BOUNDS ON THE DERIVATIVES OF THE PION ELECTROMAGNETIC FORMFACTOR*

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Using certain theorems from the theory of functions a single complex variable, we obtain upper bounds on the derivatives of the pion electromagnetic formfactor at zero momentum-transfer.

The purpose of this paper is to obtain the maximum amount of information on the pion formfactor using analyticity of $F(t)$ in the circle $|t| < 4\mu^2$.

We assume [1] that $F(t)$ is analytic in $|t| < 4\mu^2$ and continuous on $|t| = 4\mu^2$. In addition, $F(t)$ has a branch point at $t = 4\mu^2$ [1]. We make use of the following theorem [2]: If

$$G(z) = \sum_{N=0}^{\infty} a_N z^N \tag{1}$$

is analytic in $|z| < 1$, continuous on $|z| = 1$ and if $|G(z)| \leqslant M$ for $|z| \leqslant 1$, then,

$$M|a_N| < M^2 - |a_0|^2. \tag{2}$$

If we let $z = t/4\mu^2$ and define $G(z) = F(t)$, then,

$$a_N = (4\mu^2)^N F^{(N)}(0)/N! \tag{3}$$

Since $F(0) = 1$[1], we obtain the following bounds on the derivatives of $F(t)$ at $t = 0$,

$$|F^{(N)}(0)| \leqslant (N!M)/(4\mu^2)^N[1-(1/M^2)]. \tag{4}$$

The charge radius is defined as $R^2 = 6F^{(1)}(0)$, thus,

$$R^2 < (3M/2\mu^2)\,[1-(1/M^2)]. \tag{5}$$

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[1] The pion form factor, $F(t)$, is normalized such that $F(0) = 1$.

The results given in Eqs (4) and (5) are very interesting and important, in that, upper bounds on all the derivatives of the formfactor, at zero momentum transfer, have been obtained. These bounds depend on only one parameter, the maximum of the modulus of $F(t)$ on the circle $|t| = 4\mu^2$.

If we use the additional piece of information that $F(t)$ has a branch cut from $4\mu^2$ to infinity along the positive t-axis, then one may reduce the upper bound on R^2 by a factor of 1/4 [4]. Consequently, we obtain the result,

$$R^2 < (3M/8\mu^2)\,[1-(1/M^2)]. \tag{5'}$$

Since the pion is the hadron with the smallest mass, we expect the bound given in Eq. (5) to hold for the formfactors of all hadrons. This follows from the fact that a given hadron--antihadron pair can always couple to a pion-antipion. Thus, in calculating Im $F(t)$, for $t > 0$, the lowest threshold, t_0, will correspond to the two-pion state, i. e., $t_0 = 4\mu^2$.

We conclude with the following remarks: (*i*) the results of this paper do not depend on any assumptions concerning the asymptotic behaviour of the formfactor in the complex t-plane. If, in fact, $F(t)$ satisfies a dispersion relation with N-subtractions, where $N > 1$, then Eq. (4) may be used to constrain these subtraction constants. (*ii*) These bounds, also, remain true even if the formfactor does not satisfy dispersion relations. The results of this paper follow merely from analyticity in the complex cut t-plane [5].

REFERENCES

[1] M. Gourdin, in *Lectures in Theoretical Physics*, Vol. XII-B, edited by K. T. Mahanthappa and W. E. Brittin, New York 1971, p. 245–361.

[2] Zeev Nehari, *Conformal Mappings*, New York 1952, p. 172, problem 9.

[3] Zeev Nehari, *Conformal Mappings*, New York 1952, p. 101, problem 3.

[4] The author was informed of this result by Prof. Ronald Leach (Howard University). See Zeev Nehari, *Conformal Mappings*, New York 1952, p. 188, problem 3 and p. 213–214.

[5] The following papers give additional results on the charge radius of the pion: V. Baluni, Nguyen Van Hieu, V. A. Suleimanov, *Sov. J. Nucl. Phys.* **9**, 366 (1969); V. Baluni, *Teor. Mat. Fiz.* **10**, 19 (1972).