal{P}^{(\sigma\tau\lambda\varrho)} = \tfrac{1}{3}\,[R_{\mu\,.\,.\,\nu}^{\,.\tau\varrho\,.}R^{\mu\lambda\sigma\nu} + R_{\mu\,.\,.\,\nu}^{\,.\tau\lambda\,.}R^{\mu\varrho\sigma\nu} + R_{\mu\,.\,.\,\nu}^{\,.\sigma\varrho\,.}R^{\mu\tau\lambda\nu} + R_{\mu\,.\,.\,\nu}^{\,.\sigma\tau\,.}R^{\mu\varrho\lambda\nu} + R_{\mu\,.\,.\,\nu}^{\,.\lambda\varrho\,.}R^{\mu\sigma\tau\nu}$$
$$+ R_{\mu\,.\,.\,\nu}^{\,.\lambda\sigma\,.}R^{\mu\varrho\tau\nu} - \tfrac{1}{8}\,R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}(\delta^{\sigma\lambda}\delta^{\tau\varrho} + \delta^{\sigma\varrho}\delta^{\tau\lambda} + \delta^{\sigma\tau}\delta^{\lambda\varrho}) \tag{25}$$

which with the help of identities (20) can be written in the form

$$\mathcal{P}^{(\sigma\tau\lambda\varrho)} = R_{\mu\,.\,.\,\nu}^{\,.\tau\varrho\,.}R^{\mu\sigma\lambda\nu} + R_{\mu\,.\,.\,\nu}^{\,.\tau\lambda\,.}R^{\mu\sigma\varrho\nu} - \tfrac{1}{8}\,R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\delta^{\sigma\tau}\delta^{\lambda\varrho}, \tag{26}$$

i.e. the quantity $\mathcal{P}^{(\sigma\tau\lambda\varrho)}$ up to a constant factor (if the Gauss units are chosen equal to 8π) completely coincides with the Bel-Robinson superenergy tensor

$$T^{\sigma\tau\lambda\varrho} := \frac{1}{8\pi}\,(R^{\sigma\mu\lambda\nu}R^{\tau\,.\varrho\,.}_{\,.\mu\,.\nu} + R_{*}^{\sigma\mu\lambda\nu}R^{*\,\tau\,.\varrho\,.}_{\;\;.\mu\,.\nu})$$
$$\equiv \frac{1}{8\pi}\,(R^{\sigma\mu\lambda\nu}R^{\tau\,.\varrho\,.}_{\,.\mu\,.\nu} + R^{\sigma\mu\varrho\nu}R^{\tau\,.\lambda\,.}_{\,.\mu\,.\nu} - \tfrac{1}{8}\,R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}g^{\sigma\tau}g^{\lambda\varrho}). \tag{27}$$

So the Papapetrou pseudotensor posesses a number of properties distinguishing it among other pseudotensors: 1) It is a genuine tensor from the view point of the bimetric formalism. 2) It follows from the Noether theorem according to the rule (2), perfectly analogous to the definition of the ordinary symmetric energy-momentum tensor [5]. 3) It permits to give a constructive interpretation of the nonlinearity of the gravitational field as a specific feature of its sources [7,8]. 4) The symmetrization of the specific gravitatinal part of its expansion coefficient in the Riemann normal coordinates gives exactly the expression for the Bel-Robinson superenergy tensor (in vacuum) in contrast to the more complex similar connection in the case of the Einstein pseudotensor [3]. This, in its turn, could clarify the physical interpretation of the Bel-Robinson tensor.

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