A GEOMETRIC MEANING OF MANDELSTAM'S PATH DEPENDENT QUANTITIES

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By using the theory of fiber bundle, the path dependent quantities used by Mandelstam to derive the Feynman rules for the Yang-Mills and gravitational fields are shown to be just the quantities taken on a horizontal path of a fiber space. This geometric meaning enables us to construct new path dependent quantities remaining unchanged under gauge transformations.

1. Introduction

In references [1, 2] Mandelstam has developed a beautiful general gauge independent formalism for quantization of Yang-Mills and gravitational fields. The most important step in this formalism is the construction of the so called path dependent quantities. When constructing the path dependent quantities in the Yang-Mills fields [1] and gravitational fields cases [2], Mandelstam has used respectively the matrix constructed by Białynicki-Birula [3] and the tetrad technique. In this paper we shall use the theory of fiber bundle, which is very adequate for the description of gauge fields, to show that the methods of constructing path dependent quantities used in [1, 2] despite the apparent difference have a common geometric meaning. This geometric meaning enables us to construct, in the case of gravitational fields, other path dependent quantities (besides Mandelstam's ones) which remain unchanged under gauge transformations.

2. Fiber space

The structure of fiber space on E is defined by the set $E(B, F, G, p, \Phi)$ where: B is the base, F — the fiber, G — the structural group acting on the fiber F, p — the projection: $E \to B$, Φ — a family of homeomorphisms φ_α: $U_\alpha \times F \to p^{-1}(U_\alpha)$ (U_α is an open set of B) satisfying well known conditions [4].

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The case of interest is the one when F and G are the same manifold. In this case the fiber space is called the principal fiber space. We limit ourselves from now on to the principal fiber space.

Let us denote the vectorial space tangent to E and the subspace tangent to the fiber at z respectively by Θ_z and V_z; a vector is called vertical if it belongs to V_z and horizontal if it belongs to $H_z = \Theta_z \backslash V_z$; a path of E is called horizontal if its tangents are horizontal; the horizontal path remains horizontal when the fiber is subjected to the transformations $\gamma \in G$. The infinitesimal connection is then defined by a differential form of degree 1 (1-form) ω with values on the Lie algebra L of G, i.e. if $\tau \in \Theta_z$, $\omega(\tau)$ is the element of L, generated by the vector $V\tau \in V_z$, where $V\tau$ is the vertical component of τ.

The connection ω may be written in the form

$$\omega^A_B = [\omega^\Phi_\mu L_\Phi]^A_B \otimes \theta^\mu, \tag{2.1}$$

where $[L_\Phi]^A_B$ — matrix elements of generator L_Φ of the algebra L, θ^μ — corepère on the base space V_n (manifold of n dimensions).

The connection has two important properties:

1) if τ is horizontal then $\omega(\tau) = 0$, (2.2)

2) $\omega(\tau\gamma) = [\text{adj } \gamma^{-1}]\,\omega(\tau)$, $\gamma \in G$. (2.3)

Cover V_n by an open set $\{U_\alpha\}$. For $x \in U_\alpha \cap U_\beta$ we can find an element $\gamma_{\alpha\beta}(x) \in G$ so that

$$z_\beta(x) = z_\alpha(x)\gamma_{\alpha\beta}(x), \tag{2.4}$$

where z_α is the local section of E over U_α. The local section of E is the mapping μ, satisfying the condition $p\mu(x) = x$.

From (2.4) we obtain

$$dz_\beta = dz_\alpha\gamma_{\alpha\beta} + z_\beta\gamma^{-1}_{\alpha\beta}d\gamma_{\alpha\beta}.$$

Setting $\omega_\alpha(dx) = \omega(dz_\alpha)$, and using (2.3), we get

$$\omega_\beta = [\text{adj } \gamma^{-1}_{\alpha\beta}]\omega_\alpha + \gamma^{-1}_{\alpha\beta}d\gamma_{\alpha\beta}. \tag{2.5}$$

In particular when z lies on a horizontal path, using (2.2) we have instead of (2.5) the following equation

$$\omega_{H\beta} = \gamma^{-1}_{H\alpha\beta}d\gamma_{H\alpha\beta}. \tag{2.6}$$

The connection ω is related to the 2-form curvature Ω of type adj g^{-1} and the 2-form torsion Σ, respectively, by the structure equations

$$\Omega = d\omega + \omega \wedge \omega, \tag{2.7}$$

$$\Sigma = d\theta + \omega \wedge \theta, \tag{2.8}$$

where θ — arbitrary corepère.

It is easy to show that Ω and Σ transform as follows:

$$\Omega \to \gamma\Omega\gamma^{-1}, \tag{2.9}$$

$$\Sigma \to \gamma\Sigma, \tag{2.10}$$

when z is subjected to the transformations (2.4).

3. *Yang-Mills fields*

Let us consider now the Yang-Mills fields. The fiber space is then

$$E(V_4) = \bigcup_{x \in V_4} \psi(x), \tag{3.1}$$

where $\psi(x) = \{x, D\}$ is the set of internal repères over x obtained by operating matrix D belonging to a SU(2) representation on a chosen system of basic vectors. The projection: $p:E \to V_4$ maps every internal repère over x into x. The base space V_4 is the Minkowski space-time. The fiber F, to which $\psi(x)$ is homeomorphic, may be identified with the structural group SU(2).

The Yang-Mills fields are just the connection ω of the fiber space (3.1). We deduce now the transformation law for the Yang-Mills fields when the fiber $\psi(x)$ suffers the transformation of the type (2.4)

$$\psi'(x) = \psi(x)S^{-1}(x), \qquad S \in \mathrm{SU}(2). \tag{3.2}$$

In view of (2.5) we have

$$A(dx) = \mathrm{adj}\, S^{-1}A'(dx) + S^{-1}dS. \tag{3.3}$$

Using natural corepère dx^μ we have from (3.3) the transformation law for the Yang--Mills fields

$$A_\mu = S^{-1}A'_\mu S + S^{-1}\partial_\mu S. \tag{3.4}$$

Equation (2.6) in the case of Yang-Mills fields is

$$\partial_\mu S_H^{-1} = -A_\mu S_H^{-1}. \tag{3.5}$$

The solution of (3.5) satisfying the boundary condition $S_H|_{-\infty} = I$ is given by

$$S_H^{-1} = Le^{-\int_{-\infty}^{x} A_\mu dy^\mu}, \tag{3.6}$$

where L indicates that matrices A are to be ordered from the beginning to the end of the path when expanding the exponential. We can rewrite (3.6) in the following form

$$S_H^{-1} = 1 - \int_{-\infty}^{x} A_\mu(x_1)dx_1^\mu + \int_{-\infty}^{x} dx_1^\mu \int_{-\infty}^{x_1} dx_2^\nu A_\mu(x_1)A_\nu(x_2) - \ldots$$

In view of (2.4) the horizontal path is

$$\psi(x)S_H^{-1}.$$

The matrix S_H^{-1} coincides with the matrix constructed by Białynicki-Birula [3] and $\psi(x)S_H^{-1}$ is a path dependent quantity according to Mandelstam's terminology [1, 2].

Thus the Białynicki-Birula matrix and Mandelstam path dependent quantities acquire now a new geometric meaning. The matrix S_H when operating on an arbitrary path transforms it into a horizontal one, in other words the matrix S_H when operating on an arbitrary corepère transforms it into a corepère in which the connection vanishes locally.

Without any algebra we can show that S_H transforms as

$$S_H \to S_H S^{-1}$$

when ψ suffers the transformation (3.2).

In fact as a horizontal path remains horizontal under the transformations of G we have therefore

$$\psi S_H^{-1} \to \psi S^{-1} S S_H^{-1} = \psi S_H^{-1}.$$

This property may be shown directly by using (3.5) (see [3]).

Consider now the curvature of the fiber space (3.1). Using, as before, natural corepère, we obtain from (2.7)

$$\partial_\mu A^i_{j\nu} - \partial_\nu A^i_{j\mu} + A^i_{k\mu} A^k_{j\nu} - A^i_{k\nu} A^k_{j\mu} = \tfrac{1}{2} F^i_{j,\mu\nu}, \tag{3.7}$$

where latin indices i, j, k refer to SU(2) transformations. In (3.7) the curvature tensor $F^i_{j,\mu\nu}$ is connected with the curvature form Ω^i_j by the relation

$$\Omega^i_j = \tfrac{1}{2} F^i_{j,\mu\nu} dx^\mu \wedge dx^\nu. \tag{3.8}$$

Using the transformation law (2.9) one can show that the path dependent quantities $S_H \Omega S_H^{-1}$ remain unchanged under the transformations (3.2)

$$S_H \Omega S_H^{-1} \to S_H S^{-1} S \Omega S^{-1} S S_H^{-1} = S_H \Omega S_H^{-1}. \tag{3.9}$$

The quantities (3.9) are just the path dependent functions used by Mandelstam for quantization of the Yang-Mills fields [1]. Geometrically (3.9) means that on a horizontal path the curvature does not suffer any changes under the transformations (3.2).

4. Gravitational field

We shall consider different fiber spaces

$$E(R_4) = \bigcup_{x \in R_4} M(x),$$

where $M(x)$ is a set of repères with origin at x, R_4 — 4-dimensional Riemann manifold. For M we shall use natural repères or tetrads. The Riemann connection [4] Γ in every case may be found from the equations

$$\nabla g_{AB} = dg_{AB} - \Gamma^C_A g_{CB} - \Gamma^C_B g_{AC} = 0, \tag{4.1}$$

$$\Sigma = d\theta + \Gamma \wedge \theta = 0, \tag{4.2}$$

where g_{AB} is the metric tensor.

a) Using natural corepère dx^μ from (4.1) and (4.2) one can express the connection $\Gamma^\alpha_{\beta\gamma}$ and the curvature $R^\alpha_{\beta,\gamma\delta}$ in terms of $g_{\mu\nu}$ by wellknown relations. When the coordinates suffer the transformation

$$dx^{\mu'} = C^{\mu'}_\nu dx^\nu \tag{4.3}$$

one can deduce from (2.5) the transformation law for Γ and R

$$\Gamma^{\alpha'}_{\beta'\gamma'} = C^{\alpha'}_\varrho \Gamma^\varrho_{\sigma\lambda}[C^{-1}]^\sigma_{\beta'}[C^{-1}]^\lambda_{\gamma'} + C^{\alpha'}_\mu \partial_{\gamma'}[C^{-1}]^\mu_{\beta'},$$

$$R^{\alpha'}_{\beta',\gamma'\delta'} = C^{\alpha'}_\sigma R^\sigma_{\varrho,\mu\nu}[C^{-1}]^\varrho_{\beta'}[C^{-1}]^\mu_{\gamma'}[C^{-1}]^\nu_{\delta'}.$$

b) Let us consider now the corepère formed by

$$dx^a = h^a_\mu dx^\mu. \tag{4.4}$$

The structure group is now the Lorentz group. In (4.4) the latin indices refer to the local Lorentz transformations and h^a_μ, called tetrad, is a 4-vector with respect to the transformations of curved coordinates. Under the local Lorentz transformations

$$dx^{a'} = L^{a'}_b dx^b \tag{4.5}$$

the tetrads transform as follows:

$$h^{a'}_\mu = L^{a'}_b h^b_\mu. \tag{4.6}$$

From (4.1) and (4.2) we have

$$\Gamma^c_a \delta_{cb} + \Gamma^c_b \delta_{ca} = 0, \tag{4.7}$$

$$\partial_{[\nu} h^c_{\mu]} = \Gamma^a_{c[\mu} h^c_{\nu]}, \tag{4.8}$$

where δ_{ab} is the Minkowski metric. Equations (4.7) and (4.8) admit as solution the following connection

$$\Gamma^a_{b\lambda} = h^a_\mu h^\nu_b \Gamma^\mu_{\nu\lambda} + h^a_\mu \partial_\lambda h^\mu_b = h^{\mu a}\partial_{[\lambda} h_{\mu]b} + h^\mu_b \partial_{[\mu} h^a_{\lambda]} + h^{\mu a} h^\nu_b h_{\lambda c} \partial_{[\nu} h^c_{\mu]}, \tag{4.9}$$

where h^μ_a is defined by $h^a_\mu h^\mu_b = \delta^a_b$ and latin indices are raised and lowered by δ_{ab}. Using the structure equation (2.7) it is straightforward to show that

$$R^a_{b,\mu\nu} = h^a_\varrho h^\sigma_b R^\varrho_{\sigma,\mu\nu}. \tag{4.10}$$

Under the transformations (4.6) the connection and the curvature transform as follows:

$$\Gamma^{a'}_{b'\mu} = L^{a'}_c \Gamma^c_{d\mu}[L^{-1}]^d_{b'} + L^{a'}_c \partial_\mu [L^{-1}]^c_{b'}, \tag{4.11}$$

$$R^{a'}_{b',\mu\nu} = L^{a'}_c [L^{-1}]^d_{b'} R^c_{d,\mu\nu}. \tag{4.12}$$

The matrix V_H, necessary for transforming an arbitrary path into a horizontal one, is determined by an equation, analogous to (3.6)

$$dV_H^{-1} = -\Gamma V_H^{-1}, \tag{4.13}$$

where V_H may have two greek indices or one greek and one latin index depending on what corepère we want to obtain

$$dx^{\mu'} = V^{\mu'}_{H\nu} dx^{\nu}, \tag{4.14}$$

or

$$dx^{a} = V^{a}_{H\mu} dx^{\mu}. \tag{4.15}$$

In the latter case $V^{a}_{H\mu}$ form a tetrad and (4.13) can be expressed in the form of equation (4.5) of Ref. [2]:

$$\partial_a [V_H^{-1}]^{\mu}_{b} = -\Gamma^{\mu}_{\nu\lambda} [V_H^{-1}]^{\nu}_{b} [V_H^{-1}]^{\lambda}_{a}. \tag{4.16}$$

According to (2.4) the horizontal path is MV_H^{-1}. As the horizontal path remains horizontal under the transformations (4.3), it is easy to show that V_H transforms as follows:

$$V_H \to V_H C^{-1}. \tag{4.17}$$

Let us turn now to the path dependent curvature. Using (2.9) and (4.17) it is easy to construct the following quantities, which remain unchanged under the transformations (4.3):

$$[V_H^{-1}]^{\mu}_{c} [V_H^{-1}]^{\nu}_{d} V^{a}_{H\sigma} [V_H^{-1}]^{\varrho}_{b} R^{\sigma}_{\varrho,\mu\nu}, \tag{4.18}$$

$$[V_H^{-1}]^{\mu}_{\lambda'} [V_H^{-1}]^{\nu}_{\tau'} V^{\omega'}_{H\sigma} [V_H^{-1}]^{\varrho}_{\varphi'} R^{\sigma}_{\varrho,\mu\nu}. \tag{4.19}$$

The quantities (4.18) have been used by Mandelstam [2] to derive the Feynman rules for the gravitational field (cf. (4.15) in [2]). One may as well start from (4.19) which can be used as new path dependent quantities for the quantization of gravitational field in the framework of Mandelstam's coordinate independent formalism.

Thus the path dependent curvature in both cases (Yang-Mills and gravitational fields) is shown to be just the curvature taken on a horizontal path of a fiber space. The theory of fiber bundle furnishes a unique interpretation of Mandelstam's path dependent quantities.

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