THE PARADOX OF A LORENTZ INVARIANT CURRENT AND ITS SOLUTION

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The classical electromagnetic current can be a Lorentz invariant vector field. On the other hand, the electromagnetic field cannot be a Lorentz invariant field. Therefore, the field generated by a Lorentz invariant current must have a deviation from the perfect Lorentz symmetry which is not implicit in the current. It is possible to choose the deviation in such a way that its existence cannot be detected by means of experiments with classical test particles.

1. Introduction

All the known ways of quantizing the electromagnetic and the gravitational potential have — as Strocchi [1] puts it — some unpleasant features. Strocchi shows that the occurence of the unpleasant features can be proved in the framework of Wightman's theory without assuming the spectral condition, the temperedness of the fields, the uniqueness of the vacuum state, the Fock representation and the positive definitness of the metric in the Hilbert space. One feels that statements which can be proved without all those assumptions do not belong really to the field theory but concern something different. We suggest that this is indeed the case: it is impossible to have the potential both transverse and covariant because it is impossible to have a regular unit vector field tangent to a sphere.

We indicate also two simple cases in which the same phenomenon occurs: nonexistence of Lorentz invariant solutions of the classical Maxwell equations with a Lorentz invariant current and nonexistence of Lorentz invariant affine connection on the light cone.

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2. *Commutator of two transverse fields*

Let

$$A_\lambda(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{k^0} [a_\lambda(k)e^{-ikx} + a_\lambda^\dagger(k)e^{ikx}],$$

where

$$kx = k^0x^0 - k^1x^1 - k^2x^2 - k^3x^3, \quad k^0 = \sqrt{(k^1)^2+(k^2)^2+(k^3)^2},$$

be the vector potential.

$$a_\lambda(k) = \sum_{A=1}^{2} \overset{A}{e}_\lambda(k)a_A(k), \quad a_\lambda^\dagger(k) = \sum_{A=1}^{2} \overset{A}{e}_\lambda(k)a_A^\dagger(k),$$

where $\overset{1}{e}, \overset{2}{e}$ are polarization vectors i.e. unit space-like vectors mutually orthogonal and orthogonal to k while $a_A(k)$ and $a_A^\dagger(k)$ are annihilation and creation operators corresponding to two independent transverse modes of the field. Using the commutation relations

$$[a_A(k), a_B^\dagger(l)] = 2\delta_{AB}k^0\delta(\mathbf{k}-\mathbf{l})$$

one finds that

$$[A_\lambda(x), A_\beta(y)] = iD_{\lambda\beta}(x-y),$$

where

$$D_{\lambda\beta}(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{k^0}\left(-\sum_{A=1}^{2} \overset{A}{e}_\lambda \overset{A}{e}_\beta\right)\sin(kx).$$

There is exactly one null vector $m(k)$ such that

$$-\sum_{A=1}^{2} e_\lambda e_\beta = g_{\lambda\beta} - k_\lambda m_\beta - k_\beta m_\lambda, \quad km = 1.$$

Hence the commutator of two transverse fields can be written in the form

$$D_{\lambda\beta}(x) = g_{\lambda\beta}D(x) + \partial_\lambda A_\beta(x) + \partial_\beta A_\lambda(x)$$

where

$$D(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{k^0}\sin(kx) = \frac{1}{2\pi}\operatorname{sign}(x^0)\delta(xx)$$

is the Pauli–Jordan function and

$$A_\beta(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{k^0} m_\beta(k)\cos(kx).$$

One has formally

$$\Box A_\beta(x) = 0, \quad \partial^\beta A_\beta(x) = -D(x).$$

Differentiating the second equation and substracting the result from the first one we have

$$\partial^{\beta}(\partial_{\beta}A_{\lambda}-\partial_{\lambda}A_{\beta}) = \partial_{\lambda}D.$$

This means that $A_{\beta}(x)$ is a classical vector potential generated by the current $j_{\beta}(x) = \partial_{\beta}D(x)$.

3. The paradox of a Lorentz invariant current

Here an acute paradox arises. The current is clearly a Lorentz invariant vector field since it is a gradient of the invariant function $D(x)$. On the other hand, no field generated by this current can be Lorentz invariant. Indeed, in a Lorentz invariant situation the only directed quantities are the radius vector x^{λ}, the metric tensor $g_{\alpha\beta}$ and the Levi–Civita symbol $\varepsilon^{\alpha\beta\gamma\mu}$; it is clearly impossible to form an antisymmetric tensor using only x^{λ}, $g_{\alpha\beta}$ and $\varepsilon^{\alpha\beta\gamma\lambda}$.

We see that there is no Lorentz invariant commutator of two transverse fields because there is no Lorentz invariant solution of the classical Maxwell equations with a Lorentz invariant source.

The property of Lorentz invariance determines the current up to a position in space-time and a constant factor. Indeed, a Lorentz invariant current must be a gradient of a Lorentz invariant solution of the wave equation. Any such solution is a linear combination of $D(x)$ and $D_1(x) = 1/xx$. However, the gradient of $1/xx$ is clearly a vector while the electric current is not a vector but a density with the transformation rule

$$j_{\beta'} = \text{sign}\left(\frac{\partial x^{0}}{\partial x^{0'}}\right)\frac{\partial x^{\lambda}}{\partial x^{\beta'}}\, j_{\lambda}.$$

Hence the most general Lorentz invariant current has the form

$$j_{\beta}(x) = Q\partial_{\beta}D(x-c)$$

where

$$Q = \int d^3x j_0(x)$$

is the total charge. Physically the Lorentz invariant current is a spherical shell of charge imploding and exploding with the velocity of light.

4. Solution of the paradox: the deviation from the Lorentz symmetry cannot be detected, provided the symmetry is broken by a fixed null direction

Let us assume that there exists in Nature a Lorentz invariant current. We know that the field generated by this current is not Lorentz invariant, but we cannot predict how the Lorentz symmetry is broken. We can only measure — by means of scattering of test particles — the field actually produced.

It seems that the paradox of a Lorentz invariant current will be solved if it is impossible to detect how the Lorentz symmetry is broken.

Let us break the Lorentz symmetry by choosing a preferred null direction a. We need this direction to form an outer product with x. By means of the direction a, one can construct a solution of the Maxwell equations

$$\partial^{\beta}F_{\beta\lambda} = \partial_{\lambda}D,$$

$$\partial_{\lambda}F_{\beta\nu}+\partial_{\beta}F_{\nu\lambda}+\partial_{\nu}F_{\lambda\beta} = 0,$$

where

$$D(x) = \frac{1}{2\pi}\operatorname{sign}(x^0)\delta(xx),$$

in the form

$$F_{\lambda\nu}(x;a) = \frac{1}{2\pi}\left[\frac{\delta(ax)}{xx}+\frac{\delta(xx)}{|ax|}\right](a_{\lambda}x_{\nu}-a_{\nu}x_{\lambda}).$$

The field $F_{\lambda\nu}(x;a)$ is invariant with respect to the four-parameter group consisting of all Lorentz transformations which preserve the null direction a. It is impossible to have a more symmetric solution since there is no five-parameter subgroup of the Lorentz group while a Lorentz symmetric solution does not exist.

The field $F_{\lambda\nu}(x;a)$ has a remarkable property: a classical particle scattered by this field emerges with unchanged momentum and angular momentum. Hence, it is impossible to find the symmetry breaking direction a, at least from measurements involving classical test particles. This seems to be the solution of the paradox.

Proof is given in the Appendix.

5. *Affine geometry of the light-cone*

The light-cone $xx = 0$, $x^0 > 0$, where x is the radius vector in the Minkowski space, is a three-dimensional manifold, which carries a metric and a volume induced by the geometry of the Minkowski space. The metric and the volume are invariants of the Lorentz group which is therefore the group of metric motions of the light-cone. It is a legitimate geometric problem to look for an affine connection on the light-cone compatible with the metric and the volume. But, as I have shown previously [2, 3], the problem has no solution if one insists on the Lorentz invariance of the affine connection. The group of affine geometry of the light-cone turns out to be smaller than the group of metric geometry: it is the four-parameter group of transformations which preserve a fixed null direction.

In Cayley's construction of the Euclidean geometry we descend from the projective geometry to the metric geometry choosing a preferred conic section; the group of metric geometry is then the subgroup of the projective group which preserves the preferred conic section. In the case of the light-cone the order is reversed: we descend from the metric geometry to the affine geometry choosing a preferred null direction; the group of affine geometry is then the subgroup of the Lorentz group which preserves the fixed null direction.

APPENDIX

We shall find the motion of a particle in the field

$$F_{\mu\nu}(x) = 2\left[\frac{\delta(ax)}{xx} + \frac{\delta(xx)}{|ax|}\right](a_\mu x_\nu - a_\nu x_\mu), \qquad aa = 0.$$

The particle moves freely everywhere except on the surfaces $xx = 0$ and $ax = 0$, where it receives three consecutive shocks.

The notation is explained in Fig. 1. $p = mdx/ds$ is the initial kinematical (not canonical!) momentum, p_1 the momentum after the first shock which occurs at x, etc. On the

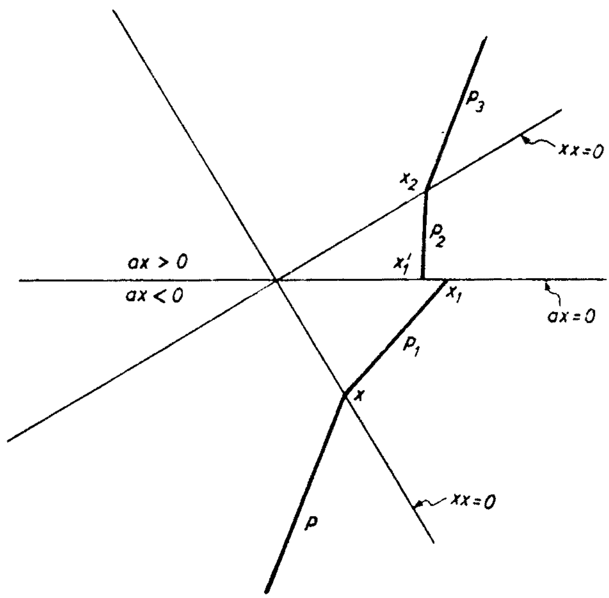

Fig. 1. Motion of a particle in the field of the Lorentz invariant current $j_\mu(x) = \partial_\mu D(x)$. It follows from the conservation laws that $p = p_3$, $x_2 - x = -2(px)p/pp$

segment $x_1 x_1'$ the particle moves with the velocity of light in the direction of a. This strange behaviour follows from the conservation of angular momentum, as explained below.

The potential of the field $F_{\mu\nu}$ has the form

$$A_\mu(x) = \frac{a_\mu}{|ax|}\,\theta(-xx) + \frac{x_\mu}{xx}\,\text{sign}\,(ax),$$

where

$$\theta(-xx) = \begin{cases} 0 \text{ for } xx > 0, \\ 1 \text{ for } xx < 0. \end{cases}$$

The potential is invariant with respect to the four-parameter group of Lorentz transformations which preserve the direction of a. Using the action principle

$$-\delta \int m \sqrt{dxdx}+eAdx = 0$$

one finds easily the corresponding conservation laws:

$$a^{\mu}\left[\left(m\frac{dx_{\nu}}{ds}+eA_{\nu}\right)x_{\mu}-\left(m\frac{dx_{\mu}}{ds}+eA_{\mu}\right)x_{\nu}\right] = \text{const},$$

$$\varepsilon^{\mu\nu\lambda\varrho}\left(m\frac{dx_{\mu}}{ds}+eA_{\mu}\right)x_{\nu}a_{\lambda} = \text{const}.$$

It is clear that p_1 is a linear combination of p, x and a. Assuming that the world line is continuous across the surface $xx = 0$ we find from the conservation laws

$$p_1 = p+e\frac{a}{ax}+\lambda x,$$

where λ remains undetermined. It can be determined, however, from the identity $p_1p_1 = pp$. In this way we obtain

$$p_1 = p+e\frac{a}{ax}-e\frac{ap}{ax(e+px)}x.$$

Similarly

$$p_3 = p_2+e\frac{a}{ax_2}-e\frac{ap_2}{ax_2(e+p_2x_2)}x_2.$$

On the surface $ax = 0$ a curious phenomenon occurs. The assumption that the world line is continuous across the surface is inconsistent with the conservation laws. It is a common point of view that whenever continuity is inconsistent with conservation laws, conservation laws prevail; this point of view leads e.g. to the hydrodynamical theory of shock waves [4].

Suppose that the particle reaches the surface $ax = 0$ at x_1 and leaves it at x'_1. The vector x'_1-x_1 is either space-like or null and then proportional to a. It cannot be space-like, however, since in this case the particle would move with a velocity exceeding the velocity of light. Thus it follows from the conservation laws that, for a short period of time, the particle is dragged along in the direction of a; it may be dragged along forward or backward in time, depending on the sign of charge.

To calculate the result of the second shock we integrate the equations of motion

$$\frac{dp_{\mu}}{ds} = eF_{\mu\nu}\frac{dx^{\nu}}{ds}$$

over a small interval around the surface $ax = 0$. We obtain

$$p_{2\mu}-p_{1\mu} = 2e\int\frac{\delta(ax)}{xx}(a_{\mu}x_{\nu}-a_{\nu}x_{\mu})dx^{\nu} = -2e\int d(ax)\delta(ax)\frac{x_{\mu}}{xx}+ea_{\mu}\int\frac{d(xx)}{xx}\delta(ax).$$

Both integrals are not well defined. In the first one x_μ is discontinuous for $ax = 0$; in the second one $d(xx)/d(ax)$ is discontinuous for $ax = 0$. However, xx is continuous for $ax = 0$. Therefore the first integral is determined up to a multiple of a_μ while the second integral is clearly proportional to a_μ. Thus we have

$$p_2 - p_1 = -2e\frac{x_1}{x_1x_1} + \lambda a,$$

where λ remains undetermined. It can be determined, however, from the identity $p_2p_2 = p_1p_1$. In this way we obtain

$$p_2 = p_1 + \frac{2e}{x_1x_1}\left(\frac{p_1x_1 - e}{ap_1}a - x_1\right).$$

The jump $x'_1 - x_1$ can be determined from the conservation laws. It turns out that

$$x'_1 = x_1 + 2e\frac{a}{ap_1}.$$

Thus the motion has been determined completely. It is easy to show that

$$p = p_3$$

and

$$x_2 - x = -2\frac{px}{pp}p.$$

This means that after three consecutive shocks the particle continues to move on the original straight line.

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