EXACTLY SOLUBLE MODEL OF THE VOLUME CONSERVING PAIRING

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An exactly soluble model for the volume conserving pairing is formulated. It is shown that the non-zero equilibrium energy gap exists only in nuclei having irregularities in spectra near the Fermi surface.

Recently a variational method [1–3] of calculating the parameters of the pairing interaction based on the condition of volume conservation was proposed. As a result of the numerical calculations [2–4] it was found that the method can be successfully applied for spherical and deformed nuclei. However, these calculations include many detailed procedures which are important for obtaining a quantitative agreement with experimental data, though at the same time they make the method complicated and unclear. In such calculations it is rather difficult to extract the essence of the mechanism which is responsible for the appearance of the energy gap in the spectra of nuclei. It would be very useful in this connection to investigate the problem in the highest degree of simplification, i.e., to formulate and to solve the model problem. In the traditional approach to the pairing interaction there is a well-known, exactly soluble model described in Ref. [5]. One investigates in this approach the equidistant single particle spectra with a density of levels high enough to replace the summation over the energy levels by integration. We will formulate the problem within this model and solve it using our method. Maximum simplicity is obtained when the total energy of the nucleus is written in the oscillator bases

$$E = [\beta \sum_{\nu} \varepsilon_{\nu} 2V_{\nu}^2 - \Delta \sum_{\nu} U_{\nu} V_{\nu}] \hbar\omega(\Delta), \tag{1}$$

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where ω is the oscillator frequency, ε_ν — the single particle energies of levels in the $\hbar\omega$ units, Δ — the energy gap parameter in the same units, U_ν and V_ν — coefficients of the Bogolyubov transformation.

The origin of the numerical factor β requires special discussion. It is clear from the H.F. theory that in Eq. (1) the expression: $\mathscr{E} = \frac{1}{2}\sum_\nu (t_\nu+\varepsilon_\nu)2V_\nu^2$, where t_ν is the mean kinetic energy in the state ν, should be written instead $\mathscr{E} = \beta\sum_\nu \varepsilon_\nu 2V_\nu^2$. For the harmonic oscillator $t_\nu = \frac{1}{2}\varepsilon_\nu$ and, therefore, $\beta = \frac{3}{4}$. If the energy in the square well potential is counted from the bottom of the well then $t_\nu = \varepsilon_\nu$ and, consequently, $\beta = 1$. A realistic potential of the S.-W. type has an intermediate shape between these two extreme cases. Therefore, the value of the shape parameter β for the S.-W. potential must lie in the interval $\frac{3}{4} < \beta < 1$. The main idea of the method described in Refs [1–4] is that in varying Δ we keep the volume of the nucleus, Ω, constant. In Refs [1–4] the conservation of Ω has been ensured by keeping the mean square radius of the nucleus constant. For the harmonic oscillator

$$\langle r^2\rangle = \frac{\hbar}{m\omega}\sum_\nu (N_\nu+\tfrac{3}{2})2V_\nu^2.$$

Using the relation $\varepsilon_\nu = N_\nu+\frac{3}{2}$ we get

$$\langle r^2\rangle = \frac{\hbar}{m\omega}\sum_\nu \varepsilon_\nu 2V_\nu^2, \tag{2}$$

where m is the mass of a nucleon. The value of $\langle r^2\rangle$ does not depend on energy for the square well potential, hence,

$$\langle r^2\rangle = \text{const}. \tag{3}$$

Formulae (2) and (3) can be combined to

$$\langle r^2\rangle = \frac{\hbar}{m\omega}\sum_\nu (a+n\varepsilon_\nu)2V_\nu^2 = \frac{\hbar}{m\omega}\left(aN+\frac{n}{\beta}\mathscr{E}\right),$$

where N is the number of particles, and a and n — two additional parameters characterizing the shape of the potential. In the case of the harmonic oscillator $a = 0$, $n = 1$. For the square well $a \neq 0$, $n = 0$. The mean square radius $\langle r^2\rangle$ is constant if

$$\omega(\Delta) = \text{const}\sum_\nu (a+n\varepsilon_\nu)2V_\nu^2 = c\left(aN+\frac{n}{\beta}\mathscr{E}\right).$$

Let us now introduce the notation $P = \Delta\sum_\nu U_\nu V_\nu$. Then, the energy is given by

$$E = (\varepsilon-P)c\left(aN+\frac{n}{\beta}\mathscr{E}\right).$$

By differentiating the energy E with respect to Δ, we get the equation for the equilibrium energy gap

$$\left(aN+\frac{n}{\beta}\mathscr{E}\right)\frac{d}{d\Delta}(\mathscr{E}-P)+(\mathscr{E}-P)\frac{n}{\beta}\frac{d\mathscr{E}}{d\Delta}=0. \tag{4}$$

The quantity P can be neglected in comparison with E. Further simplification of the equation (4) can be obtained in two extreme cases: for the harmonic oscillator,

$$\frac{d}{d\Delta}(2\mathscr{E}-P)=0 \tag{5}$$

and for the square well,

$$\frac{d}{d\Delta}(\mathscr{E}-P)=0. \tag{6}$$

In order to retain simplicity for a more general case, we introduce a parameter $\alpha = 2\beta = \frac{3}{2}$ for the oscillator, and $\alpha = \beta = 1$ for the square well. For more realistic potentials the parameter α will take intermediate values. Therefore, the equation for the equilibrium energy gap can be written for a general case as

$$\frac{d}{d\Delta}(\mathscr{E}'-P)=0, \tag{7}$$

where

$$\mathscr{E}'=\alpha\sum_{\nu}\mathscr{E}_{\nu}2V_{\nu}^{2}.$$

Changing the summation in Eq. (1) into the integration we get

$$\mathscr{E}'-P=\varrho\frac{\alpha}{2}\Big[\mu^{2}-(\mu+\lambda)\left((\mu-\lambda)^{2}+\Delta^{2}\right)^{1/2}+\lambda(\lambda^{2}+\Delta^{2})^{1/2}$$
$$+\left(1-\frac{1}{\alpha}\right)\Delta^{2}\ln\frac{\mu-\lambda+((\mu-\lambda)^{2}+\Delta^{2})^{1/2}}{-\lambda+(\Delta^{2}+\lambda^{2})^{1/2}}\Big], \tag{8}$$

where ϱ is the density of levels and μ is the upper limit of the spectrum. From the requirement of conservation of the particle number we get

$$((\mu-\lambda)^{2}+\Delta^{2})^{1/2}=\mu-2\lambda_{0}+(\lambda^{2}+\Delta^{2})^{1/2}, \tag{9}$$

where $\lambda_0=\dfrac{N}{\varrho}$ is the chemical potential at $\Delta = 0$. By squaring both sides of Eq. (9) a very useful relation is obtained:

$$(\lambda^{2}+\Delta^{2})^{1/2}=[2\lambda_{0}(\mu-\lambda_{0})-\mu\lambda]\,(\mu-2\lambda_{0})^{-1},$$
$$\Delta^{2}=4\lambda_{0}(\mu-\lambda_{0})\,(\lambda_{0}-\lambda)\,(\mu-\lambda-\lambda_{0})\,(\mu-2\lambda_{0})^{-2}. \tag{10}$$

Further simplification of the equation (8) can be obtained, using relations (9) and (10).

$$\mathscr{E}' - P = \varrho \frac{\alpha}{2} \left\{ \frac{2\lambda_0}{\mu - 2\lambda_0} [2\lambda(\mu - \lambda_0) - \mu\lambda_0] + \left(1 - \frac{1}{\alpha}\right) \Delta^2 \ln \frac{\mu - \lambda - \lambda_0}{\lambda_0 - \lambda} \right\}. \tag{11}$$

Hence, the derivative takes the form

$$\frac{d}{d\Delta} (\mathscr{E}' - P) = \varrho \Delta \left[(\alpha - 1) \ln \frac{\mu - \lambda - \lambda_0}{\lambda_0 - \lambda} - (2\alpha - 1) \frac{\mu - 2\lambda_0}{\mu - 2\lambda} \right]. \tag{12}$$

It is clear from (7) and (12) that the energy E always has an extremum at $\Delta = 0$. In order to find the remaining extrema it is necessary to solve the equation:

$$\ln x = B \frac{x-1}{x+1} \tag{13}$$

where: $x = \dfrac{\mu - \lambda - \lambda_0}{\lambda_0 - \lambda}$ and $B = \dfrac{2\alpha - 1}{\alpha - 1}$. In terms of x

$$\lambda = [\lambda_0(x+1) - \mu] (x-1)^{-1},$$

$$\Delta^2 = bx(x-1)^{-2}, \quad b = 4\lambda_0(\mu - \lambda_0).$$

The functions $\ln x$ and $B \dfrac{x-1}{x+1}$ are shown schematically in Fig. 1. It is easy to see that there exist three crossing points, namely $0 < x_1 < 1$, $x_2 = 1$, and $x_3 > 1$. The solution

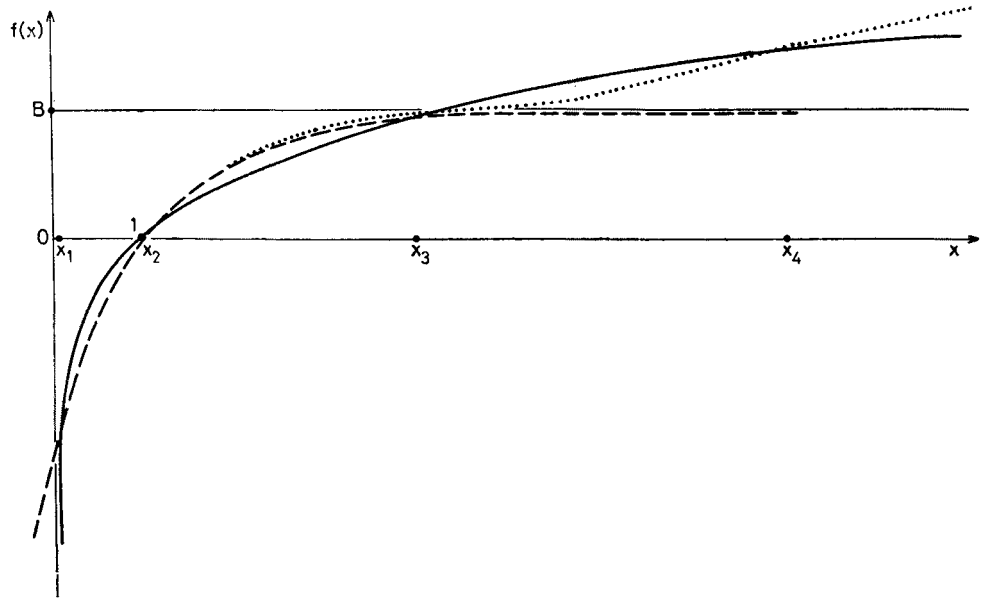

Fig. 1. Graphical solution of Eq. (13) and (18). Thick curve — $f(x) = \ln x$, broken curve — $f(x) = B \dfrac{x-1}{x+1}$,

dotted curve — $f(x) = \dfrac{x-1}{x+1} \left\{ B + \dfrac{3}{2} \dfrac{e}{b} \dfrac{(\mu + (\mu^2 + b(x-1))^{1/2})^2}{(\mu^2 + b(x-1))^{1/2}} \right\}$

x_1 corresponds to the negative value of $(\lambda^2+\Delta^2)^{1/2}$ and has no physical significance. It appears as a result of squaring the equation (9). The solution x_2 corresponds to $\Delta^2 = \infty$ and is therefore of no interest to us. What remains is x_3. It is easy to see from Fig. 1 that the derivative (12) is positive for $\Delta < \Delta(x_3)$ and negative for $\Delta > \Delta(x_3)$. Consequently, the extremum at $\Delta = \Delta(x_3)$ is the maximum. Fig. 2 shows the dependence of the energy E

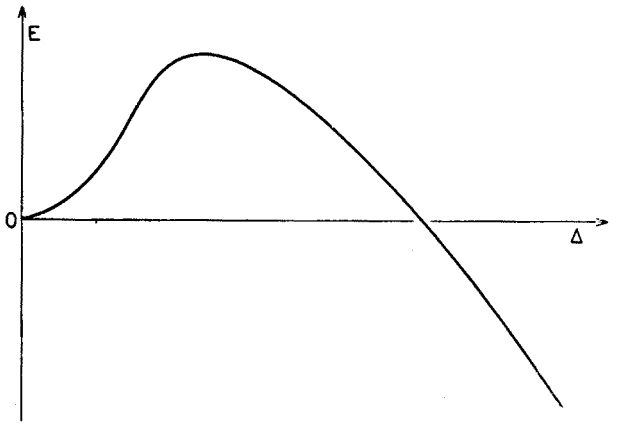

Fig. 2. Dependence of the total energy of the nucleus E on Δ in the case when the equilibrium energy gap is equal zero

on Δ for this last case. The first non-trivial result of the model can be formulated in the following way: If the nucleus has an equidistant spectrum and a smooth dependence of $\langle r_\nu^2 \rangle$ on ε_ν in the vicinity of the Fermi surface, then the equilibrium energy gap is equal to zero. For real nuclei this situation is very improbable. First of all, there exist fluctuations of the level density which create shells and subshells. Besides this, the *ls*-force and the deformation can push down some levels with small $\langle r_\nu^2 \rangle$. Thus, the dependence of $\langle r_\nu^2 \rangle$ on ε_ν near the Fermi surface can be quite irregular. Both factors may lead to a non-zero equilibrium energy gap. In the case of the shell effect this was shown in Ref. [1]. We examine here the second case. Let us investigate the dependence of $\langle r_\nu^2 \rangle$ on ε_ν shown in Fig. 3.

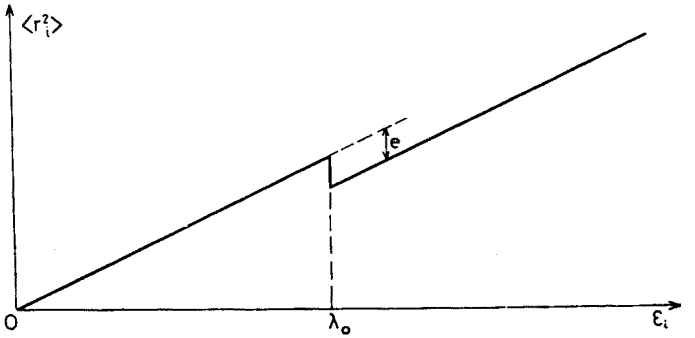

Fig. 3. The imitation of the irregularity of $\langle r_i^2 \rangle$ as a function of ε_i

It imitates irregularity in the $\langle r_\nu^2 \rangle$ distribution by introducing a single jump at the Fermi surface

$$\langle r_\nu^2 \rangle = \begin{cases} a + n\varepsilon_\nu, & \varepsilon_\nu \leqslant \lambda_0 \\ a + n(\varepsilon_\nu - e), & \varepsilon_\nu > \lambda_0. \end{cases}$$

Instead of Eq. (4) we now have

$$\omega(\Delta) = c\left[aN + \frac{n}{\beta}(\mathscr{E} - \beta e\psi)\right],$$

where $\psi = \sum_\nu 2V_\nu^2$. Therefore

$$E = (\mathscr{E} - P)c\left[aN + \frac{n}{\beta}(\mathscr{E} - \beta e\psi)\right],$$

and varying E on Δ we get

$$\left[aN + \frac{n}{\beta}(\mathscr{E} - \beta e\psi)\right]\frac{d}{d\Delta}(\mathscr{E} - P) + (\mathscr{E} - P)\frac{n}{\beta}\left(\frac{d\mathscr{E}}{d\Delta} - \beta e\frac{d\psi}{d\Delta}\right) = 0. \tag{14}$$

Neglecting here P and $\beta e\psi$ in comparison with $\mathscr{E}$, we get the equation for the equilibrium energy gap which simplifies considerably in the case of the harmonic oscillator

$$\frac{d}{d\Delta}(2\mathscr{E} - P - \beta e\psi) = 0 \tag{15}$$

and in the case of the square well,

$$\frac{d}{d\Delta}(\mathscr{E} - P) = 0. \tag{16}$$

Introducing once more the shape parameter α one can replace Eq. (14) by a simple equation containing (15) and (16) as the extreme cases

$$\frac{d}{d\Delta}(\mathscr{E}' - P - \tfrac{3}{2}(\alpha - 1)e\psi) = 0. \tag{17}$$

This equation differs from (7) by the term $\frac{3}{2}e(\alpha - 1)\frac{d\psi}{\Delta d}$. Carrying out the calculation similarly as before, we see that the solution $\Delta = 0$ always exists. The extrema at $\Delta \neq 0$ are given by the equation

$$\ln x = (x-1)(x+1)^{-1}\left[B + \tfrac{3}{2}\frac{e}{b}\frac{(\mu - (\mu^2 + b(x-1))^{1/2})^2}{(\mu^2 + b(x-1))^{1/2}}\right]. \tag{18}$$

The right-hand side of Eq. (18) is schematically presented in Fig. 1 (by the dotted line). If e is small (as it should be) and if the remaining parameters have reasonable values, then $\frac{e}{b} \ll B$. Hence, it is clear that for small values of x the whole picture practically does

not change and the point x_3 is not shifted. With increasing x the second component in the square bracket becomes important. At large values of x this component increases as $x^{1/2}$, therefore the right-hand side of Eq. (18) becomes larger than $\ln x$. Thus, it is clear that there exists one more intersection x_4. It is easy to see that the extremum at Δ_4 corresponds to the minimum of E. The dependence of E on Δ for this case is shown in Fig. 4. One should also mention the case in which the value of e is so large that for any $x > 1$

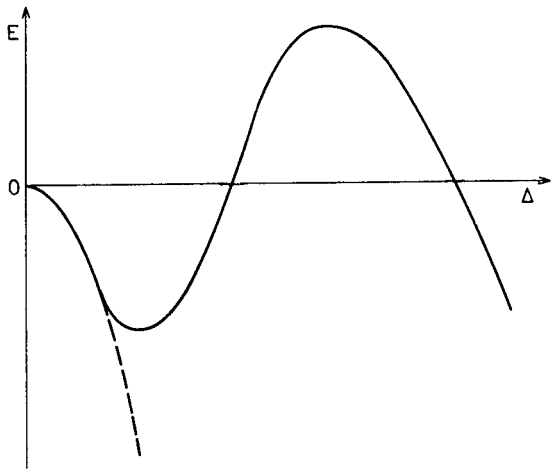

Fig. 4. Dependence of the total energy of the nucleus E on Δ in the case of a non-zero equilibrium energy gap (full line). The broken line represents the energy $E(\Delta)$ for the unphysical values of the model parameters

the right-hand side of Eq. (18) is larger than $\ln x$. This means that the solution at $x > 1$ does not exist. The curve $E(\Delta)$ is then given by a broken line in Fig. 4. However, this case does not occur in real nuclei. If an anomalous decrease in $\langle r_\nu^2 \rangle$ at the Fermi surface really occurs, it is always compensated by an increase in $\langle r_\nu^2 \rangle$ in the neighbouring part of the spectra. In order to prove the physical significance of the model let us estimate its results for reasonable values of the parameters. For the harmonic oscillator the minimal value of e should be of the order $\hbar\omega$. For $A \sim 150$ nuclei $\lambda_0 \simeq 5\,\hbar\omega$ and therefore $e \simeq 0.2\,\lambda_0$. For $\mu = 3\lambda_0$ we get: $x_3 \simeq 155$, $\Delta_3 \simeq 0.23\,\lambda_0$, $\lambda_3 \simeq 0.994\,\lambda_0$, $x_4 \simeq 670$, $\Delta_4 \simeq 0.11\,\lambda_0$, $\lambda_4 \simeq 0.998\,\lambda_0$.

Let us investigate an intermediate case between the harmonic oscillator and the square well potential: $\alpha = 1.3$. The mean square radius $\langle r_\nu^2 \rangle$ now changes much more slowly and therefore the minimal value of e should be chosen small. For $e = 0.1\,\lambda_0$ one gets: $x_3 \simeq 890$, $\Delta_3 \simeq 0.095\,\lambda_0$, $\lambda_3 \simeq 0.9989\,\lambda_0$, $x_4 \simeq 2120$, $\Delta_4 \simeq 0.061\,\lambda_0$, $\lambda_4 \simeq 0.9995\,\lambda_0$.

One may therefore conclude that for reasonable values of the parameters the model gives reasonable results and can be used as a simple qualitative way of understanding what happens in more complicated, realistic situations.

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