ON THE QUANTIZATON OF YANG-MILLS AND GRAVITATIONAL FIELDS

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An approach to the quantization problem of gauge fields, based on the external source method combined with the quantum dynamical equation is presented.

1. Introduction

After the discovery of the Feynman rules for Yang-Mills and gravitational fields by Feynman himself [1] the quantization problem of these fields have been solved by many authors using quite different techniques, adequate to different points of departure [2–7].

The present paper is aimed at clarifying the connection between the methods of above references by presenting an approach to the quantization problem of gauge fields, based on the external source method combined with the DeWitt [8] quantum dynamical equation.

2. Yang-Mills field

We shall use indices from the beginning of the Greek alphabet to denote components in isotopic spaces and indices from the middle of the Greek alphabet to denote components in the space-time.

As in the electrodynamics we consider the following Lagrangian

$$L = -\tfrac{1}{4} F^{\alpha}_{\mu\nu}(x)F^{\alpha}_{\mu\nu}(x)+iA^{\alpha}_{\mu}(x)J^{\alpha}_{\mu}(x). \tag{2.1}$$

Here

$$F^{\alpha}_{\mu\nu}(x) = \partial_{\mu}A^{\alpha}_{\nu}(x)-\partial_{\nu}A^{\alpha}_{\mu}(x)+fc^{\alpha\gamma\beta}A^{\gamma}_{\mu}(x)A^{\beta}_{\nu}(x),$$

A^{α}_{μ} — Yang-Mills field, $C^{\alpha\gamma\beta}$ — group structure constants.

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From (2.1) it follows that

$$\frac{\delta S}{\delta A_\mu^\alpha(x)} = -iJ_\mu^\alpha(x). \tag{2.2}$$

In equation (2.2) S is the action of Yang-Mills field and is equal to

$$S = -\int F_{\mu\nu}^\alpha(x) F_{\mu\nu}^\alpha(x) dx \equiv \int L_{YM}(x) dx.$$

Because of the invariance of L_{YM} under arbitrary gauge transformations

$$\begin{aligned} A_\mu^\alpha(x) &\to A_\mu^\alpha(x) - f^{-1}\nabla_\mu^{\alpha\beta}(x) u^\beta(x), \\ \nabla_\mu^{\alpha\beta}(x) &= \delta^{\alpha\beta}\partial_\mu + f c^{\alpha\gamma\beta} A_\mu^\gamma(x), \end{aligned} \tag{2.3}$$

the corresponding variation of S must vanish

$$\delta S = \int \frac{\delta S}{\delta A_\mu^\alpha(x)} \delta A_\mu^\alpha(x) dx = f^{-1} \int u^\beta(x) \nabla_\mu^{\beta\alpha}(x) \frac{\delta S}{\delta A_\mu^\alpha(x)} dx = 0.$$

The arbitrariness of functions $u^\beta(x)$ yields

$$\nabla_\mu^{\beta\alpha}(x) \frac{\delta S}{\delta A_\mu^\alpha(x)} = 0. \tag{2.4}$$

The identity (2.4) and the equation (2.2) impose the following condition on the external source

$$\nabla_\mu^{\beta\alpha}(x) J_\mu^\alpha(x) = 0. \tag{2.5}$$

Thus the external source $J_\mu^\alpha(x)$ must depend on the A_μ field; this condition cannot be satisfied by arbitrary external source; this difficulty of external source method has been pointed out by DeWitt. To avoid this difficulty DeWitt has used the free background field in his wellknown paper [2].

Here we propose a new approach which allows us to overcome the mentioned difficulty on one hand, and on the other to clarify the connection between the approaches of other authors. For this purpose we introduce a new field $K^\alpha(x)$ which in the case of Lorentz gauge

$$\partial_\mu A_\mu^\alpha(x) = 0$$

enters the Lagrangian under the following form

$$\mathscr{L} = -\tfrac{1}{4} F_{\mu\nu}^\alpha(x) F_{\mu\nu}^\alpha(x) + iA_\mu^\alpha(x) J_\mu^\alpha(x) + i\partial_\mu A_\mu^\alpha(x) K^\alpha(x). \tag{2.6}$$

The scalar field $K^\alpha(x)$ will generate the ghost particle as it will be shown in the following. In some sense the scalar field plays the role of the Lagrange multiplier used in [6], but the approach presented here differs from the approach of [6]. We shall construct $K^\alpha(x)$ so that the condition (2.5) will be satisfied by any arbitrary external source $J_\mu^\alpha(x)$. We shall limit ourselves from now on to the Lorentz gauge case, but it is easy to deal analogously with other cases.

From the Lagrangian (2.6) we obtain

$$\frac{\delta S}{\delta A^\alpha_\mu(x)} + i\left(J^\alpha_\mu(x) + \int \frac{\partial}{\partial y^\nu}\frac{\delta A^\beta_\nu(y)}{\delta A^\alpha_\mu(x)} K^\beta(y)dy\right) = 0. \tag{2.7}$$

Our next step is to convert this equation into the quantum dynamical equation [8] that has the form

$$T(Y) = 0,$$

where T denotes chronological product and Y is obtained from the l.h.s. of (2.7) by replacing all classical fields $A^\alpha_\mu(x)$ by quantized ones $A^\alpha_\mu(x)$. Taking the vacuum mean value of this equation we obtain

$$\langle 0|T(Y)|0\rangle = 0.$$

This equation can be shown to be equivalent to the following functional differential equation for the vacuum-to-vacuum amplitude Z

$$\left(\frac{\delta S}{\delta A^\alpha_\mu(x)} + iJ^\alpha_\mu(x) + i\!:\!\int \frac{\partial}{\partial y^\nu}\frac{\delta A^\beta_\nu(y)}{\delta A^\alpha_\mu(x)} K^\beta(y)dy\!:\right)_{A^\alpha_\mu \to \frac{\delta}{\delta J^\alpha_\mu}} Z = 0. \tag{2.8}$$

In (2.8) the symbol :...: indicates that in the expression represented by three points, the external source J^α_μ and the field A^α_μ are to be ordered so that J^α_μ appears before A^α_μ. The ordering of J^α_μ and A^α_μ is necessary because of the commutation relation $[\delta/\delta J^\alpha_\mu(x), J^\beta_\nu(y)] = \delta_{\alpha\beta}\delta_{\mu\nu}\delta(x-y)$.

We take for Z the form suggested by DeWitt in [8]

$$Z = \int_\Sigma F[A] e^{i\int A^\alpha_\mu J^\alpha_\mu dx} \prod_{x,\mu,\alpha} dA^\alpha_\mu(x), \tag{2.9}$$

where Σ is the hypersurface $\partial_\mu A_\mu = 0$.

By acting $\nabla^{\beta\alpha}_\mu(x)$ on (2.8) and using (2.4) we have as a result the important equation serving to define $K^\alpha(x)$

$$\nabla^{\beta\alpha}_\mu(x) J^\alpha_\mu(x)|_{A^\alpha_\mu \to \frac{\delta}{\delta J^\alpha_\mu}} Z = -\nabla^{\beta\alpha}_\mu(x)\!:\!\partial_\mu K^\alpha(x)\!:\!|_{A^\alpha_\mu \to \frac{\delta}{\delta J^\alpha_\mu}} Z. \tag{2.10}$$

Note that (2.10) is analogous to equation (4.39) in Mandelstam's paper [4] for the path independent Green functions.

One can verify that the solution of (2.10) has the following form (for details of proof see Appendix B in [4]):

$$:\partial_\mu K^\alpha(x): = -:\partial_\mu \int \nabla^{\gamma\delta}_\nu(y) J^\delta_\nu(y) G^{\alpha\gamma}(x, y)dy: - fc^{\alpha\beta\gamma}\partial_\mu G^{\beta\gamma}(x, y)|_{y=x}, \tag{2.11}$$

where $G^{\alpha\beta}(x, y)$ is the Green function defined by

$$\nabla^{\alpha\gamma}_\mu(x)\partial_\mu G^{\gamma\delta}(x, y) = \delta^{\alpha\delta}\delta(x-y). \tag{2.12}$$

Inserting (2.11) into (2.8) we obtain the functional equation for Z

$$\left(\frac{\delta S}{\delta A^\alpha_\mu} + iJ^\alpha_\mu - i\!:\!\partial_\mu \int \nabla^{\gamma\delta}_\nu(y) J^\delta_\nu(y) G^{\alpha\gamma}(x, y)dy\!:\! - ifc^{\alpha\beta\gamma}\partial_\mu G^{\beta\gamma}(x, y)|_{y=x}\right)_{A^\alpha_\mu \to \delta/\delta J^\alpha_\mu} Z = 0.$$

The expression in the parentheses in the l.h.s. of the preceding equation is by itself an equation for A^α_μ when one sets it equal to zero and impose on A^α_μ the gauge condition $\partial^\mu A^\alpha_\mu = 0$. Thus the third term of this expression may be omitted as it is a pure divergence. As a result we have for Z

$$\left(\frac{\delta S}{\delta A^\alpha_\mu(x)} + iJ^\alpha_\mu(x) - ifc^{\alpha\beta\gamma}\partial_\mu G^{\beta\gamma}(x, y)|_{x=y}\right)_{A^\alpha{}_\mu \to \delta/\delta J^\alpha{}_\mu} Z = 0. \tag{2.13}$$

Substituting (2.9) for Z in (2.13) we obtain after an integration by parts the following equation for determining F on Σ:

$$(\delta S/\delta A^\alpha_\mu - ifc^{\alpha\beta\gamma}\partial_\mu G^{\beta\gamma}(x, y)|_{x=y} - \delta/\delta A^\alpha_\mu)F = 0. \tag{2.14}$$

It is straightforward to verify the identity

$$\frac{\delta}{\delta A^\alpha_\mu} \operatorname{Sp} \ln (\delta_{\alpha\gamma} + fc^{\alpha\beta\gamma} A^\beta_\nu \partial_\nu \Box^{-1}) = -fc^{\alpha\beta\gamma}\partial_\mu G^{\beta\gamma}(x, y)|_{y=x} \tag{2.15}$$

by expanding the left and right sides as a perturbation series in f [7].

Inserting (2.15) into (2.14) we can easily integrate (2.14) and using (2.9) we obtain the result

$$Z = \int_\Sigma \exp [S + i \operatorname{Sp} \ln (\delta_{\alpha\gamma} + fc^{\alpha\beta\gamma} A^\beta_\mu \partial_\mu \Box^{-1}) + i \int J^\alpha_\mu A^\alpha_\mu dx] \prod_{x,\mu,\alpha} dA^\alpha_\mu(x). \tag{2.16}$$

The multiplier

$$\exp (i \operatorname{Sp} \ln (\delta_{\alpha\gamma} + fc^{\alpha\beta\gamma} A^\beta_\mu \partial_\mu \Box^{-1}))$$

is the wellknown Faddeev-Popov measure Δ_L, describing the interaction of Yang--Mills fields A^α_μ with ghost scalar field, which appears only in closed loops of Feynman diagrams.

3. *Gravitational field*

We shall take for the gravitational field the Lagrangian

$$L = \tfrac{1}{2}\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu} + i\sqrt{-g}\, g^{\mu\nu} J_{\mu\nu} \equiv \tfrac{1}{2}\mathfrak{g}^{\mu\nu} R_{\mu\nu} + i\mathfrak{g}^{\mu\nu} J_{\mu\nu}, \tag{3.1}$$

where $g = \det g^{\mu\nu}$, $R_{\mu\nu}$ — curvature tensor, $J_{\mu\nu}$ — external source.

From (2.1) we obtain

$$\frac{\delta S}{\delta \mathfrak{g}^{\mu\nu}} = -iJ_{\mu\nu}, \quad S = \tfrac{1}{2}\int \mathfrak{g}^{\mu\nu} R_{\mu\nu} dx. \tag{3.2}$$

The infinitesimal gauge transformation of $\mathfrak{g}^{\mu\nu}$ is of the form

$$\mathfrak{g}^{\mu\nu} \to \mathfrak{g}^{\mu\nu} - u^\varrho \partial_\varrho \mathfrak{g}^{\mu\nu} + \mathfrak{g}^{\mu\varrho}\partial_\varrho u^\nu + \mathfrak{g}^{\nu\varrho}\partial_\varrho u^\mu - \mathfrak{g}^{\mu\nu}\partial_\varrho u^\varrho. \tag{3.3}$$

The invariance of S under (3.3) leads to the following identity analogous to (2.4)

$$\nabla_{\lambda}^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} = 0, \tag{3.4}$$

where $\nabla_{\lambda}^{\mu\nu}(x) = g^{\mu\nu}\partial_{\lambda} - 2\delta_{\lambda}^{\mu}\partial_{\varrho}g^{\varrho\nu} - 2\delta_{\lambda}^{\mu}g^{\nu\varrho}\partial_{\varrho}$.

Thus (3.4) and (3.2) impose

$$\nabla_{\lambda}^{\mu\nu} J_{\mu\nu} = 0. \tag{3.5}$$

This condition cannot be satisfied by an arbitrary external source. Thus instead of (3.1) we have to take the following Lagrangian:

$$\mathscr{L} = \tfrac{1}{2}\, g^{\mu\nu}(x)R_{\mu\nu}(x) + ig^{\mu\nu}(x)J_{\mu\nu}(x) + i\partial_{\mu}g^{\mu\nu}(x)K_{\nu}(x). \tag{3.6}$$

The Lagrangian (3.6) is written for the case of harmonic gauge condition $\partial_{\mu}g^{\mu\nu} = 0$.

From the Lagrangian (3.6) we obtain

$$\frac{\delta S}{\delta g^{\mu\nu}(x)} + i\left(J_{\mu\nu}(x) + \int \frac{\partial}{\partial y^{\lambda}} \frac{\delta g^{\lambda\varrho}(y)}{\delta g^{\mu\nu}(x)} K_{\varrho}(y)dy\right) = 0. \tag{3.7}$$

The functional differential equation for Z in the case of gravitational field is

$$\left(\frac{\delta S}{\delta g^{\mu\nu}(x)} + iJ_{\mu\nu}(x) + i{:}\int \frac{\partial}{\partial y^{\lambda}} \frac{\delta g^{\lambda\varrho}(y)}{\delta g^{\mu\nu}(x)} K_{\varrho}(y)dy{:}\right)_{g^{\mu\nu} \to \delta/\delta J_{\mu\nu}} Z = 0. \tag{3.8}$$

The form of Z is analogous to (2.9)

$$Z = \int_{\Sigma} F[g^{\mu\nu}]e^{i\int g^{\mu\nu}J_{\mu\nu}dx} \prod_{x,\mu,\nu} dg^{\mu\nu}(x), \tag{3.9}$$

where the hypersurface Σ is determined by $\partial_{\mu}g^{\mu\nu} = 0$. To find $K_{\varrho}(x)$ we have the following equation, obtained by acting $\nabla_{\lambda}^{\mu\nu}(x)$ on (3.8)

$$\nabla_{\lambda}^{\mu\nu}(x)J_{\mu\nu}(x)|_{g^{\mu\nu} \to \delta/\delta J_{\mu\nu}}Z = -\nabla_{\lambda}^{\mu\nu}(x){:}\int \frac{\partial}{\partial y^{\lambda}} \frac{\delta g^{\lambda\varrho}(y)}{\delta g^{\mu\nu}(x)} K_{\varrho}(y)dy{:}|_{g^{\mu\nu} \to \delta/\delta J_{\mu\nu}}Z. \tag{3.10}$$

After some algebra (which can be performed following the steps of Appendix B in [4]) we can verify that the solution of (3.10) is

$$\begin{aligned} {:}\int \frac{\partial}{\partial y^{\lambda}} \frac{\delta g^{\sigma\varrho}(y)}{\delta g^{\mu\nu}(x)} K_{\varrho}(y)dy{:} = -{:}\int \nabla_{\lambda}^{\varrho\sigma}(y)J_{\varrho\sigma}(y)\left[\int \frac{\partial}{\partial z^{\xi}} \frac{\delta g^{\xi\lambda}(z)}{\delta g^{\mu\nu}(x)} G(z,y)dz\right]dy{:} \\ -\partial_{\lambda}\int \frac{\partial}{\partial z^{\varrho}} \frac{\delta g^{\varrho\lambda}(z)}{\delta g^{\mu\nu}(x)} G(z,y)dz|_{x=y}, \end{aligned} \tag{3.11}$$

where $G(x, y)$ is defined by

$$\nabla_{\mu}^{\nu\lambda}(x)\int \frac{\partial}{\partial y^{\sigma}} \frac{\delta g^{\sigma\varrho}(y)}{\delta g^{\nu\lambda}(x)} G(y,z)dy = \delta_{\mu}^{\varrho}\delta(x-z). \tag{3.12}$$

Omitting, as in the Yang-Mills case, a pure divergence in the equation obtained by inserting (3.8) into (3.11), we have then

$$\left(\frac{\delta S}{\delta \mathfrak{g}^{\mu\nu}(x)}+iJ_{\mu\nu}(x)-i\partial_\lambda\int\frac{\partial}{\partial z^\varrho}\frac{\delta\mathfrak{g}^{\varrho\lambda}(z)}{\delta\mathfrak{g}^{\mu\nu}(x)}G(z,y)dz|_{x=y}\right)_{\mathfrak{g}^{\mu\nu}\to\delta/\delta J_{\mu\nu}} Z=0. \tag{3.13}$$

An integration by parts allows us to deduce the equation determining $F[\mathfrak{g}^{\mu\nu}]$ on Σ

$$\left(\frac{\delta S}{\delta\mathfrak{g}^{\mu\nu}}-i\partial_\mu\partial_\nu G(x,y)|_{x=y}-\frac{\delta}{\delta\mathfrak{g}^{\mu\nu}}\right)F=0. \tag{3.14a}$$

Let us write $\mathfrak{g}^{\mu\nu}$ in the form

$$\mathfrak{g}^{\mu\nu}=\eta^{\mu\nu}+\lambda h^{\mu\nu},$$

where $\eta^{\mu\nu}$ is the Minkowskian metric. Equation (3.14a) then acquires the form

$$\left(\frac{\delta S}{\delta h^{\mu\nu}}-i\lambda\partial_\mu\partial_\nu G(x,y)|_{x=y}-\frac{\delta}{\delta h^{\mu\nu}}\right)F=0. \tag{3.14b}$$

Using the identity (see Appendix)

$$\frac{\delta}{\delta h^{\mu\nu}}\operatorname{Sp}\ln(1+\lambda h^{\varrho\sigma}\partial_\varrho\partial_\sigma\Box^{-1})=\lambda\partial_\mu\partial_\nu G(x,y)|_{y=x} \tag{3.15}$$

we can easily integrate (3.14b) and obtain finally

$$Z=\int_\Sigma\exp\left[S-i\operatorname{Sp}\ln(1+\lambda h^{\varrho\sigma}\partial_\varrho\partial_\sigma\Box^{-1})+i\int\mathfrak{g}^{\mu\nu}J_{\mu\nu}dx\right]\prod_{x,\mu,\nu}dg^{\mu\nu}(x). \tag{3.16}$$

The functional Z contains the wellknown Faddeev-Popov measure

$$\nabla_H=\exp(i\operatorname{Sp}\ln(1+\lambda h^{\varrho\sigma}\partial_\varrho\partial_\sigma\Box^{-1}))$$

describing the interaction of gravitational field $h^{\mu\nu}$ with ghost vector field.

Thus, basing on the external source method combined with the DeWitt quantum dynamical equation we have presented a new approach to the problem of quantization of the Yang-Mills and gravitational fields. This approach may be immediately applied to the quantization of any gauge field. The main advantage of this approach is the developing of a procedure of quantization in which the connection between the methods used by Fradkin-Tyutin, DeWitt, Mandelstam and Faddeev-Popov is made transparent.

APPENDIX

In this appendix we shall establish the validity of (3.15). From (3.12) using the harmonic gauge condition we can obtain

$$\mathfrak{g}^{\sigma\lambda}\partial_\sigma\partial_\lambda G(x-y)=(\Box+\lambda h^{\sigma\lambda}\partial_\sigma\partial_\lambda)G(x-y)=\delta(x-y).$$

Expanding $G(x-y)$ as a perturbation series in λ we have

$$G(x-y)$$
$$=\sum_0 (-1)^n\lambda^n \int \Box^{-1}(x-x_1)h^{\sigma_1\lambda_1}\partial_{\sigma_1}\partial_{\lambda_1}\Box^{-1}(x_1-x_2)\ldots h^{\sigma_n\lambda_n}\partial_{\sigma_n}\partial_{\lambda_n}\Box^{-1}(x_n-y)\,dx_1\ldots dx_n.$$

Then

$$\lambda\partial_\mu\partial_\nu G(x,y)|_{y=x} = -\sum_1 (-1)^n\lambda^n \int \partial_\mu\partial_\nu\Box^{-1}(x-x_1)h^{\sigma_1\lambda_1}\partial_{\sigma_1}\partial_{\lambda_1}\Box^{-1}(x_1-x_2)\ldots$$
$$\ldots h^{\sigma_{n-1}\lambda_{n-1}}\partial_{\sigma_{n-1}}\partial_{\lambda_{n-1}}\Box^{-1}(x_{n-1}-x)dx_1\ldots dx_{n-1}. \qquad \text{(A1)}$$

Let us consider now

$$\frac{\delta}{\partial h^{\mu\nu}(x)}\,\mathrm{Sp}\ln(1+\lambda h^{\varrho\sigma}\partial_\varrho\partial_\sigma\Box^{-1})$$
$$=\frac{\delta}{\delta h^{\mu\nu}(x)}\left[-\sum_1\frac{(-1)^n}{n}\lambda^n\int h^{\varrho_1\sigma_1}\partial_{\varrho_1}\partial_{\sigma_1}\Box^{-1}(x_1-x_2)\ldots\right.$$
$$\ldots h^{\varrho_n\sigma_n}\partial_{\varrho_n}\partial_{\sigma_n}\Box^{-1}(x_n-x_1)dx_1\ldots dx_n = -\sum_1(-1)^n\lambda^n\int\partial_\mu\partial_\nu\Box^{-1}(x-x_1)h^{\sigma_1\lambda_1}$$
$$\times\partial_{\sigma_1}\partial_{\lambda_1}\Box^{-1}(x_1-x_2)\ldots h^{\sigma_{n-1}\lambda_{n-1}}\partial_{\sigma_{n-1}}\partial_{\lambda_{n-1}}\Box^{-1}(x_{n-1}-x)dx_1\ldots dx_{n-1}. \qquad \text{(A2)}$$

Comparing (A1) and (A2) we obtain (3.15).

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