ON THE RELATIVISTIC HYDROGEN ATOM*

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A new quasipotential two-particle theory is presented for the description of the spinless relativistic hydrogen atom. Recoil effects are fully taken into account, but are shown to have no effect on the fine structure of the spectrum. For an infinitesimal value of the fine-structure constant α the theory has O(4) symmetry. Apart from the recoil effects the energy levels in this case coincide with those of the Dirac atom with maximum spin value. For a nonvanishing value of α the O(4) symmetry is broken in the same order in α in which the Lamb-shift occurs. The Klein paradox probably does not exist.

1. Introduction

In a recent paper [1] a Lorentz invariant Lippmann-Schwinger equation was introduced for the description of particle production in high energy scattering. In addition to choosing a special form for the phenomenological potential describing the transitions between different many particle states, the main difference with other unitarization schemes consisted of a special choice for the Green's function of the Lippmann-Schwinger equation. Our scattering amplitude $M_{\beta\alpha}(P)$ for the transition from the state α to the state β was taken to satisfy the equation

$$M_{\beta\alpha}(P) = V_{\beta\alpha} - \int_{\gamma} V_{\beta\gamma} L_{\gamma}(P) M_{\gamma\alpha}(P), \tag{1}$$

where

$$L_{\gamma}(P) = \int_{0}^{\infty} \frac{ds'}{s'-s} \delta^4(P_{\gamma}-P')$$

$$= \frac{2}{P_{\gamma}^2(P_{\gamma}^2-s)} (1-v_{\gamma}^2)(1-v_{\alpha}^2)\delta^3(\vec{v}_{\gamma}-\vec{v}). \tag{2}$$

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The symbol $\int_\gamma$ indicates an invariant sum over intermediate states, such that the particles are on their mass shells. The intermediate energy and momentum can be off-shell, but subject to the condition that the total velocity of the intermediate state is conserved.

The homogeneous equation derived from Eq. (1) describes bound systems and can be cast into the form

$$(P_\alpha^2 - M_b^2)\phi_\alpha + \frac{2}{P_\alpha^2}\int_\gamma V_{\alpha\gamma} L^3_{\alpha\gamma}\phi_\gamma = 0, \tag{3}$$

where

$$L^3_{\gamma\alpha} = (1-v_\gamma^2)\,(1-v_\alpha^2)\delta^3(\vec{v}_\gamma - \vec{v}_\alpha) \tag{4}$$

and

$$\vec{v}_\gamma = \frac{\vec{P}_\gamma}{P_{\gamma 0}} = \vec{v}_\alpha = \frac{\vec{P}_\alpha}{P_{\alpha 0}} = \vec{v} \tag{5}$$

is an arbitrary, but fixed velocity of the whole system. Eq. (3) is the eigenvalue equation for the mass M_b of the bound state and must be such that the wave function can be normalized, which means that

$$\int_\gamma |\phi_\gamma|^2 L^3_{\gamma\alpha} < \infty. \tag{6}$$

From Eq. (2) follows that $L^3_{\gamma\alpha}$ and therefore also this norm is invariant under Lorentz transformations. For a system of two spinless particles 1 and 2 with masses m and M respectively, Eq. (3) takes the form

$$\begin{aligned}&((p_1+p_2)^2 - M_b^2)\phi(p_1, p_2) + \\ &+ \frac{2}{(p_1+p_2)^2}\int d^4k_1 d^4k_2 \delta(k_1^2 - m^2)\theta(k_{10})\delta(k_2^2 - M^2)\theta(k_{20}) \\ &\times V(p_1 p_2 | k_1 k_2)\,(1-v^2)^2 \delta_3(\vec{v}_\gamma - \vec{v})\phi(k_1, k_2) = 0,\end{aligned} \tag{7}$$

with

$$\vec{v}_\gamma = \frac{\vec{k}_1 + \vec{k}_2}{k_{10} + k_{20}} \quad \text{and} \quad \vec{v}_\alpha = \frac{\vec{p}_1 + \vec{p}_2}{p_{10} + p_{20}} = \vec{v} \text{ fixed.} \tag{8}$$

So far Eq. (7) very much resembles a quasipotential equation, for bound states of two particles about which the literature can be traced, e. g. from Ref. [2]. Yaes [3] has even shown that by modifying the propagator in the Kadyshevsky equation, one can obtain an infinite set of quasipotential equations which satisfy both Lorentz covariance and elastic unitarity. The difference with Eq. (7) arises, however, when choosing a Lorentz covariant form for the potential $V(p_1 p_2 | k_1 k_2)$. An obvious choice would be to take for V some real function of the square of the four-momentum transfer. Since, however, the three-momentum is not conserved in the intermediate states, the quantities $t_1 = (p_1 - k_1)^2$ and

$t_2=(p_2-k_2)^2$ are not equal and it is not clear which expression to take for the momentum transfer. Instead one can show by using the conservation of the total velocity (cf. Eq. (8)), that the four-vectors

$$\frac{k_1+k_2}{\sqrt{s'}} \quad \text{and} \quad \frac{p_1+p_2}{\sqrt{s}}, \quad \text{with} \quad s'=(k_1+k_2)^2 \quad \text{and} \quad s=(p_1+p_2)^2$$

are equal. This implies that

$$\tilde{t} \equiv \sqrt{ss'}\left(\frac{p_1}{\sqrt{s}}-\frac{k_1}{\sqrt{s'}}\right)^2 = \sqrt{ss'}\left(\frac{p_2}{\sqrt{s}}-\frac{k_2}{\sqrt{s'}}\right)^2. \tag{9}$$

Whith this variable the symmetry between the particles 1 and 2 has been restored and since, moreover, in the nonrelativistic limit $s, s' \to (m+M)^2$ the variables t_1, t_2 and $\tilde{t}$ become equal, we will consider the interaction potential $V(\tilde{t})$ to be a function of $\tilde{t}$ only [4]. An additional symmetric dependence on s and s' could be included, but for the present this will not be done. In analogy with the Bethe-Salpeter equation it would be very instructive to show in which sense our Eq. (7) could in an approximate way be derived from field theory, so that we would have a systematic way of calculating the potential $V(\tilde{t})$. This, however, we have not succeeded to do. Eq. (7) therefore, must be considered as a phenomenological equation, the proof of which is in the comparison with experiment.

One extra condition we can impose on the potential $V(\tilde{t})$ is the requirement that in the non-relativistic limit $s, s' \to (m+M)^2$ and in the reference frame for which $\vec{v}_\alpha = 0$, Eq. (7) reduces to the Schrödinger equation with a given form of the non-relativistic potential

$$V_{\mathrm{NR}}(\vec{q}) = \frac{1}{(2\pi)^3}\int e^{-i\vec{q}\cdot\vec{r}} V_{\mathrm{NR}}(\vec{r})d\vec{r}. \tag{10}$$

Here $\vec{q} = \vec{k}-\vec{k}'$ and $\vec{k}$ and $\vec{k}'$ are the relative momenta

$$\vec{k} = \frac{m\vec{p}_2 - M\vec{p}_1}{m+M} \quad \text{and} \quad \vec{k}' = \frac{m\vec{k}_2 - M\vec{k}_1}{m+M}.$$

By keeping only the lowest order terms in k^2 and k'^2 it can indeed be shown that the above mentioned requirement is satisfied, provided

$$V(\tilde{t}) = 4mM\, V_{\mathrm{NR}}(\vec{q}) \tag{11}$$

in the limit $k^2, k'^2 \ll Mm$.

The remaining part of this paper is devoted to the study of Eq. (7), especially for the case of Coulombic interaction, foı which the potential, satisfying Eq. (11), is taken as

$$V(\tilde{t}) = \frac{2\alpha mM}{\pi^2\tilde{t}}, \tag{12}$$

with $\alpha = e^2$ equal to the fine structure constant. Although this potential has a form which is different from the usual Coulomb interaction — because of the variable $\tilde{t}$ — it

must be kept in mind that it also appears in an equation which differs from the usual equations in so far that not the momentum, but the velocity is conserved. The best justification, however, for our assumption that the Coulomb interaction is properly described by the potential of Eq. (12), is the close agreement of the spectrum with that obtained from the Dirac equation. This will be explained in the next two sections.

2. Expansion in partial waves and the non-relativistic limit

A first reduction of Eq. (7) can be obtained by introducing instead of k_2 the four integration variables $\vec{v}_\gamma$ (given by Eq. (8)) and k_2^2. These integrations can be performed using four of the five δ-functions. The fifth δ-function is used to remove k_1^2 as an integration variable. Of the three remaining variables one is an azimuthal angle, just giving a factor 2π, while for the last two variables we choose s' and $\tilde{t}$ of Eq. (9). When keeping track of the Jacobians of these transformations we obtain the following form for the eigenvalue problem

$$(s-M_b^2)\phi(s,0)+\frac{\pi}{2\sqrt{\lambda}}\int_{(m+M)^2}^{\infty} ds'\sqrt{\frac{s'}{s}}\int_{t_-}^{t_+} d\tilde{t}\,V(\tilde{t})\phi(s',\tilde{t})=0, \tag{13}$$

with

$$\lambda=\lambda(s,m^2,M^2)=s^2+m^4+M^4-2sm^2-2sM^2-2m^2M^2. \tag{14}$$

The integration limits $t_\pm$ depend on s and s' and are given by $t_\pm=t_0\pm\tau$, with

$$t_0=\frac{1}{2\sqrt{ss'}}\{-ss'+(s+s')(m^2+M^2)-(M^2-m^2)^2\} \tag{15}$$

and

$$\tau=\tfrac{1}{2}\sqrt{\frac{\lambda\lambda'}{ss'}}, \tag{16}$$

where λ' is obtained from λ by replacing s with s'. From (15) and (16) it can be shown that

$$t_-<t_+\leqslant 0. \tag{17}$$

The condition (6) for the wave function to be normalizable becomes

$$\int_{(m+M)^2}^{\infty} ds'\sqrt{s'}\int_{t_-}^{t_+} d\tilde{t}\,|\phi(s',\tilde{t})|^2<\infty. \tag{18}$$

Expanding in partial waves

$$\phi(s',\tilde{t})=\frac{m+M}{\sqrt{s'}}\sum_{l=0}^{\infty}(l+\tfrac{1}{2})\phi_l(s')P_l\left(\frac{\tilde{t}-t_0}{\tau}\right) \tag{19}$$

and defining

$$V_l(s, s') = 2\pi^2(\lambda\lambda')^{-1/4} \int_{t_-}^{t_+} d\tilde{t}\, V(\tilde{t}) P_l\left(\frac{\tilde{t}-t_0}{\tau}\right), \tag{20}$$

the equations (13) and (18) become, respectively

$$(s-M_b^2)\phi_l(s)+\frac{1}{4\pi}\int_{(m+M)^2}^{\infty} ds'\left(\frac{\lambda'}{\lambda}\right)^{1/4} V_l(s, s')\phi_l(s') = 0 \tag{21}$$

and

$$\int_{(m+M)^2}^{\infty} \frac{\sqrt{\lambda'}}{s'} |\phi_l(s')|^2 ds' < \infty. \tag{22}$$

If we now specialize the interaction and take for $V(\tilde{t})$ the Coulomb potential of Eq. (12), formula (20) gives

$$V_l(s, s') = -\frac{8\alpha mM}{(\lambda\lambda')^{1/4}} Q_l\left(-\frac{t_0}{\tau}\right), \tag{23}$$

where Q_l is the Legendre function of the second kind. In order to simplify the equations we introduce dimensionless variables as follows

$$x^2 = \frac{s-(m+M)^2}{v\alpha^2(m+M)^2}; \quad x'^2 = \frac{s'-(m+M)^2}{v\alpha^2(m+M)^2}; \quad x_b^2 = \frac{(m+M)^2-M_b^2}{v\alpha^2(m+M)^2}, \tag{24}$$

where now x_b^2 is the eigenvalue that must be found. As a measure for the mass ratio we have taken

$$v = \frac{mM}{(m+M)^2} = \frac{m/M}{(1+m/M)^2}, \tag{25}$$

while the reduced mass will be indicated by $\mu = \dfrac{mM}{m+M}$. The formula for the (positive) binding energy $E_b = (m+M)-M_b$, when written with these new dimensionless quantities, now becomes

$$E_b = (m+M)\left[1-\sqrt{1-v\alpha^2 x_b^2}\right], \tag{26}$$

or in expanded form

$$E_b = \tfrac{1}{2}\mu\alpha^2\left[x_b^2+\tfrac{1}{4}v\alpha^2 x_b^4+\tfrac{1}{8}v^2\alpha^4 x_b^6+\ldots\right]. \tag{27}$$

The terms containing v are called recoil terms. They disappear when the proton is held

fixed. The eigenvalue equation from which x_b^2 must be determined now becomes

$$(x^2+x_b^2)\phi_l(x) = \frac{2}{\pi x\sqrt{1+\frac{1}{4}\alpha^2x^2}}\int_0^\infty x'dx'Q_l\left(-\frac{t_0}{\tau}\right)\phi_l(x'). \tag{28}$$

In terms of the new variables the argument of the Q_l-function is

$$-\frac{t_0}{\tau} = \tfrac{1}{2}\left(\frac{y}{y'}+\frac{y'}{y}\right) \geqslant 1 \tag{29}$$

with

$$y = \frac{x}{\sqrt{1+\frac{1}{4}\alpha^2x^2}} \quad \text{and} \quad y' = \frac{x'}{\sqrt{1+\frac{1}{4}\alpha^2x'^2}}. \tag{30}$$

The condition for the normalizability reads

$$\int_0^\infty \frac{x^2\sqrt{1+\frac{1}{4}\alpha^2x^2}}{1+v\alpha^2x^2}|\phi_l(x)|^2dx < \infty. \tag{31}$$

We observe that Eq. (28) does not contain v consequently and also x_b^2 is independent of this parameter. This implies that, at least for the Coulomb interaction, all recoil effects indeed occur only in those terms of Eq. (27) which were indicated as such. For a Yukawa interaction with "exchange mass" κ the only change in Eq. (28) consists of an additional term to the argument of the Q_l-function. This term is equal to $\frac{\kappa^2}{\tau}$ and depends explicitly on v. Recoil effects therefore occur already in the first term of Eq. (27) for the binding energy.

The non-relativistic limit of Eq. (28) is obtained by neglecting α^2x^2 and $\alpha^2x'^2$ with respect to unity. If we then introduce the angular variables u and v by

$$x = x_b \operatorname{tg}\frac{v}{2} \quad \text{and} \quad x' = x_b \operatorname{tg}\frac{u}{2} \tag{32}$$

the eigenvalue equation becomes

$$x_b\phi_l(v) = \frac{1}{\pi}\int_0^\pi Q_l\left[\frac{1-\cos v\cos u}{\sin v\sin u}\right]\phi_l(u)du. \tag{33}$$

This equation has been solved by Eriksen [5] in the following way. He introduces a new set of orthonormal (Gegenbauer) functions

$$c_k^\nu(u) = 2^\nu\Gamma(\nu)\left[\frac{(k+\nu)k!}{2\pi\Gamma(k+2\nu)}\right]^{1/2}(\sin u)^\nu C_k^\nu(\cos u), \tag{34}$$

which satisfy

$$\int_0^\pi c_k^\nu(u)c_{k'}^\nu(u)du = \delta_{kk'} \tag{35}$$

and for which he proves the formula

$$\frac{1}{\pi} Q_l\left[\frac{1-\cos v \cos u}{\sin v \sin u}\right] = \sum_{k=0}^{\infty} \frac{1}{k+l+1} c_k^{l+1}(v)c_k^{l+1}(u). \tag{36}$$

Substituting this representation of the Q_l-function into Eq. (33) immediately shows that the normalized eigenfunctions are given by

$$\phi_{kl}(u) = c_k^{l+1}(u). \tag{37}$$

The corresponding eigenvalues are

$$x_b = \frac{1}{n} \qquad \text{with } n \equiv k+l+1 = l+1, \quad l+2, \ldots$$

Substituted into Eq. (27) and keeping only the first term in this non-relativistic limit, gives the usual values for the spectrum of the hydrogen atom. This is no surprise, but only a check on the calculations, since we knew already that in this limit our equation had to reduce to the Schrödinger equation.

3. O(4) symmetry and its breaking

By taking as new variables (cf. Eq. (30))

$$z = \tfrac{1}{2}\alpha y \quad \text{and} \quad z' = \tfrac{1}{2}\alpha y'$$

and defining

$$\chi_l(z) = \frac{\varphi_l(x)}{(1-z^2)^2} \tag{38}$$

the eigenvalue equation (28) can be written as

$$(z^2+z_0^2)\chi_l(z) = \frac{\beta}{\pi z}\int_0^1 z' Q_l\left[\tfrac{1}{2}\left(\frac{z}{z'}+\frac{z'}{z}\right)\right]\chi_l(z')dz', \tag{39}$$

with

$$z_0^2 = \frac{\tfrac{1}{4}\alpha^2 x_b^2}{1-\tfrac{1}{4}\alpha^2 x_b^2} \quad \text{and} \quad \beta = \frac{\alpha}{1-\tfrac{1}{4}\alpha^2 x_b^2}. \tag{40}$$

The condition of Eq. (31) becomes

$$\int_0^1 z^2(1-z^2)|\chi_l(z)|^2 dz < \infty. \tag{41}$$

In the next section we will return to Eq. (39), but here we first investigate the possible O(4) symmetry of our equation.

For that purpose we introduce the variables

$$p = \frac{2z}{\beta}, \quad p' = \frac{2z'}{\beta} \quad \text{and} \quad p_0 = \frac{2z_0}{\beta}. \tag{42}$$

Using the formula

$$\frac{1}{|\vec{p}-\vec{p}'|^2} = \frac{2\pi}{pp'} \sum_{lm} Q_l\left[\tfrac{1}{2}\left(\frac{p}{p'} + \frac{p'}{p}\right)\right] Y_{lm}(\theta, \phi) Y_{lm}^*(\theta', \phi') \tag{43}$$

one can show from Eq. (39) that the function $\chi(\vec{p}) = \chi_l(p)\, Y_{lm}(\theta, \phi)$ satisfies the equation

$$(p^2 + p_0^2)\chi(\vec{p}) = \frac{1}{\pi^2} \int_* \frac{\chi(\vec{p}')}{|\vec{p}-\vec{p}'|^2}\, dp', \tag{44}$$

where $\int_*$ indicates that the integration is to be performed over the volume of a three dimensional sphere with radius $2/\beta$. We now apply a stereographic projection onto the

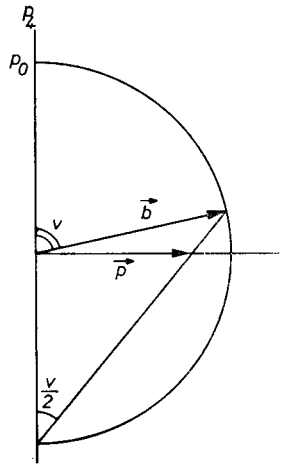

Fig. 1. Stereographic projection

surface of a four dimensional sphere with radius p_0, as first used by Fock [6]. In Figure 1 it is shown how the polar coordinates (p, θ, ϕ) of $\vec{p}$ are transformed into the variables (v, θ, ϕ) of the four dimensional vector $\vec{b}$ with length p_0. Doing the same for $\vec{p}' = (p', \theta', \phi')$ going to $\vec{b}' = (u, \theta', \phi')$ it can be shown [7] that the equation for the function $\psi(v, \theta, \phi) = (p^2 + p_0^2)^2\, \chi(\vec{p})$ becomes

$$\psi(v, \theta, \phi) = \frac{p_0}{2\pi^2} \int_0^{\bar{u}} \sin^2 u\, du \int_0^{\pi} \sin\theta'\, d\theta' \int_0^{2\pi} d\phi' \, \frac{\psi(u, \theta', \phi')}{|\vec{b}-\vec{b}'|^2}, \tag{45}$$

where

$$\bar{u} = 2 \operatorname{arc\,tg} \frac{1}{z_0}. \tag{46}$$

For $\alpha \to 0$ the parameter $\bar{u} \to \pi$ and $p_0 \to x_b$, so that Eq. (45) transforms into the manifestly O(4)-symmetric equation for the non-relativistic hydrogen atom (see e. g. Eq. (4.14) of Ref. [7]). In our equation the O(4)-symmetry is slightly broken, because $\bar{u}$ is almost, but not quite equal to π: a small hole is left near the south pole of the four dimensional globe. If, however, in a first approximation we ignore this hole and thereby restore the symmetry, the spectrum is given by

$$p_0 = \frac{1}{n} \quad \text{with } n = l+1, l+2,... \tag{47}$$

From Eqs (40) and (42) this implies

$$x_b^2 = \frac{2}{\alpha^2}\left[1 \pm \sqrt{1 - \frac{\alpha^2}{n^2}}\right] \equiv x_n^2(\pm). \tag{48}$$

The solution $x_n(-)$ is the one which in the limit $\alpha \to 0$ corresponds to the non-relativistic hydrogen atom. When substituted into Eq. (27) for the binding energy and when the recoil terms are omitted, we obtain

$$E_b = \mu\left[1 - \sqrt{1 - \frac{\alpha^2}{n^2}}\right]. \tag{49}$$

This is exactly equal to the corresponding value for the binding energy of the Dirac atom with the largest possible value of the spin, i. e., with $j = n - \frac{1}{2}$. It is hoped, of course, that when we succeed in extending this theory to include the spin of the electron, the degeneracy of the levels given by Eq. (49) will be lifted to give the fine structure of the Dirac atom, but that the energy of the state with $j = n - \frac{1}{2}$ will still be given by the above expression.

The other solution $x_n(+)$ gives rise to states with a mass M_b just above $M - m$. When we write $M_b = M - m + E_n$ we find with Eq. (24) that to lowest order in α

$$E_n \simeq \tfrac{1}{2}\mu\alpha^2 \frac{1}{n^2}. \tag{50}$$

This is the usual spectrum reflected with respect to the proton mass M. We want to emphasize that this second solution has arisen by treating our original equations in an approximate way and it is by no means clear that such a second solution also exists for the exact equation (39). Actually, when substituting $x_n(+)$ into the expression (40) for β we find for small α

$$\beta \simeq \frac{4n^2}{\alpha} \geqslant \frac{4}{\alpha}. \tag{51}$$

Like in the Dirac equation it is very plausible that there is no solution to Eq. (39) for such a large effective coupling. In this case we do not have to worry about the interpretation of these states. If, nevertheless, they do exist we have to accept the Klein paradox [8] and explain it away by using the idea of the Dirac sea. In the next section the reason given for the absence of a second solution will be made somewhat more rigorous.

4. Perturbative solution

We cast Eq. (39) into still another form by introducing the variables v and u as follows

$$z = z_0 \operatorname{tg} \frac{v}{2} \quad \text{and} \quad z' = z_0 \operatorname{tg} \frac{u}{2} \tag{52}$$

and defining

$$\psi_l(v) = z(z^2 + z_0^2)\chi_l(z). \tag{53}$$

The integral equation then becomes

$$\frac{1}{\pi} \int_0^{\bar{u}} Q_l \left[\frac{1 - \cos v \cos u}{\sin v \sin u} \right] \psi_l(u) du = \lambda \psi_l(v), \tag{54}$$

where

$$\lambda = \frac{\sin \bar{u}}{\alpha}. \tag{55}$$

If the eigenvalues $\bar{u}$ and the eigenfunctions $\psi_l(u)$ have been found, subject to the normalization condition, which now reads

$$\int_0^{\bar{u}} \left(\operatorname{tg}^2 \frac{\bar{u}}{2} - \operatorname{tg}^2 \frac{u}{2} \right) \cos^2 \frac{u}{2} \, |\psi_l(u)|^2 du < \infty, \tag{56}$$

then the value of x_b^2 can be found via equations (40) and (46). Expressed in terms of $\varepsilon = \pi - \bar{u}$ we obtain

$$\lambda = \frac{\sin \varepsilon}{\alpha} \quad \text{and} \quad x_b^2 = \frac{2}{\alpha^2}(1 - \cos \varepsilon). \tag{57}$$

The non-relativistic limit Eq. (33) is recovered by writing $\varepsilon = \alpha \, x_b^{NR}$ and taking $\alpha \to 0$.

Before applying perturbation theory in the small parameter ε we change Eq. (54) into a matrix equation by using Eriksen's formula Eq. (36). Provided a summation and integration may be interchanged it is straightforward to show that the quantity

$$X_k^l = \frac{1}{\sqrt{k+l+1}} \int_0^{\bar{u}} c_k^{l+1}(u) \psi_l(u) du \tag{58}$$

must satisfy the equation

$$\frac{X_k^l}{k+l+1} - \sum_{k'=0}^{\infty} M_{kk'}^l X_{k'}^l = \lambda X_k^l, \tag{59}$$

where the matrix M^l is given by

$$M_{kk'}^l = \frac{(-1)^{k-k'}}{\sqrt{(k+l+1)(k'+l+1)}} \int_0^{\varepsilon} c_k^{l+1}(u) c_{k'}^{l+1}(u) du. \tag{60}$$

Once the eigenvalue problem (59) has been solved the corresponding wavefunction may be found from

$$\psi_l(v) = \frac{1}{\lambda} \sum_k \frac{X_k^l c_k^{l+1}(v)}{\sqrt{k+l+1}}. \tag{61}$$

To lowest order in ε we find

$$\lambda \simeq \frac{\varepsilon}{\alpha} = \frac{1}{k+l+1} = \frac{1}{n}. \tag{62}$$

If this is substituted into the formula for x_b^2 of Eq. (57) we find to the same order $x_b^2 = \frac{1}{n^2}$, which corresponds to the non-relativistic hydrogen atom.

Neglecting the interaction M^l in Eq. (59) gives

$$\frac{\sin \varepsilon}{\alpha} = \frac{1}{n} \tag{63}$$

and therefore

$$x_b^2 = \frac{2}{\alpha^2}\left[1 - \sqrt{1 - \frac{\alpha^2}{n^2}}\right]. \tag{64}$$

Since this is the same as $x_n^2(-)$ of Eq. (48) we conclude that M^l is the term which breaks the O(4) symmetry.

To first non vanishing order the matrix M^l is given by

$$M_{kk'}^l = D_{kk'}^l \varepsilon^{2l+3} + \ldots \tag{65}$$

with

$$D_{kk'}^l = \frac{(-1)^{k+k'} \cdot 2^{2l+2}}{2\pi(2l+3)} \left(\frac{l!}{(2l+1)!}\right)^2 \sqrt{\frac{(k+2l+1)!(k'+2l+1)!}{k!k'!}}. \tag{66}$$

From this we find

$$\lambda = \frac{\sin \varepsilon}{\alpha} = \frac{1}{k+l+1} - D_{kk}^l \varepsilon^{2l+3} + \ldots \tag{67}$$

Since $\varepsilon = O(\alpha)$ we obtain from this x_b^2 at least to order α^{2l+4}. We do not bother to give the resulting formula for the binding energies. Let it be sufficient to remark again that the recoil terms only depend on the principal quantum number n and that the lifting of the degeneracy in l, due to the breaking of the original O(4)-symmetry, does not occur in terms of lower order than α^{2l+3}. The Lamb-shift between a P- and S-state with the same n is in this way calculated to be

$$\Delta E_n(P, S) = \tfrac{1}{2}\mu\alpha^2\left[\frac{2D^0_{n-1,n-1}\alpha^3}{n^4} + O(\alpha^5)\right]. \tag{68}$$

With $D^0_{n-1,n-1} = \dfrac{2n}{3\pi}$ this becomes

$$\Delta E_n(P, S) = \frac{4\alpha^3}{3\pi n^3}\tfrac{1}{2}\mu\alpha^2. \tag{69}$$

When this is compared with the result of a simple QED calculation [8, 9],

$$\Delta E_n^{\mathrm{QED}}(P, S) = \frac{4\alpha^3}{3\pi n^3}\ln\left(\frac{1}{\alpha^2}\right)\tfrac{1}{2}\mu\alpha^2, \tag{70}$$

we observe that our result is too small by a factor $\ln\left(\dfrac{1}{\alpha^2}\right) \simeq 9.8$. Our theory actually should not be expected to give the correct value for the Lamb-shift, since the latter is due to the interaction of the electron with the quantum fluctuations of the electromagnetic field and these are not included in our quasipotential theory. It must be remarked, however, that the possibility to obtain a singularity in the binding energy of the type of $\ln 1/\alpha^2$ was excluded when we started our perturbative calculation. There are actually indications that the exact function $E_n(\alpha)$ for the energy levels is not analytic in $\alpha = 0$. On physical grounds one would expect this, since the bound states cease to exist when the sign of α is reversed. The mathematical indication for the non-analyticity derives from the fact that the kernel of the integral equation (54) is of the Hilbert-Schmidt type. This is shown as follows. By making the substitutions

$$e^{t-\xi} = \operatorname{tg}\frac{u}{2}, \quad e^{t-\eta} = \operatorname{tg}\frac{v}{2} \quad \text{and} \quad e^t = \operatorname{tg}\frac{\bar{u}}{2}, \tag{71}$$

the square of the Hilbert-Schmidt norm becomes

$$N^2 = \frac{1}{\pi^2}\int_0^{\bar{u}} du dv \left|Q_l\left(\frac{1-\cos u\cos v}{\sin u \sin v}\right)\right|^2$$

$$= \frac{1}{\pi^2}\int_{-t}^{\infty}\frac{d\xi d\eta}{\cosh\xi\cosh\eta}\,|Q_l(\cosh(\xi-\eta))|^2. \tag{72}$$

This is certainly less than the number obtained by replacing the lower limit by $-\infty$. With the new variables $y = \xi - \eta$ and $y' = \xi + \eta$ and after performing the y'-integration we obtain in this way

$$N^2 < \frac{2}{\pi^2} \int_{-\infty}^{\infty} \frac{y}{\sinh y} |Q_l(\cosh y)|^2 dy < \infty, \tag{73}$$

which proves the Hilbert-Schmidt type of the integral equation. From this follows that for real ε also the matrix M^l of Eqs (59) and (60) is Hilbert-Schmidt. If on the other hand M^l is expanded in a power series in ε

$$M^l = \sum_{j=2l+3}^{\infty} \varepsilon^j M_j^l, \tag{74}$$

which can be shown to converge for all finite complex values of ε, it turns out that the matrices M_j^l are no longer of the Hilbert-Schmidt type. This means that, when restricting oneself to a finite number of terms in the expansion of Eq. (74), results may be obtained which do not exist when the full matrix is taken into account. If in particular we only take the first term in Eq. (74) we obtain the matrix of Eq. (65), which according to Eq. (66) is separable. It is then easy to show that this M_j^l for $j = 2l+3$ is not bounded and therefore not even compact. We therefore cannot expect to obtain the exact eigenvalues by first approximating M^l in Eq. (74) by a finite number say J terms and then taking the limit $J \to \infty$.

This all goes to show that it is certainly not excluded that in an exact calculation our expression (69) for the Lamb-shift still has to be multiplied by a singular factor, which may be of the type $\ln 1/\alpha^2$. If this is really going to happen we have to face the problem of how on earth the vacuum fluctuations of the electromagnetic field can be incorporated in such a simple quasipotential. A possible clue may be found from the argument [10–13] that a vector potential is required to break the O(4)-symmetry. Since in the present theory breaking occurs without the explicit occurrence of such a vector potential it may be included implicitly. In that case the appearance of the Lamb-shift is no longer a great mystery.

The reason given in the previous section for the probable absence of the Klein paradox, i. e., of a second solution to the eigenvalue problem, can now be made slightly more rigorous. This is done by using the fact that the kernel of the integral equation (54) is of the Hilbert-Schmidt type. The eigenvalues $\lambda_n(\varepsilon)$ therefore have only one accumulation point, which is zero. If now we write Eq. (57) as

$$x_b^2 = \frac{2}{\alpha^2} [1 \pm \sqrt{1 - \alpha^2 \lambda_n^2}], \tag{75}$$

we see that with $n \to \infty$ either $x_b^2 \to 0$ or $x_b^2 \to \frac{4}{\alpha^2}$. The first case corresponds to $\beta \to \alpha$ (cf. Eq. (40)) and for α sufficiently small Eq. (39) probably has a proper solution. In the second case, however, $\beta \to \infty$ and our guess is that Eq. (39) has no solution in that limit.

5. Summary

Many papers have been devoted to the problem of how a relativistic invariant description of the hydrogen atom can be given in such a way that a) the effects of the recoiling proton [2] are taking into account, b) the correct Dirac spectrum for the bound states is obtained, c) no Klein paradox and negative energy states occur, d) an approximate O(4)-symmetry is apparent [10–13] and e) an extension to other potentials is simple. Yukawa-type interactions, e. g., should also be treated in a relativistic invariant way, although it has been claimed [14] that in general these do not allow bound states, when put into the Dirac equation. Here an effort is made to include all the above mentioned requirements, by constructing a new quasi-potential theory, which differs from previous suchtheories [2] in one important respect. External potentials are functions not of the usual square of the momentum transfer, but of a new transfer variable $\tilde{t}$, which is suggested by the conservation of the total velocity instead of the total momentum. An obvious form for the Coulomb interaction in terms of this new variable then leads to a formulation of the eigenvalue problem, which serves as the basis for a discussion of the problems mentioned above. Recoil effects on the binding energy, at least for the Coulomb interaction, turn out to depend only on the principal quantum number and not on the angular momentum, so that they cannot be found in the fine structure of the atomic levels. Their effect on the distance between levels with different values of the principle quantum number is of order α^2 in units of Rydbergs and is therefore beyond the present experimental accuracy. In contrast with this it is found that for a Yukawa interaction the fine structure degeneracy is lifted by these recoil effects. So far no relativistic invariant theory existed, which combined O(4)-symmetry with the correct (Dirac) formula for the energy spectrum. With the present theory energy levels are obtained which, in the limit where the O(4)-symmetry is not broken, are given by the Dirac formula for states with the highest possible spin value. In the full theory this symmetry is slightly broken. The result is that the degeneracy of e. g. the $2S$ and $2P$ levels is lifted and the level shift is comparable to the value of the Lamb-shift as calculated in quantum electrodynamics. How this result is to be explained on the basis of quantum electrodynamics, especially when a missing logarithmic singularity can be found, is a complete mystery. Zero progress in this direction has been made so far. One last aspect to be mentioned is the probable absence of negative energy states and therefore the non existence of the Klein paradox. Since it has not yet been possible to give a rigorous proof of this statement a cautious attitude must be taken. If, however, it turns out to be true, it will have been shown that a relativistic theory for a finite number of particles is possible. All further problems connected with locality, causality and crossing symmetry should then either be solved in this context or relegated to that part of physics where particle production becomes important, i. e., to quantum field theory.

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