HOW TO EVALUATE CROSS-SECTIONS IN MODELS WHERE THE S-MATRIX IS UNITARY BUT DOES NOT CONSERVE ENERGY

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(Received June 4, 1975)

The standard time-dependent description of the scattering processes is used to explain that, when the S-matrix does not conserve energy, the coefficient relating the squared modulus of the S-matrix element to the cross-section becomes model-dependent, and the optical theorem does not necessarily follow from the unitarity of the S-matrix. It is suggested that, if one insists on using such models, the optical theorem should be imposed as a constraint and used to fix the model-dependent coefficient.

Recently models have been proposed (cf. e.g. [1], [2] and references quoted there) where the S-matrix is unitary, but does not enforce energy-momentum conservation. Such models are attractive, because of their formal simplicity. There are, however, some problems, when one works without energy-momentum conservation. One of the problem is: how to calculate absolute values of cross-sections?

In standard scattering theory, the cross-section for the transition from the initial state $|i\rangle$ to the final state $|f\rangle$ is

$$\sigma(f\rightarrow i) = K|\langle f|S-1|i\rangle|^2, \tag{1}$$

where K is a known kinematical coefficient. Moreover the unitarity of the S-matrix implies the optical theorem

$$(d\sigma^{\mathrm{el}}/dt)_0 \sin^2\varphi = 16\pi(\sigma^{\mathrm{tot}})^2, \tag{2}$$

where φ is the phase of the forward scattering amplitude.

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We show that, if the S-matrix does not enforce energy conservation[1], the coefficient K cannot be generally calculated, and the optical theorem does not necessarily follow from the unitarity of the S-matrix. We suggest that in models with unitary, energy non-conserving, S-matrices one should *assume* the optical theorem as an additional constraint. This additional constraint fixes the coefficient K and makes the calculation of absolute cross-sections possible.

Let us recall the standard [3] time dependent description of the scattering process. The incident particles are described as wave packets well localized in momentum space around the momentum of state $|i\rangle$. They are distributed at random in the impact parameter plane $\vec{b}$ perpendicular to the beam direction, and do not interfere with each other. For simplicity it is assumed that the shapes of all these packets are identical. Thus the wave function for a single wave packet is

$$\psi_{\text{in}}(\vec{p}) = e^{-i\vec{b}\vec{p}}\varphi(\vec{p}), \tag{3}$$

where the function $\varphi(\vec{p})$ does not depend on $\vec{b}$. The initial state is mixed. Its density matrix is

$$\varrho_{\text{in}} = \int d^2b \int d^3p \int d^3p' |\vec{p}\rangle \varphi^*(\vec{p})\varphi(\vec{p}')e^{-i(\vec{p}-\vec{p}')\vec{b}}\langle \vec{p}'|, \tag{4}$$

or after integration over $\vec{b}$ and $\vec{p}_{\text{T}}$:

$$\varrho_{\text{in}} = 4\pi^2 \int d^3p \int dp_z' |\vec{p}_{\text{T}}, p_z\rangle \varphi^*(\vec{p}_{\text{T}}, p_z)\varphi(\vec{p}_{\text{T}}, p_z') \langle \vec{p}_{\text{T}}, p_z'|. \tag{5}$$

The cross-section is given in terms of the S-matrix by

$$\sigma(f \leftarrow i) = \langle f|(S-1)\, \varrho_{\text{in}}\, (S^+ - 1)|f\rangle \tag{6}$$

or, substituting (5)

$$\sigma(f \leftarrow i) = 4\pi^2 \int d^3p \int dp_z' \langle f|S-1|\vec{p}_{\text{T}}, p_z\rangle \langle \vec{p}_{\text{T}}, p_z'|S^+ - 1|f\rangle \times \varphi^*(\vec{p}_{\text{T}}, p_z)\varphi(\vec{p}_{\text{T}}, p_z'). \tag{7}$$

Up to this point there has been no difference between the cases with and without energy conservation. Now comes the distinguishing assumption. Since the wave packets $\varphi(\vec{p})$ are well localized in momentum space, and since there is no reason to expect that the matrix elements vary particularly rapidly when momentum changes, it is legitimate to replace both $|\vec{p}_{\text{T}}, p_z\rangle$ and $|\vec{p}_{\text{T}}, p_z'\rangle$ by the same fixed state $|i\rangle$. Then formula (1) follows with

$$K = 4\pi^2 \int d^3p \int dp_z' \varphi^*(\vec{p}_{\text{T}}, p_z)\varphi(\vec{p}_{\text{T}}, p_z'). \tag{8}$$

For an energy conserving S-matrix the assumption that the matrix elements vary slowly can be made only after the δ-functions for energy conservation have been factored out. One of these δ-functions can be converted into $\delta(p_z - p_z')$, which reduces the momentum

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[1] For simplicity, we discuss scattering in the centre of mass frame, or on a fixed centre, so that momentum conservation is irrelevant.

integration to a simple normalization integral, equal one whatever the exact shape of the initial wave packets. In models without the δ-functions the coefficient K depends on the shape of the initial wave packets and therefore both: cannot be generally calculated and confuses the physical interpretation of the cross-section.

Les us define the effective slope α_{eff} by the relation

$$(d\sigma^{\text{el}}/dt)_0 = \alpha_{\text{eff}}\sigma^{\text{el}}. \tag{9}$$

The optical theorem (2) can be rewritten as

$$\sigma^{\text{tot}} = 16\pi\alpha_{\text{eff}}(\sigma^{\text{el}}/\sigma^{\text{tot}}) \sin^2 \varphi. \tag{10}$$

Since the right-hand side does not depend on the coefficient K, and the left-hand side is linear in K, the optical theorem is satisfied for one and only one value of K. It seems that, when choosing a unitary S matrix, it is usually understood that the optical theorem should also be valid. Therefore, relation (10) can be used to calculate the coefficient K.

From the phenomenological point of view this is probably the optimal procedure. We would like to stress, however, that it does not make the calculation free from logical objections. For instance the question: why all the shapes of the initial packet, which do not lead to (10), are forbidden, has no answer. Such difficulties plague models without energy conservation whether or not one calculates absolute cross-sections. Otherwise (10) could be used instead of (1) to normalize the cross-sections without ever mentioning the coefficient K. Therefore, calculations of cross-section ratios and of absolute cross-sections from models without energy conservation seem to be on the same plausibility level.

REFERENCES

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