LETTERS TO THE EDITOR

THE POLYA PROBABILITY DISTRIBUTION AND MEASURE RELATIVE INFORMATION

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(Received August 10, 1976)

The Polya distribution is derived from the minimum information principle, using the measure relative information defined by Kossakowski.

Many important probability distributions (e. g. the equal probability distribution, the geometrical and the Gauss distribution) may be derived from the minimum-information or maximum-entropy principle [1] by maximizing the Shannon missing information under suitable constraints.

Also the Poisson distribution can be obtained, as done by Ingarden and Kossakowski [2], from the maximum entropy principle, if one considers instead of the usual Shannon definition, a special case of the general information defined by Kossakowski [3]: the measure relative information.

The aim of the present note is to show that, following Ingarden and Kossakowski, also the Polya distribution [4] can be easily derived.

Let us consider a probability distribution p_n (where n indicates the n^{th} event). The measure relative information reads

$$S(p, m) = -\sum_{n=0}^{\infty} p_n \ln \frac{p_n}{m_n}. \tag{1}$$

The measures m_n in (1) are „weights" or a priori probabilities assigned to the n^{th} event. (We note that when all m_n are equal to 1, this expression reduces to Shannon missing information.)

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Ingarden and Kossakowski assumed

$$m_n = \frac{1}{n!}, \tag{2}$$

and the Poisson distribution was then obtained as the one which maximizes (1), subject to the constrains

$$\sum_{n=0}^{\infty} p_n = 1, \tag{3}$$

(which of course follows from the definition of probability) and

$$\sum_{n=0}^{\infty} n\, p_n = \langle n \rangle. \tag{4}$$

Let us take, instead of (2)

$$m_n = \frac{(g+n-1)!}{(g-1)!\, n!}. \tag{5}$$

The physical meaning of our assumption can be understood as follow: let us consider as the n^{th} event registration by a counter or the emission (say in a high energy multiproduction) of n identical particles which are distributed among g „cells" or subspaces of their phase space. Expression (5) gives the number of different ways in which the n indistinguishable particles can be distributed among the cells, i. e. the number of ways (complexions) in which the n^{th} event can be realized.

It is then natural, regarding all the complexions as equiprobable, to assume (5) as „weights" or a priori probability to assign to the n^{th} event. Let us now look for a probability distribution p_n which maximizes (1) with m_n defined as in (5), and requiring the constraints (3) and (4) to be satisfied.

It can be done easily using Lagrange multipliers and maximizing the expression

$$I = -\sum_{n=0}^{\infty} p_n \ln \frac{p_n}{m_n} + \lambda \sum_{n=0}^{\infty} n p_n + \mu \sum_{n=0}^{\infty} p_n. \tag{6}$$

By elementary calculation one obtains

$$p_n = \frac{(g+n-1)!}{(g-1)!\, n!} \exp\,[\mu - 1 + \lambda n], \tag{7}$$

where the values of the Lagrange multipliers λ and μ are to be chosen so as to satisfy (3) and (4). The final result is

$$p_n = \frac{g^g \langle n \rangle^n}{(g+\langle n \rangle)^{g+n}} \frac{(g+n-1)!}{(g-1)!\, n!}, \tag{8}$$

i. e. the Polya (or partially degenerate-boson-gas) distribution.

We observe that the expression (8) approaches a Poisson distribution when g is very large and an exponential distribution when $g \to 1$.

We note moreover that the Polya formula (8) proved to be very useful for example as a photoelectric counting distribution [5] and also as a multiplicity distribution for multiparticle production processes at high energies [6]. In conclusion we remark that, though the result we obtained may appear rather academic, the method of Ingarden and Kossakowski followed here seems very fruitful and can be easily extended to other physical situations.

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