NON-STATIC CHARGED FLUID SPHERE IN GENERAL RELATIVITY

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An exact solution corresponding to a relativistic charged fluid sphere is found. It is a generalisation of the metric found previously by Banerjee and Banerji for a spherically symmetric distribution of an electrically neutral fluid. The behaviour of the model is studied from the conditions of fit at the boundary with the Reissner Nordström metric. It is found that in some cases the model collapses, while in other cases a bounce occurs at a certain epoch of an initially contracting model.

1. *Introduction*

While the isotropic irrotational expansion or contraction of charged incoherent matter is not permitted in general relativity (De and Raychaudhuri [1]) it is of some interest to study such motion in the case of a spherically symmetric charged fluid, where the pressure gradient force is not negligible. There are only a few such exact solutions in the literature (Faulkes [2], Vaidya and Shah [3], Omote [4]) corresponding to static or nonstatic relativistic charged fluid spheres. In the present paper we give a new exact solution for a charged perfect fluid sphere undergoing shear free motion with the possibilities of collapse as well as bounce. This charged case is a generalisation of a previously obtained solution (Banerjee and Banerji [5]) for a spherically symmetric distribution consisting of an electrically neutral fluid. The behaviour of the model is not much different from that in the corresponding uncharged case. The matter distribution in this case is also inhomogeneous in the sense that the density and pressure of the fluid are functions of both the radial co-ordinate and time. The interesting feature of the metric is that when one of the constants appearing

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in the solution is zero, the charge also vanishes and it reduces to the solution previously obtained by Banerjee and Banerji. From the conditions of fit with the exterior Reissner--Nordström metric at the boundary the behaviour of the model with respect to collapse or bounce is studied.

2. *Solutions of the field equations*

We consider an isotropic form of the line element

$$ds^2 = e^\nu dt^2 - e^\omega (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2), \tag{1}$$

where ν, ω are functions of r — the radial co-ordinate and t — the time co-ordinate. Since the fluid is assumed to be perfect

$$T_1^1 = T_2^2 = T_3^3 = -p, \quad T_4^4 = \varrho,$$

where we have used co-moving co-ordinates. One can then obtain from the field equations, considering that the pressure is isotropic, the following relation (see Faulkes)

$$\frac{\partial^2 R}{\partial x^2} = A(x)R^2 + B(x)R^3, \tag{2}$$

where $x = r^2$ and $R = e^{-\omega/2}$. $A(x)$, $B(x)$ are arbitrary functions of x. A special solution of the differential Eq. (2) can be given as

$$e^{\omega/2} = \left[\frac{(T+a/y)^2 - \alpha}{y}\right], \tag{3}$$

where T is an arbitrary function of time, $y = (1+kr^2)$ and a, k, α are constants. Again $T_4^1 = 0$, which in turn gives $e^{\nu/2}$ with a suitable time co-ordinate in the form

$$e^{\nu/2} = \frac{T+a/y}{[(T+a/y)^2 - \alpha]}. \tag{4}$$

The matter density ϱ, the pressure p, and the charge density σ are then calculated from the field equations and are given in the form

$$8\pi p = -\frac{1}{S^2}\left[4k + \frac{4ak}{(S+\alpha)^{1/2}} \frac{(1-kr^2)}{(1+kr^2)}\right] - \frac{2S\ddot{S}}{(S+\alpha)} + \frac{S\dot{S}^2}{(S+\alpha)^2} - \frac{3\dot{S}^2}{(S+\alpha)}, \tag{5}$$

$$8\pi\varrho = \frac{3\dot{S}^2}{(S+\alpha)} + \frac{12k}{S^2} + \frac{24ak(S+\alpha)^{1/2}}{S^3} \cdot \frac{(1-kr^2)}{(1+kr^2)}, \tag{6}$$

$$4\pi\sigma = \pm \frac{12ak\alpha^{1/2}}{S^3} \frac{(1-kr^2)}{(1+kr^2)}, \tag{7}$$

where

$$S(r, t) = [(T+a/y)^2 - \alpha]. \tag{8}$$

It can be seen from solutions (3) and (4) that the explicit form for $B(x)$ in (2) is represented by $B(x) = \frac{8a^2k^2\alpha}{y^6}$ and one can immediately write the total charge $q(r)$ up to the co--moving radius r as (see Faulkes, also Bekenstein [6])

$$[q(r)]^2 = \frac{16a^2k^2\alpha r^6}{(1+kr^2)^6}. \tag{9}$$

The constant α can therefore assume only positive values and when $\alpha = 0$, the solutions (3) and (4) reduce to those of the corresponding uncharged fluid sphere found earlier by Banerjee and Banerji. Whereas, when one puts $a = 0$ the line element reduces to that of the Robertson–Walker cosmological model and the condition $k = 0$ leads to the open model of Einstein–deSitter where space-time becomes spatially flat and spatially infinite in extent.

3. *Boundary conditions for matching with the exterior metric*

The interior metric obtained above can be matched with the exterior Reissner–Nordström metric

$$ds^2 = (1-2m/\bar{r}+\varepsilon^2/\bar{r}^2)d\bar{t}^2-(1-2m/\bar{r}+\varepsilon^2/\bar{r}^2)^{-1}d\bar{r}^2-\bar{r}^2(d\theta^2+\sin^2\theta d\phi^2), \tag{10}$$

provided at the moving boundary ($r = r_0$) (Faulkes)

$$p(r_0, t) = 0, \qquad q^2(r_0) = \frac{16a^2k^2\alpha r_0^6}{(1+kr_0^2)^6} = \varepsilon^2 \tag{11}$$

and

$$-4kS_0^2+\left(\frac{16a^2k^2r_0^2}{y_0^2}+\frac{2my_0^3}{r_0^3}\right)S_0-8ak\frac{(1-kr_0^2)}{(1+kr_0^2)}S_0(S_0+\alpha)^{1/2} = \frac{S_0^4\dot{S}_0^2}{(S_0+\alpha)}. \tag{12}$$

Again since $\dot{\omega}/2 = \dot{S}/S$, the situation $\dot{S} = 0$ corresponds to the turning point in the motion of the sphere.

Now from (5), (12) and remembering that $p = 0$ at $r = r_0$ one gets immediately for $\dot{T} = \dot{S}_0 = 0$ the relation

$$2\ddot{S}_0 = -\frac{1}{S_0^3}\left[2k(S_0+2\alpha)+1/2\left(\frac{16a^2k^2r_0^2}{y_0^2}+\frac{2my_0^3}{r_0^3}\right)\right]. \tag{13}$$

When $k > 0$, $\ddot{S}_0 < 0$ at the turning point. This shows that S_0 has a maximum and no minimum. The charged sphere in this case with an initial outward motion will expand to a maximum of the proper volume followed by a collapse to a singularity $S_0 \to 0$ i.e. to a singular state of infinite density.

A condition necessary but not sufficient for bounce from an initially contracting state may be given by the negative value of the constant k.

4. *Case of collapse to zero proper volume*

Let $a > 0$, $k > 0$, $(1-kr_0^2) > 0$. Writing

$$4k = P, \quad \left(\frac{16a^2k^2r_0^2}{y_0^2} + \frac{2my_0^3}{r_0^3}\right) = Q, \quad \frac{8ak(1-kr_0^2)}{(1+kr_0^2)} = R,$$

where P, Q, R all are positive constants, one can obtain the following relation from the matching condition (12) at the boundary of the distribution

$$-PS_0^2 + QS_0 - RS_0(S_0+\alpha)^{1/2} = \frac{4\dot{S}_0^2}{(S_0+\alpha)} S_0^3 \geqslant 0. \tag{14}$$

The equality sign holds only at the turning point in the motion (i.e. $\dot{S}_0 = 0$) corresponding to

$$S_0 = \frac{(2PQ+R^2) \pm \{(2PQ+R^2)^2 - 4P^2(Q^2-\alpha R^2)\}^{1/2}}{2P^2}. \tag{15}$$

It follows from Eq. (14) that the quantity $\Delta = (2P^2S_0 - 2PQ)$ is always non-positive, and since from Eq. (15) we have $\Delta = [R^2 \pm \text{Square root term}]$ the overall non-positive character of Δ can be guaranteed only by the minus sign before the square root, and this reduces the ambiguity of sign in Eq. (15), hence we have

$$(S_0)_{\max} = \frac{1}{2P^2}[(2PQ+R^2) - \{(2PQ+R^2)^2 - 4P^2(Q^2-\alpha R^2)\}^{1/2}] \tag{16}$$

and this is greater than zero because $Q > R\alpha^{\frac{1}{2}}$ as is evident from the relation (14). There is thus a collapse to a singularity of zero proper volume. It can now be shown that the matter density ϱ and the pressure p remain positive in all regions in the interior of the model. The matter density ϱ turns out to be always greater than zero from Eq. (6). Again differentiating Eq. (5) with respect to the radial co-ordinate and remembering that S' is negative in the present case, one finds that the pressure is a monotonically decreasing function of the radial co-ordinate r. Again since the pressure vanishes at the boundary, it is greater than zero everywhere in the interior region of the distribution. In this particular case so long as $S > 0$ as it is here, $\varrho > |\sigma|$ from (5) and (6).

5. *Bouncing models*

Let $a < 0$, $k < 0$ and $y_0 = (1-kr_0^2) > 0$. The last condition is necessary here for the regularity of the metric in all regions of the interior.

Now putting

$$P_1 = 4|k|, \tag{17}$$

$$Q_1 = \left(\frac{16a^2k^2r_0^2}{y_0^2} + \frac{2my_0^3}{r_0^3}\right), \tag{18}$$

$$R_1 = 8|a||k|\frac{(1+|k|r_0^2)}{(1-|k|r_0^2)}, \tag{19}$$

the boundary condition (12) gives the relation

$$P_1S_0+Q_1-R_1(S_0+\alpha)^{1/2} \geqslant 0 \tag{20}$$

which in turn goes over to the relation

$$P_1^2S_0^2-(R_1^2-2P_1Q_1)S_0+(Q_1^2-R_1^2\alpha) \geqslant 0. \tag{21}$$

At the turning point in the motion $\dot{S}_0 = 0$ and the equality sign holds in (21). Thus the turning point corresponds to

$$S_0 = \frac{1}{2P_1^2}[(R_1^2-2P_1Q_1)\pm\{(R_1^2-2P_1Q_1)^2-4P_1^2(Q_1^2-R_1^2\alpha)\}^{1/2}]. \tag{22}$$

When $R_1^2 > 2P_1Q_1$ and $Q_1 > R_1\alpha^{\frac{1}{2}}$ both roots are real and positive. In view of (21) one finds that with a positive sign in (22) the boundary reaches a minimum volume and bounces back, while the other case with a negative sign corresponds to the maximum volume reached by the boundary after which the collapse starts.

When $R_1^2 > 2P_1Q_1$ and $Q_1 \leqslant R_1\alpha^{\frac{1}{2}}$ there can only be bounces of the models. The physical significance of the relation $Q_1 \gtreqless R_1\alpha^{\frac{1}{2}}$ is that it is equivalent to the relation $\varrho_0 \gtreqless |\sigma_0|$ where ϱ_0 and $|\sigma_0|$ stand for the matter density and the magnitude of the charge density at the boundary. This result may be easily verified from (6) and (7) applying the boundary conditions (11) so that

$$8\pi\varrho_0 = \left(\frac{48a^2k^2r_0^2}{y_0^2}+\frac{6my_0^3}{r_0^3}\right)\frac{1}{S_0^3} = \frac{3Q_1}{S_0^3} \tag{23}$$

and

$$4\pi|\sigma_0| = \frac{12|a|\ |k|\alpha^{1/2}}{S_0^3}\ \frac{(1+|k|r_0^2)}{(1-|k|r_0^2)} = \frac{3}{2}\frac{R_1\alpha^{1/2}}{S_0^3}. \tag{24}$$

Lastly, we make some remarks on the nature of the physical quantities such as pressure and density of the fluid in such cases. It is possible to say something definitely only at the boundary $r = r_0$. ϱ_0 from (23) is always positive. Further, in view of (5) and (11) remembering that the pressure vanishes at the boundary one can obtain for the pressure gradient

$$8\pi p'(r_0, t) = \frac{3Q_1}{2}\frac{S_0'}{(S_0+\alpha)}\frac{(S_0+2\alpha)}{S_0^4} - \frac{3R_1\alpha S_0'}{(S_0+\alpha)^{1/2}}\frac{1}{S_0^4}. \tag{25}$$

It can be written also in the form

$$8\pi p'(r_0, t)/S_0' = \frac{3}{2(S_0+\alpha)S_0^4}[Q_1(S_0+2\alpha)-2R_1\alpha(S_0+\alpha)^{1/2}]. \tag{26}$$

Again when $Q_1 \geqslant R_1\alpha^{\frac{1}{2}}$

$$[Q_1(S_0+2\alpha)-2R_1\alpha(S_0+\alpha)^{1/2}] \geqslant R_1\alpha^{1/2}[(S_0+\alpha)^{1/2}-\alpha^{1/2}]^2, \tag{27}$$

so that from (26) and (27) it is evident that the ratio $p'(r_0, t)/S_0'$ is always positive, which in view of $S_0' < 0$ gives a negative value of the pressure gradient. On the other hand when $Q_1 < R_1\alpha^{\frac{1}{2}}$ $[Q_1(S_0+2\alpha)-2R_1\alpha(S_0+\alpha)^{\frac{1}{2}}]$ is less than a positive quantity so that the ratio p'/S_0' cannot exceed a certain positive constant. Thus with $S_0' < 0$, the latter condition cannot exclude positive pressure gradient corresponding to internal tensions.

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