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SYMMETRIES AND SOLUTIONS TO THE GRAVITATIONAL FIELD EQUATIONS*

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Investigations of exact solutions to the Einstein gravitational field equations are described. A systematic search for solutions with certain symmetries leads to several interesting properties of the gravitational field equations.

The Einstein gravitational field equations are now sixty years old. The extreme richness of these equations is exemplified by the fact that important new solutions to these equations are still being discovered. These equations, in the usual notation, may be written

$$G_{ij}+\Lambda g_{ij} = 8\pi T_{ij}.$$

Here the Einstein tensor G_{ij} is built from the geometry (metric tensor) whereas the energy-momentum tensor T_{ij} is built from the geometry and other fields which are the source of the gravitational field.

We list a few energy-momentum tensors that have been useful in discussing solutions: perfect fluid, electromagnetic, scalar field, vector field, spinor field, etc. In general the energy momentum tensor will be a sum of these various types. Thus, in order to discuss solutions to the Einstein equations one must specify the source terms in these equations. Solving outside the sources $T_{ij} = 0$ is referred to as the vacuum problem.

The most important solution is the Schwarzschild solution for the gravitational field outside a spherically symmetric source.

In order to find solutions one must make some simplifying assumptions. Several different types of assumptions can be made but the one we shall consider is to assume the geometry has certain symmetries. For example, in the Schwarzschild solution one assumes spherical symmetry. Petrov [1] has given an exhaustive classification of possible gravita-

* Dedicated to my mother on her sixty eighth birthday.

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tional fields based on their symmetries. These different types are usually referred to as Bianchi type gravitational fields. In order to completely specify the Bianchi type we must give the number of independent symmetry transformations and the nature of the transformations. For example, the Bianchi type G_4V means there are four symmetry transformations and they are Bianchi type V. Finally in order to further specify the gravitational field one must specify the type of subspaces that the group of symmetry transformations generates. For example G_4V on V_3 means that G_4V generates three dimensional hypersurfaces V_3.

We have started a detailed study of all spacetimes of the form G_4 — on V_3 and G_4 — on V_3^* where V_3^* means a null hypersurface. One reason for this study is to see how the physical properties of gravitational fields are associated with their symmetries.

One of the first solutions of G_4V on V_3 was discovered by Farnsworth [2], who found exact solutions for the case where the source is dust (pressureless perfect fluid). Farnsworth showed his solution was an expanding anisotropic cosmological solution. One of our first results showed that different Bianchi types do sometimes have very different physical properties. The Bianchi type G_4IV on V_3 allows only spacelike (tachyon) dust solutions [3, 4]. We now know that Farnsworth's case may have either timelike (ordinary) or spacelike dust solutions[5]. Apparently this property of the Farnsworth case is associated with the possibility of a „whimper singularity" for this model [6]. (In a „whimper singularity" the four-velocity changes from spacelike to timelike).

A detailed investigation of the possible types of dust for all Bianchi types of the form G_4 — on V_3 yields the results shown in Table I [5]. In this table S means spacelike four-

TABLE I

Form of the four-velocity for G_4 on V_3 gravitational fields

Bianchi type of the gravitational field	Form of the four-velocity
G_4I	S
G_4II	(No gravitational field)
G_4III	T
G_4IV	S
G_4V	S, T
G_4VI$_1$	S
G_4VI$_2$	(No gravitational field)
G_4VI$_3$	S
G_4VI$_4$	S, T
G_4VII	S, T
G_4VIII	S, T

-velocity and T means timelike four-velocity. Table I shows that for the Farnsworth G_4V we have both spacelike and timelike cases whereas for G_4IV we have only the spacelike case. One of the interesting results from this study is the preponderance of spacelike cases. Tachyons have not been found in laboratory experiments, however, from our study of the gravitational field equations we find many such solutions. Our results do not prove that tachyons will be found to be important in physics, however, they do show that from one point of view, (Table I) they are preferred.

An investigation of several types of sources in Bianchi types G_4 — on V_3^* shows that often these spacetimes often do not have solutions to the Einstein equations for most of the simple sources mentioned earlier [7]. For example G_4VII on V_3^* has no solutions for: perfect fluids, massless scalar fields, massive scalar fields, massless vector fields, massive vector fields, massless spinor fields, massive spinor fields, electromagnetism, massless scalar fields plus dust, electromagnetism plus dust, massless scalar fields plus electromagnetism, and any other source with a traceless energy-momentum tensor. Other Bianchi types in this same class also do not have solutions for these cases [8]. The reasons that these Bianchi types do not allow solutions to the Einstein equations are not clear. It is possible that other gravitational field equations would allow solutions for these cases.

It is interesting to compare these results with electromagnetism. Maxwell's equations can be written in the form

$$F^{ij}_{;j} = 4\pi j^i, \qquad F_{ij} = A_{j,i} - A_{i,j}.$$

For any choice of the potential A_i we can calculate the current j^i which will produce this electromagnetic field. Thus, Maxwell's equations will always be compatible with any symmetry assumptions on the potential A_i. Thus, the property of the gravitational field equations discussed in the previous paragraph does not have a counterpart in electromagnetism.

It is interesting that a certain symmetry type G_4 — on V_3^* is apparently not compatible with the Einstein equations. It is possible that this same type of behaviour could also be true for "quantum gravity". Since we have found the classical equations are incompatible with certain symmetries, the equations of "quantum gravity" might also have this property. Thus, one might worry about a priori assumptions on the geometry in trying to develop "quantum gravity", in particular in "quantum cosmology" where one starts with particular (but reasonable) Bianchi types.

In general the spacetimes we have considered in detail in this paper are simple enough to correspond to cosmological solutions. Our interest in these simple Bianchi types is, however, to try to understand the inherent possibilities in the gravitational field equations. As long as the general solution to these equations in unknown one must proceed in a manner similar to that described in this paper.

When we look for solutions of the Einstein equations with a massive vector field as the source we are asking how gravity fits together with massive vector fields. (Of course, nature knows the answer since both vector mesons and gravity do exist.) In our work to date on Bianchi type gravitational fields we believe we have found some interesting heretofore unknown results. Of these results the preponderance of tachyon solutions and the incompatibility of the Einstein equations with certain symmetries deserve special mention.

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