CANONICAL QUANTIZATION ON GENERAL SPACELIKE HYPERSURFACES

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A Schroedinger type quantization of free fields on arbitrary spacelike hypersurfaces in curved space-time is considered. The gravitational field is assumed to be a classical background. In the field representation the state functional depends on the field configuration on the hypersurface. A generalized Tomonaga-Schwinger equation describes the dynamical evolution of the quantum field. The expectation values of the field operators (e. g., energy-momentum tensor) are defined.

1. Introduction

Using the Heisenberg picture we quantized the Maxwell field in an external gravitational field [9]. In the present paper we are treating the quantum field theory in a prescribed curved space-time from a different point of view. Schweber [1] writes: "The fact that second-quantized formulation can be considered as a quantized field theory suggests that besides the particle description there should also be a *field* description". Tomonaga [2], Schwinger [3], and Dirac [4] formulated quantum theory with states defined on general spacelike hypersurfaces in the Minkowski space-time. In the canonical quantization of gravity (for a survey see [5]) this concept of states defined on hypersurfaces has fundamental importance. In this spirit we introduce a state functional which gives us the probability to measure a certain configuration of the considered matter field on a particular hypersurface in the nonflat background space-time. From the principle of path independence [6] the commutation rules (8) are derived. By means of these relations and the dynamical law (15) it follows that the appropriately defined expectation values of the field operators fulfil the corresponding classical equations. As an example we consider the Hermitian scalar field influenced by a Friedman metric.

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2. *Spacelike hypersurfaces*

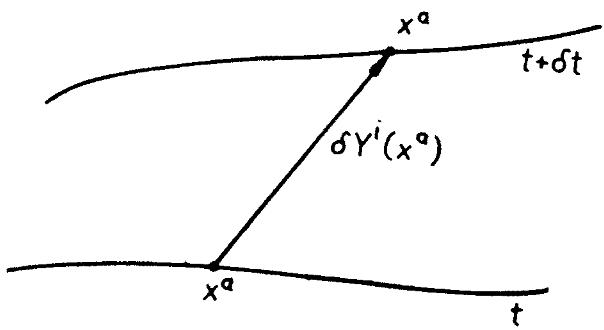
A spacelike hypersurface S (with unit normal vector n^i) embedded in space-time is given by prescribing the space-time coordinates Y^i as functions of the intrinsic coordinates x^a [1]

$$Y^i = Y^i(x^a).$$

The metric tensors are correlated by

$$g_{ab} = Y^i_a Y^j_b g_{ij}, \quad g^{ij} = Y^i_a Y^j_b g^{ab} - n^i n^j, \quad Y^i_a \equiv Y^i_{,a}, \quad n_i Y^i_a = 0.$$

An infinitesimal deformation of the surface is described by a vector field $\delta Y^i(x^a)$ pointing from one point x^a on the original surface to the point with the same label x^a on the slightly deformed surface (fig.).



Choosing a special coordinate system we can parametrize some family of hypersurfaces by $x^4 = t =$ constant;

$$n_i = -n\delta^4_i, \quad n \equiv \frac{1}{\sqrt{-g^{44}}}, \quad g_{ij} = \begin{pmatrix} g_{ab} & N_a \\ N_b & -n^2 + N_a N^a \end{pmatrix}. \tag{1}$$

All covariant equations can be written in a form containing only "kinemetrical invariants" [11] with respect to the restricted group of transformations

$$x^{a'} = x^{a'}(x^b, x^4), \quad x^{4'} = x^{4'}(x^4).$$

Expressed by its components in the adapted coordinate system (1) the infinitesimal deformation reads

$$\delta Y^i = (Y_a{}^i N^a + n^i n)\delta t. \tag{2}$$

The functions n and N^a are connected with the normal and tangential deformations, respectively.

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[1] Indices: $i, j = 1$–4 (space-time), $a, b, c = 1$–3 (hypersurface).

3. *Canonical formalism*

We consider classical fields U_A with the Lagrangian

$$\mathscr{L} = \mathscr{L}(U_A, U_{A,i}, g_{ij}). \tag{3}$$

Defining the momentum π^A canonically conjugate to U_A in the usual manner, we obtain for the action of the matter field[2]

$$S = \int L dt, \quad L = \int (\pi^A \dot{U}_A - N^a \mathscr{T}_a - n\mathscr{T}) d^3x, \quad \pi^A = \frac{\partial \mathscr{L}}{\partial \dot{U}_A}, \tag{4}$$

$$\mathscr{T}_a \equiv Y^i_a \mathscr{T}_i, \quad \mathscr{T} \equiv n^i \mathscr{T}_i, \quad \mathscr{T}_i \equiv -\sqrt{g}\, T_i^{\,j} n_j,$$

where T_{ij} is the symmetrical energy-momentum tensor. In the case of the Lagrangian (3) the following relations hold:

$$\frac{\partial \mathscr{T}_a}{\partial g_{bc}} = 0, \quad \frac{\partial \mathscr{T}}{\partial g_{ab}} = \tfrac{1}{2} \mathscr{T}^{ab}, \quad \mathscr{T}_{ab} \equiv \sqrt{g}\, T_{ab},$$

provided that $\mathscr{T}$ and $\mathscr{T}_a$ are considered as functions of the canonical momenta. The field equations have the Poisson bracket form

$$U_A = [U_A, H], \quad \pi^A = [\pi^A, H], \tag{5}$$

and the infinitesimal change of some functional F under (2) is

$$\delta F = [F, H]\delta t + \int \frac{\delta F(x)}{\delta g_{ab}(x')} \delta g_{ab}(x') d^3x', \quad F = F(U_A, U_{A,a}, \pi^A, g_{ab}). \tag{6}$$

The Hamiltonian H is immediately obtainable from (4),

$$H \equiv \int (n\mathscr{T} + N^a \mathscr{T}_a) d^3x = \int T^4_4 \sqrt{g}\, n d^3x, \tag{7}$$

in accordance with [8]. Starting with given values U_A and π^A on an initial hypersurface S_1 and solving the dynamical equations (5), we get the values U_A and π^A on a final hypersurface S_2 independently of the special slicing of space-time between S_1 and S_2. From this "principle of path independence" one can derive the commutation rules for $\mathscr{T}$ and $\mathscr{T}_a$:[3]

$$[\mathscr{T}(x), \mathscr{T}(x')] = \mathscr{T}^a(x)\delta_{,a}(x, x') + \mathscr{T}^a(x')\delta_{,a}(x, x'),$$

$$[\mathscr{T}_a(x), \mathscr{T}_b(x')] = \mathscr{T}_a(x')\delta_{,b}(x, x') + \mathscr{T}_b(x)\delta_{,a}(x, x'),$$

$$[\mathscr{T}_a(x), \mathscr{T}(x')] = \mathscr{T}(x)\delta_{,a}(x, x') + \mathscr{T}^b_a(x')\delta_{,b}(x, x'). \tag{8}$$

The external field g_{ab} is not included among the dynamical variables which are used in evaluating the Poisson brackets. In order to derive the relations (8) we have therefore to

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[2] Apart from a sign in $\mathscr{T}$ the same definitions as in [9] are used.

[3] $F(x) = \int F(x')\, \delta(x, x')\, d^3x'$: The δ-function is by definition a 3-density with respect to its second argument.

generalize the method in [7]. Two infinitesimal deformations D_1 and D_2 are applied one after the other. If they are performed in reversed order, one arrives in general at another hypersurface. That difference has to be compensated by a further deformation D_3. For the both paths (characterized by D_2D_1 and $D_3D_1D_2$) leading from the same initial hypersurface to the same final hypersurface we compute the dynamical evolution according to the law (6) up to terms of second order in the deformations. Because of the path independence the results must agree. This consistency between field dynamics and kinematics of sliced space-time gives rise to the commutation rules (8). They are automatically fulfilled in the classical theory and should also be valid in an acceptable quantum theory (with commutators in place of Poisson brackets). The "structure constants" on the right-hand side of (8) are universal in the sense that they are the same ones for all fields. However, they depend on the spatial metric.

4. *Integral conserved quantities*

If the space-time admits a group of motions G_r with r Killing vectors $K^i_\mu(\mu = 1-r)$ and the structure constants $C^\varrho_{\mu\nu}$,

$$K^i_\mu K^j_{\nu,i} - K^i_\nu K^j_{\mu,i} = C^\varrho_{\mu\nu} K^j_\varrho, \tag{9}$$

we have r integral conserved quantities

$$E_\mu = -\int K^i_\mu T_i^j df_j.$$

Splitting the Killing vectors into parts normal and tangent to the surface,

$$K_\mu^{\ i} = Y^i_a K^a_\mu - n^i K_\mu,$$

we get from the definitions in equation (4)

$$E_\mu = \int (K^a_\mu \mathscr{T}_a - K_\mu \mathscr{T}) d^3x. \tag{10}$$

The relation

$$[E_\mu, E_\nu] = C^\varrho_{\mu\nu} E_\varrho \tag{11}$$

has been derived in [10]. We obtain it without lengthy calculations when we start from the basic relations (8). For this purpose we write down the kinemetrically invariant form of the Killing equations,[4]

$$K_{\mu(i;j)} = 0 : \begin{cases} K_{\mu(a;b)} = \frac{1}{2}\, \partial_4 g_{ab} K_\mu \\ \partial_4 K^a_\mu = -n(n^{-1}K_\mu)^{,a} \\ \partial_4 K_\mu = n_{,a} n^{-1} K_\mu^{\ a}, \end{cases} \tag{12}$$

[4] In the 3-covariant form, metric operations are performed with g_{ab}. The symbol ∂_4(= invariant time derivative) has been defined in [9].

with the aid of which we verify the relations

$$2(K^a_{[\mu}K^b_{\nu],a} + K_{[\mu}K_{\nu]}{}^{,b}) = C^\varrho_{\mu\nu}K^b_\varrho,$$
$$2K^a_{[\mu}K_{\nu],a} = C^\varrho_{\mu\nu}K_\varrho \tag{13}$$

following from the structure relations (9). In order to verify the result (11) we evaluate the Poisson brackets of the integrals (10) using the equations (8), (12), and (13).

5. *Transition to the quantum field theory*

We choose Schroedinger picture and field representation. In the present paper, we restrict ourselves to boson fields. In analogy to the quantum mechanics we substitute ($\hbar = 1$)

$$[,] \to -i[,]_-, \quad U_A(x), \pi^A(x) \to U_A(x), -i\frac{\delta}{\delta U_A(x)} \tag{14}$$

representing the usual canonical commutation relations for boson fields. We introduce the probability amplitude or state functional Ψ which is a functional of the field configuration $U_A(x)$ and depends on the selected hypersurface, symbolically

$$\Psi = \Psi\{U_A(x), Y^i(x)\}.$$

The state is defined on an arbitrary spacelike hypersurface. When we deform the hypersurface by δY^i, the state functional changes its value. We postulate the fundamental dynamical law (generalized Tomonaga–Schwinger equation)

$$\boxed{i\frac{\delta\Psi}{\delta Y^i(x)} = \mathcal{T}_i(x)\Psi} \tag{15}$$

which describes the change of state associated with *local* deformations. Choosing a family of hypersurfaces t = constant we derive from (15) the Schroedinger equation

$$i\frac{\partial\Psi}{\partial t} = H\Psi, \quad H = \int \mathcal{T}_4 d^3x \tag{16}$$

with the Hamiltonian (7). The constraints in the parametrized theory [4] suggest the structure of the dynamical law (15). To test this equation one has first of all to check the compatibility condition

$$\frac{\delta\mathcal{T}_i(x)}{\delta Y^j(x')} - \frac{\delta\mathcal{T}_j(x')}{\delta Y^i(x)} - i[\mathcal{T}_i(x), \mathcal{T}_j(x')]_- = 0 \tag{17}$$

following from the permutability of the functional derivatives. The condition (17) is proved to be identically satisfied because of the commutation rules (8), in which the Poisson brackets are now substituted by commutators according to (14). For the calculations we use the formulas [7]:

$$\frac{\delta Y_a^i(x)}{\delta Y^j(x')} = \delta_j^i \delta_{,a}(x, x'),$$

$$\frac{\delta g_{ab}(x)}{\delta Y^i(x')} = g_{jk,i} Y_a^j Y_b^k \delta(x, x') + g_{ij}(Y_a^j \delta_{,b}(x, x') + Y_b^j \delta_{,a}(x, x')),$$

$$\frac{\delta n^j(x)}{\delta Y^i(x')} = g^{ab} Y_b^j n_i \delta_{,a}(x, x') - g_{kl,i} n^k (g^{jl} + \tfrac{1}{2} n^j n^l)\delta(x, x'),$$

the rules for the delta function, and the equations (4). The non-commuting field operators have to be ordered in such a way that the basic relations (8) are valid in the quantum field theory. This can be achieved by symmetrizing the products of operators in linear field theories.

6. *Expectation values*

In accordance with the probability interpretation we normalize Ψ in the usual manner,[5]

$$\int \Psi^* \Psi dU = 1.$$

The norm is conserved under hypersurface deformations because of the generalized Tomonaga–Schwinger equation (15). The expectation values of the field operators are defined like in quantum mechanics, but the infinite number of degrees of freedom in field theory is reflected by functional integration instead of ordinary integration over the configuration space. The expectation values

$$\overline{U}_A \equiv \int \Psi^* U_A \Psi dU, \quad \overline{\pi}^A \equiv -i \int \Psi^* \frac{\delta \Psi}{\delta U_A} dU$$

satisfy the classical field equations (e.g., Klein–Gordon equation). This statement is analogous to Ehrenfest's theorem in quantum mechanics. Thus, the classical field equations are derivable from the basic equations of the theory. The expectation values of the energy--momentum tensor are important because the quantum field could influence the classical gravitational field via Einstein's equations

$$R_{ij} - \tfrac{1}{2} R g_{ij} = \kappa \overline{T}_{ij}, \tag{18}$$

if $\overline{T}_{ij}$ fulfills the condition

$$\overline{T}^{ij}{}_{;j} = 0: \begin{cases} -\partial_4 \overline{\mathscr{T}} + n^{-2}(n^2 \overline{\mathscr{T}}^a)_{,a} + \tfrac{1}{2} \overline{\mathscr{T}}^{ab} \partial_4 g_{ab} = 0 \\ \partial_4 \overline{\mathscr{T}}_a + n^{-1}(n \overline{\mathscr{T}}_a{}^b)_{;b} - n^{-1} n_{,a} \overline{\mathscr{T}} = 0. \end{cases} \tag{19}$$

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[5] The notation $\int \ldots dU$ means functional integration.

The kinemetrically invariant equations (19) have been given in [9]. The space-time components of the tensor $\overline{T}_{ij}$ are

$$\overline{T}^{ij} = Y_a^i Y_b^j \overline{T}^{ab} + 2n^{(i} Y_a^{j)} \overline{T}^a - n^i n^j \overline{T}.$$

Using the Schroedinger equation (16) we have, for instance,

$$\partial_4 \overline{\mathcal{T}} = -in^{-1} \int \Psi^* [\mathcal{T}, H]_- \Psi dU + \int \Psi^* \partial_4 \mathcal{T} \Psi dU.$$

With the aid of the commutation relations (8) we are then able to verify the energy-momentum law (19) explicitly. The expectation values $\overline{T}_{ij}$ on a fixed space-time point P are not changed by deformations of the hypersurface through P on other points, for in that case the local operators $\mathcal{T}_i(x')$ in (15) commute with $T_{ij}(P)$.

7. *Scalar field*

As an example we consider the Hermitian massless scalar field U with the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2} \sqrt{^4 g}\, U_{,i} U^{,i}.$$

Expressed by the canonical momenta, the operator $\mathcal{T}$ and $\mathcal{T}_a$ are

$$\mathcal{T} = \tfrac{1}{2} \left(\frac{\pi^2}{\sqrt{g}} + \sqrt{g}\, U_{,a} U^{,a} \right), \qquad \mathcal{T}_a = \tfrac{1}{2} (\pi U_{,a} + U_{,a} \pi).$$

They have real expectation values,

$$\overline{\mathcal{T}} = \frac{\sqrt{g}}{2} \int \Psi^* \left(U_{,a} U^{,a} - \frac{\delta^2}{g \delta U^2} \right) \Psi dU, \qquad \overline{\mathcal{T}}_a = \frac{i}{2} \int \left(\frac{\delta \Psi^*}{\delta U} U_{,a} \Psi - \Psi^* U_{,a} \frac{\delta \Psi}{\delta U} \right) dU. \quad (20)$$

The classical Klein–Gordon equation is completely equivalent to the first order system of equations

$$\partial_4 U = \frac{\pi}{\sqrt{g}}, \qquad \partial_4 \pi = \sqrt{g}\, n^{-1} (n U_{,a})^{,a},$$

which is also valid for the expectation values $\overline{U}$ and $\overline{\pi}$ in the quantum theory.

We calculate the expectation values (20) for a special state of the scalar field in an external Friedman metric

$$ds^2 = -dt^2 + b^2(t) h_{ab} dx^a dx^b$$

with spaces t = constant of constant positive curvature. The function $U(x)$ can be expanded in terms of real spherical harmonics S_{nl}[6]

$$U = \sum_{n,l} a_{nl} S_{nl}, \qquad \Delta S_{nl} = -n(n+2) S_{nl}, \qquad n = 0, 1, 2, \ldots, \qquad l = 1 \ldots (n+1)^2.$$

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[6] n, l are no tensor indices here; l labels the various eigenfunctions which belong to a fixed n (degeneracy). Δ = Laplacian with respect to h_{ab}.

From a general theorem for spherical harmonics [12] follow the useful formulas

$$2\pi^2(n+1)^{-2}\sum_l S_{nl}^2 = 1, \qquad 2\pi^2(n+1)^{-2}\sum_l S_{nl,a}S_{nl,b} = \frac{n(n+2)}{3}\, h_{ab}.$$

Owing to the orthonormalization of the functions S_{nl} the expansion of the functional derivative is given by

$$\frac{\delta}{\delta U} = \sqrt{h}\sum_{n,l} S_{nl}\frac{\partial}{\partial a_{nl}}.$$

The probability amplitude is now a function (not a functional) of all the expansion coefficients a_{nl} and of the cosmic time t. The Schroedinger equation is separable; Ψ is assumed to be an infinite product of functions Ψ_{nl} satisfying the partial differential equation

$$i\frac{\partial \Psi_{nl}}{\partial t} = -\frac{1}{2b^3}\frac{\partial^2 \Psi_{nl}}{\partial a_{nl}^2} + \frac{b}{2}\, n(n+2)a_{nl}^2\Psi_{nl}$$

for each pair (n, l). The state should have the same symmetry as the background space--time, consequently, Ψ_{nl} (n fixed) must depend on all coefficients a_{nl}, $l = 1 \ldots (n+1)^2$, in the same manner. The form of a Gauss distribution

$$\Psi_{nl} = w_n \exp\left(-\tfrac{1}{2}\lambda_n a_{nl}^2\right), \qquad \lambda_n + \lambda_n^* > 0 \tag{21}$$

(w_n normalization factor) is maintained in the course of time, but the functions λ_n and w_n are time-dependent. The Schroedinger equation demands

$$i\dot{\lambda}_n - b^{-2}\lambda_n^2 + bn(n+2) = 0. \tag{22}$$

For the special state (21) the expectation values (20) are

$$\overline{\mathscr{T}} = \sqrt{g}\,(2\pi b)^{-2}\sum_n \frac{(n+1)^2}{\lambda_n+\lambda_n^*}\left[n(n+2)+b^{-4}\lambda_n\lambda_n^*\right], \qquad \overline{\mathscr{T}}_a = 0, \tag{23}$$

$$\overline{\mathscr{T}}_{ab} = \sqrt{g}\, g_{ab}(2\pi b)^{-2}\sum_n \frac{(n+1)^2}{\lambda_n+\lambda_n^*}\left[\frac{n(n+2)}{3} - \frac{\lambda_n\lambda_n^*}{b^4}\right].$$

The energy-momentum law (19), reducing in the Friedman metric simply to

$$\dot{\overline{\mathscr{T}}} = \tfrac{1}{2}\,\dot{g}_{ab}\overline{\mathscr{T}}^{ab},$$

can be verified by means of the equation (22).

The simultaneous system (22) and

$$\dot{b}^2+1 = \frac{\kappa}{3}\, b^2\overline{T}$$

with $\overline{T}$ given in (23) is equivalent to the Einstein–Schroedinger equations (16), (18) in the case under consideration. The infinite sum of positive expressions in $\overline{T}$ diverges whatever may be the behaviour of λ_n for large n. Regularization or renormalization methods [13] are therefore necessary in order to obtain regular energy-momentum expectation values.

8. Conclusions

The consistent basic relations governing the quantum field theory in curved background space-time are the commutation rules (8) and the generalized Tomonaga–Schwinger equation (15). From these postulates we derived:

— the relations (11) in a space-time admitting a group of motions,
— the Schroedinger equation (16) with the Hamiltonian (7) for the time-dependence of the probability amplitude,
— the validity of the compatibility conditions (17),
— the statement that the expectation values satisfy the corresponding classical equations, especially the energy-momentum law (19),
— the statement that the local expectation values do not change when one deforms the hypersurface on other points.

Finally we investigated an example: scalar field in a Friedman metric. The simultaneous system of equations (16) and (18), which we called Einstein–Schroedinger equations, describes the interaction of the classical gravitational field and the quantum field via the energy-momentum expectation values which must be regularized.

The application of the quantization procedure to fermion fields needs further investigations.

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