THE EINSTEIN-CARTAN EQUATIONS IN ASTROPHYSICALLY INTERESTING SITUATIONS. II. HOMOGENEOUS COSMOLOGICAL MODELS OF AXIAL SYMMETRY

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(*Received March 24, 1975*)

The three classes of axially symmetric space-times, corresponding to homogeneous cosmological models which are filled with perfect fluid are studied in the framework of the Einstein-Cartan theory of gravitation. A general set of equations is given with the admissible non-vanishing torsion and spin components compatible with the symmetry. This is further specialized to the "generalized classical description" of spin, and the "classical description" with only one surviving spin tensor component. Closed and open models have to be necessarily shearing while Euclidean models admit also a zero value of shear. A general expression for shear in all classes of models is derived; it is helpful in proving that the singularity may be prevented in the Einstein-Cartan theory thanks to the repulsive spin-spin interaction. This is possible even in those semiclosed models which do not permit any simply transitive group of motions; they may constitute authentically pulsating models with shear and torsion.

1. *On the departure from spherical symmetry in cosmology*

It is our aim to give an account of a systematic application of the Einstein-Cartan theory (Hehl 1973, 1974; Trautman 1973a; Kuchowicz 1975a) to cosmology in order to show explicitly how the singularity can be prevented in it. We deal with spatially homogeneous cosmological models; these models may be characterized by an isometry group with 3, 4 or 6 parameters. The symmetries following from the existence of the Killing vector field in the space-time under study refer not only to the metric tensor g_{ij}, but also to the torsion tensor $Q^{i}{}_{jk}$ and spin angular momentum density tensor $s^{i}{}_{jk}$. This is the standard approach adopted by the Warsaw group (e. g. Kopczyński 1973; Tafel 1973). We adopted it in our three previous studies (Kuchowicz 1975 b, c, d), and we call cosmological models based on it the aligned spin models. The difference between the aligned spin models and the random spin models is explained in two other recent studies (Kuchowicz 1975e, f). Let us mention here that the random spin models are based on the microscopic approach to the Einstein-Cartan equations which is vividly advocated by Hehl's

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group (Hehl et al. 1974; Hehl 1975; von der Heyde 1975). In the latter approach, these equations are looked upon as describing correctly the gravitational properties of matter at a microphysical level, while from some averaging procedure one arrives at Einstein's equations with some correction terms resulting from microscopically fluctuating spin and torsion.

When the isometry group has 6 parameters, this gives us the Robertson-Walker models. Spherical symmetry characteristic for such models cannot be reconciled with the "classical description" of spin for aligned spin models (Kopczyński 1972; Kuchowicz 1975b, c). The Robertson-Walker models admit only a random distribution of spin which is able to prevent a singularity (Kuchowicz 1975f). But when we are interested in aligned spin models, we have to start in the Einstein-Cartan theory with models admitting a 4-parameter isometry group as the models of highest symmetry. Such models have already some degree of anisotropy. When we look at the list of all possible 4-parameter Lie algebras (e. g. in Petrov 1966), we find that each of them has at least one three-parametric subgroup. The latter subgroup may now act either in 3-dimensional subspaces, being thus simply transitive, or in two-dimensional subspaces, being multiply transitive. The second case (of 2-dimensional surfaces of transitivity of the 3-parameter group) corresponds just to axial symmetry which is of interest for us in this paper.

The case of axial symmetry corresponds to the most simple extension beyond the common assumption of a complete homogeneity and isotropy. Now the isotropy is partially violated: two spatial directions are locally equivalent while the third is not so. Such models have been studied in the framework of general relativity by Doroshkevich (1965), Shikin (1966) and Thorne (1967). Exact solutions of the Einstein equations for dust matter have been found by Kantowski and Sachs (1966), and compared with the Friedmann solutions both of open and closed types. Further solutions are given in the thesis of Kantowski (1966). A systematic investigation of the models in case has been made, finally, in general relativity by Vajk and Eltgroth (1970) for the case of a γ-law equation of state: $p = (\gamma-1)\,\varrho$, with $1 \leqslant \gamma < 2$, where p is pressure, and ϱ — energy density; the reader is referred to their paper for further references.

In the most general case, it is possible to resign from any relic of isotropy, and to study completely anisotropic models, as it was done by Jacobs (1968) for Bianchi type I cosmologies. Yet in our systematic search for exact non-singular solutions of the Einstein-Cartan equations, and taking into account the relation between the concept of axial symmetry and of spin, it seems appropriate to start our investigations with the spacetimes of axial symmetry. The number of the components of the torsion and spin density tensors corresponds to the symmetry of the space-time. Later, to provide some practical estimates we reduce their number to the only non-vanishing component which remains in the "classical description" of spin. All our notation, terminology etc. follows closely that of the preceding part (Kuchowicz 1975c), and we are to define only those specifically new quantities which appear for the first time in this part of the paper.

2. *Three types of axially symmetric space-times*

In our study we are dealing with those spatially homogeneous cosmological models which allow for a 4-parametric group of isometries with a multiply transitive 3-parametric subgroup. The properties of these models, within the framework of general relativity, are listed most completely in the thesis of Kantowski (1966). Let us summarize them briefly for our aims.

The two-dimensional orbits of the three-parametric subgroup are locally the sphere, plane and pseudosphere, corresponding to the Lie algebras which are usually applied to describe the Bianchi types IX, VII, and VIII, respectively. Let us denote the three different situations by the symbols: S — spherical geometry, F — flat geometry, H — hyperbolic geometry. Then we have the following four-parametric Lie algebras for the three cases:

$$\mathrm{S}\colon [X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \quad [X_i, X_4] = 0, \tag{2.1}$$

$$F\colon [Z_1, Z_2] = Z_3, \quad [Z_2, Z_3] = 0, \quad [Z_3, Z_1] = -Z_2, \quad [Z_i, Z_4] = 0, \tag{2.2}$$

$$\mathrm{H}\colon [Y_1, Y_2] = Y_1, \quad [Y_2, Y_3] = -Y_3, \quad [Y_3, Y_1] = 2Y_2, \quad [Y_i, Y_4] = 0. \tag{2.3}$$

In each case, the first three generators of the group are acting on a two-dimensional surface. But only in case S the resulting four-parametric algebra (2.1) has no other subalgebra than the initial one (consisting of X_1, X_2, and X_3). In the two other cases there exist three--parametric subalgebras different from the initial one; they are: of Bianchi type I in case F, and of Bianchi type III in case H. Onle the case S is left outside the Bianchi types, as we have to do with such spatially homogeneous models which do not allow for a simply transitive group of motions. In general relativity, the formalism of Heckmann and Schücking (1962) cannot be applied to case S, which was studied extensively by Kantowski (1966) and Kantowski and Sachs (1966).

In spite of the fact that with respect to group properties there exists an essential difference between the "closed" (or rather semiclosed) models of type S, and the "flat" and "open" models (types F and H), it is possible to go forward with quite analogous formulae for these three cases. The common feature of all the models to be studied here is that they have two equivalent "tangential" directions, and one inequivalent "longitudinal" direction at each point in space-time. To be able to treat all these models at equal footing, we attribute the following set of orthogonal basis 1-forms to the three models under study:

	S	F	H
θ^1	$= Xdx$	Xdx	Xdx
θ^2	$= Yd\theta$	Ydy	$Yd\theta$
θ^3	$= Y\sin\theta\, d\varphi$	Ydz	$Y\,\mathrm{sh}\,\theta\, d\varphi$
θ^4	$= dt$	dt	dt

(2.4)

The corresponding Killing vector fields are given in Appendix I. From now on numerical indexes refer to the forms. The independent non-zero components of the torsion tensor $Q^i{}_{jk}$ are (for the three cases under study):

$$A = Q^1{}_{14}, \quad B = Q^1{}_{23}, \quad C = Q^2{}_{12} = Q^3{}_{13}, \quad D = Q^2{}_{13} = -Q^3{}_{12}, \tag{2.5}$$
$$E = Q^2{}_{24} = Q^3{}_{34}, \quad F = Q^2{}_{34} = -Q^3{}_{24}, \quad G = Q^4{}_{14}, \quad H = Q^4{}_{23}.$$

These are the components with the same numerical indexes which are non-vanishing in the case of spherical symmetry (Kuchowicz 1975c), of course, for the identification of the basis 1-forms as given there. The fact that we have no more components now, points to the fundamental role of the axial symmetry while dealing with spin and torsion.

From the algebraic part of the set of the Einstein-Cartan equations we are able to express the independent, non-vanishing components of the spin angular momentum density tensor $s^i{}_{jk}$ in terms of the torsion components:

$$\begin{gathered} 8\pi\tilde{G}s^1{}_{14} = -2E, \quad 8\pi\tilde{G}s^1{}_{23} = B, \\ 8\pi\tilde{G}s^2{}_{12} = 8\pi\tilde{G}s^3{}_{13} = -C-G, \quad 8\pi\tilde{G}s^2{}_{13} = -8\pi\tilde{G}s^3{}_{12} = D, \\ 8\pi\tilde{G}s^2{}_{24} = 8\pi\tilde{G}s^3{}_{34} = -A-E, \quad 8\pi\tilde{G}s^2{}_{34} = -8\pi\tilde{G}s^3{}_{24} = F, \\ 8\pi\tilde{G}s^4{}_{14} = -2C, \quad 8\pi\tilde{G}s^4{}_{23} = H. \end{gathered} \tag{2.6}$$

This set of relations has the same form as in the spherically symmetric case. $\tilde{G}$ denotes here the gravitational constant (in order to distinguish it from the torsion component G). When, for practical aims, we go over later to the "classical description" of spin, there remains only the non-zero component $s^4{}_{23}$, and correspondingly, the torsion component H.

3. The general set of equations

With the 1-forms given by Eq. (2.4) we are going to calculate the connection 1-forms and curvature 2-forms in the same way as we have done it in preceding papers (Kuchowicz 1975b, c). The resulting, rather long expressions are collected in the Appendixes. We use them to give here the final expressions for the remaining Einstein-Cartan equations, apart from the algebraic set (2.6). The underlying physics is much the same as in our preceding investigations, i. e. the symmetric energy-momentum tensor is that of a perfect fluid of energy density ϱ and pressure p, and comoving coordinates are used for matter (with only $u^4 \neq 0$ as the non-vanishing component of the 4-velocity of matter). It is important to emphasize that though the fluid motion is anisotropic, the pressure is isotropic in our models.

With these assumptions, Eq. (1.6) of the preceding part of the paper does not vanish trivially for the following sets of indexes "ij": 11, 22, 33, 44, 14, 41, 23, 32, and we arrive at a resulting set of 6 equations. Three of them are the "diagonal" equations (from the diagonal terms of the energy-momentum tensor):

$$\begin{aligned} 8\pi\tilde{G}\varrho = {} & \frac{\dot{Y}}{Y}\left(2\frac{\dot{X}}{X}+\frac{\dot{Y}}{Y}\right)+\frac{\varepsilon}{Y^2}+2\frac{\dot{Y}}{Y}(A+E)+2\frac{\dot{X}}{X}E \\ & +2AE+\tfrac{1}{4}B^2+BD-C^2+E^2+\tfrac{1}{2}FH+\tfrac{1}{4}H^2, \end{aligned} \tag{3.1}$$

$$\begin{aligned} 8\pi\tilde{G}p = {} & -2\frac{\ddot{Y}}{Y}-\left(\frac{\dot{Y}}{Y}\right)^2-\frac{\varepsilon}{Y^2}-2\left(\dot{E}+2E\frac{\dot{Y}}{Y}\right) \\ & +\tfrac{1}{4}B^2+C^2+2CG-E^2+\tfrac{1}{2}FH+\tfrac{1}{4}H^2, \end{aligned} \tag{3.2}$$

$$8\pi\tilde{G}p = -\frac{\ddot{Y}}{Y} - \frac{\ddot{X}}{X} - \frac{\dot{X}\dot{Y}}{XY} - \dot{A} - \dot{E} - (A+E)\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y}\right)$$

$$-AE - \tfrac{1}{4}B^2 - \tfrac{1}{2}BD + CG + \tfrac{1}{2}FH + \tfrac{1}{4}H^2. \tag{3.3}$$

By subtracting from each other the expressions for the "14" and "41" components we get the algebraic condition

$$EG = AC \tag{3.4}$$

while by summing up these expressions we get the following relation

$$-\dot{C} + C\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right) + G\frac{\dot{Y}}{Y} + 2AC + \tfrac{1}{2}BF + \tfrac{1}{2}BH + \tfrac{1}{2}DH = 0, \tag{3.5}$$

Another algebraic relation, finally, is obtained from the equations for the "23" and "32" components

$$B(2C+G) = H(2E+A). \tag{3.6}$$

The conditions (3.4) ... (3.6) have the same form as in the case of spherical symmetry in the preceding part of the paper. The only difference with the equations (2.11), (2.13) and (2.14) of that paper is the present independence of the quantities in the equations on the radial variable, which is natural in a homogeneous cosmology.

It is natural to demand that the two formulae (3.2) an (3.3) express indetically the same physical quantity — the pressure; from the equality of the right-hand sides of these two formulae we get the pressure isotropy condition

$$\frac{\ddot{Y}}{Y} - \frac{\ddot{X}}{X} + \left(\frac{\dot{Y}}{Y}\right)^2 - \frac{\dot{X}\dot{Y}}{XY} + \frac{\varepsilon}{Y^2} - \dot{A} + \dot{E} - A\left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y}\right)$$

$$+E\left(3\frac{\dot{Y}}{Y} - \frac{\dot{X}}{X}\right) - AE - \tfrac{1}{2}B^2 - \tfrac{1}{2}BD - C^2 - CG + E^2 = 0. \tag{3.7}$$

It is easy to see that for vanishing torsion, our formulae for the S and H cases reduce to the closed and open models of general relativity (Kantowski and Sachs 1966). When we restrict ourselves to the "classical description" of spin, with H as the only surviving torsion component, our general expressions for the flat metrics (case F) reduce to those of Kopczyński (1973) for Bianchi type I models.

4. The set of equations with the condition $s^i{}_{ki} = 0$

The most general set of equations (3.1) ... (3.7) is of low practical use as there are too many torsion components in it to make a physical interpretation easy. Now, a significant reduction of these components which is sufficient for our aims occurs when we take into account one of the conditions which are applied in the "classical description" of spin which we used earlier (Kuchowicz 1975b). This is the reasonable condition

$$s^i{}_{ki} = 0 \tag{4.1}$$

which, with the use of the relations (2.6), makes the following four torsion components to vanish: $A = C = E = G = 0$. Now the conditions (3.4) and (3.6) are automatically satisfied, and the remaining equations may be given the relatively simple form, in terms of the characteristic kinematic quantities of the motion of the cosmological fluid: expansion Θ and shear σ. These quantities are defined in the standard way (see e.g. Ellis 1971), and for our three axially symmetric space-times they turn out to be equal to

$$\Theta = \frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}, \qquad \sigma^2 = \tfrac{1}{3}\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right)^2. \tag{4.2}$$

The density and pressure are expressed in terms of them and of the remaining torsion components

$$8\pi\tilde{G}\varrho = \tfrac{1}{3}\Theta^2 - \sigma^2 + \tfrac{1}{4}B^2 + BD + \tfrac{1}{2}FH + \tfrac{1}{4}H^2, \tag{4.3}$$

$$8\pi\tilde{G}p = -\tfrac{1}{3}(2\dot{\Theta} + \Theta^2) - \sigma^2 + \frac{2}{\sqrt{3}}(\dot{\sigma} + \Theta\sigma) - \frac{\varepsilon}{Y^2} + \tfrac{1}{4}B^2 + \tfrac{1}{2}FH + \tfrac{1}{4}H^2. \tag{4.4}$$

The pressure isotropy condition (3.7) reads now

$$\sqrt{3}\,[\Theta\sigma + \dot{\sigma}] = \frac{\varepsilon}{Y^2} - \tfrac{1}{2}B^2 - \tfrac{1}{2}BD, \tag{4.5}$$

and the remaining algebraic condition (3.5) is now reduced to

$$BF + BH + DH = 0. \tag{4.6}$$

If either B or H is zero, then Eq. (4.6) shows us that there are at most two non-vanishing torsion tensor components, and these may be arbitrary. When either B or H is different from zero, this equation gives us one of the other torsion components in terms of the remaining ones. This equation does not give us any condition in the classical description of spin, when only H differs from zero. In the last case we have an especially simple form of the contracted Bianchi identities

$$\frac{d}{dt}\left[\left(8\pi\tilde{G}\varrho - \frac{H^2}{4}\right)R^3\right] + \left[8\pi Gp - \frac{H^2}{4}\right]\frac{dR^3}{dt} = 0, \tag{4.7}$$

where the length scale R is related through its time derivative to the expansion $\Theta = 3\dfrac{\dot{R}}{R}$ and in view of Eq. (4.2) we have simply $R^3 = XY^2$. R corresponds to the radius function in the Robertson–Walker metrics, and characterizes a direction-averaged rate of change of mutual distances in cosmology at a given instant of time.

When only H remains as the single non-vanishing torsion component, both equations (4.6) and (4.7) are identically fulfilled, and imply no additional constraint. The situation is a little more involved in the general case considered in this section. We do not write out in length the generalization of Eq. (4.7) to this situation, as it is sufficient to point only to some conclusions concerning the admissible number of independent torsion components. Both the contracted Bianchi identities and the algebraic condition (4.6) are fulfilled identically by the following choice of the torsion components (valid within our initial condition $s^i{}_{ki} = 0$): $F \equiv 0$, H and $B = -D$ — arbitrary. In the following we call this the "*generalized classical description*" of spin, as it corresponds just to a generalization of the "classical description" which we considered earlier (Kuchowicz 1975 c). While the torsion component H is proportional to the density of aligned (along the distinguished axis No. 1) spin angular momentum, the other possibly non-vanishing torsion component is locally proportional to the flux of spin angular momentum perpendicular to a 2-surface; this interpretation is evident from the set of equations (2.6).

It might be possible to insist on non-zero F, but this would bring us too far from the classical model of spinning point particles. Within the special relativistic treatment of angular momentum (see e.g. Box 5.6 in Misner et al. 1973), the angular momentum 3-tensor components $s^2{}_{34}$ and $s^3{}_{24}$ (which are proportional to F) are related not to the intrinsic angular momentum but to the centre of mass motion.

In the following we restrict ourselves mostly to the "classical description" of spin, with only $H \neq 0$, as this will be sufficient for our aim which is to look for the possibility of a prevention of the singularity by the H^2 term, resulting from an alignment of spins. Some of the results (e.g. those from the subsequent section) are valid also in the wider context of the "generalized classical description" of spin.

5. *Pressure isotropy condition, and the behaviour of the shear*

Some general features of the axially symmetric models under study follow from a consideration of the pressure isotropy condition (4.5), in which the last two terms (with torsion components only) cancel for our "generalized classical description" of spin. This equation may be now integrated easily to get the following form of the shear

a) for flat models ($\varepsilon = 0$):

$$\sigma = \frac{\sigma_0}{R^3}, \qquad \sigma_0 \text{ — constant,} \tag{5.1}$$

b) for closed or open models ($\varepsilon = \pm 1$):

$$\sigma = \frac{\varepsilon}{\sqrt{3}\, R^3} \int X(t)dt. \tag{5.2}$$

These expressions do not differ in form what may be obtained in general relativity. While the Universe expands, the shear can only gradually diminish in flat models. The behaviour of the shear in other classes of models is complicated by the presence of the integral over the metric function X (corresponding to the spin alignment axis in the Einstein–Cartan cosmology).

Only for the flat models under study we are able to assume that the shear may vanish at all. This would give us the subclass of metrically isotropic models, possessing an axis of spin alignment. Incidentally, the expressions for energy density and pressure of these flat, non-shearing models coincide with the expressions for flat Robertson–Walker models of the third class of extension to the Einstein–Cartan theory, which were studied in the first part of this paper (Kuchowicz 1975 c). Of course, the components of the torsion (though having formally the same indexes in the two papers) are related in each case to the corresponding metric. The length scale X (which equals Y for this subclass) replaces the radius R from the Robertson–Walker metric.

The assumption of vanishing shear would lead to an essential contradiction in Eq. (4.5) with $\varepsilon = \pm 1$. Open and closed models of this paper have to be necessarily shearing. In the framework of general relativity, they have been studied by many authors, where they were shown to be singular. The initial (and, for closed models, also the final) state corresponds to a zero volume and infinite density. Very often, an infinite value of the expansion Θ is accompanied by an infinite value of the shear σ. Flat models with $\sigma_0 \neq 0$ have, within general relativity, of course an infinite initial shear. In general relativity attempts have been made to classify the singularity in anisotropic flat models (Thorne 1967; Jacobs 1968). One distinguishes between point, cigar, and barrel or pancake singularities for the flat metric (2.4): As we approach the singularity, we may have $X \to 0$, $Y \to 0$ (point singularity), or $X \to \infty$, $Y \to 0$ (cigar singularity), $X \to 0$, $Y \to Y_0$ (pancake singularity), or $X \to X_0$, $Y \to 0$ (barrel singularity). We will use these concepts when studying in the following section the behaviour of the integral in the expression (5.2) in relation to the possibility of a prevention of singularities in the Einstein-Cartan theory. It is possible to express here the following statement concerning the behaviour of the shear while approaching the singularity: The integral $\int X(t)dt$ is always finite provided the singularity constitutes an extension (to the non-Euclidean space-time in case) of the poin, pancake or barrel singularity. This feature cannot be generalized straightforward to approaching all possible cigar singularities.

6. *May the singularity be prevented in axially symmetric models?*

It was pointed out by Tafel (1973) in the framework of the "classical description" of spin, that non-singular models of the Bianchi I to VIII types are possible. All these Bianchi types are characterized by either zero or negative curvature scalar, and one may wonder whether the same mechanism of preventing a singularity to occur in the Einstein–Cartan theory could work in a space of positive curvature for which the proof did not apply. Now, our case S corresponds just to a positive curvature scalar (which, for the space part of the metric (2.4) with signature $(+++)$ is equal to $2/Y^2$). It is our aim to show that the case S allows for non-singular cosmological models. The possibility of such models for the two other cases with the "classical description" of spin results from Tafel's proof because the homogeneous spaces of these models are among the Bianchi types I to VIII. It is possible, of course, to prove directly this possibility. We refer here to the proof for flat models which is given elsewhere together with some exact solutions (Kuchowicz 1975 d). In the following, we restrict ourselves to the case of non-Euclidean models which might be treated in parallel.

Let us consider a linear equation of state of the type used by Vajk and Eltgroth (1970)

$$p = (\gamma - 1)\varrho, \qquad \text{with} \qquad 1 \leqslant \gamma < 2. \tag{6.1}$$

For $\gamma = 1$ this gives pressureless dust, for $\gamma = 4/3$ — radiation; the limiting case of $\gamma = 2$ (which we do not consider in this paper) gives us Zel'dovich's stiff matter. We restrict ourselves to the classical description of spin, i.e. we put $F = B = D = 0$ in the expressions for density and pressure. When we insert Eq. (6.1) into the contracted Bianchi identities, we get by a simple integration the two conservation relations (for energy density and for spin density, because H is actually proportional to the only spin component S_{23}):

$$8\pi\tilde{G}\varrho = \frac{A}{R^{3\gamma}}, \quad H = \frac{H_0}{R^3}, \tag{6.2}$$

where A and H_0 are constants. With the use of Eq. (5.2) and (6.2), and of the relation: $\Theta = 3\dot{R}/R$, we get from Eq. (4.3) the following expression

$$A = (3\dot{R}^2R+\varepsilon X)R^{3(\gamma-1)}+\left[\frac{H_0^2}{4}-\tfrac{1}{3}(\textstyle\int X(t)dt)^2\right]R^{3(\gamma-2)}. \tag{6.3}$$

Since A is a positive constant, the right-hand side of this equation should be necessarily constant, at an arbitrary instant of time. Were it not the last term which is due to the spin--spin interaction, such a constancy of the right-hand side could be achieved even at the instant of a singularity (because even in the worst case we had to subtract from each other two infinite quantities, and this could yield us a finite value). Now, the role of the spin-spin term is to make such a procedure impossible, and to show us in this way that a singularity cannot be admitted.

Let us concentrate on the case with $\varepsilon = +1$. If we could succeed in proving that the term in square brackets is always positively defined, this would be sufficient to demonstrate the inadmissibility of the singularity (i.e. of the value $R = 0$), otherwise the whole right-hand side of Eq. (6.3) had to go to infinity with R approaching zero. No possibility of subtracting infinite values remains, as in this case all the other terms in the right-hand side of Eq. (6.3) are non-negative.

The value of the integral $\int X dt$ turns thus to be decisive in the problem of singularity. In the much simpler case of the flat space-time (Kuchowicz 1975 d), we had a shear--induced constant instead of this integral, and we were able to predict that a singularity cannot occur if the constant $H_0^2/4$ arising from the spin-spin interaction is larger than the shear-induced constant. With the latter constant being sufficiently large, the effect of shear could overwhelm that of the spin (and torsion), and a singularity could be possible. An analogous situation can occur for our "closed" models. The expression in the square bracket in Eq. (6.3) is always positively definite provided the inequality holds

$$\frac{H_0}{2}\sqrt{3} > \left|\int_{t_0}^{t} X(t)dt\right| \tag{6.4}$$

which means that the constant H_0^2 should be sufficiently large (but finite). This occurs provided we have $\alpha > -1$ in the leading term of the asymptotic expansion of $X(t)$ near the initial point of the time scale: $X(t) = X_0t^{\alpha}+\ldots$ (as the integral $\int t^{\alpha}dt$ goes then to zero with $t \to 0$, and only the integration constant in the right-hand side of Eq. (6.4) remains). Without the knowledge of any exact "closed" model of axial symmetry in the Einstein–Cartan theory, we are able only to look at the exact solutions of general relativity, and we may say that if some kind of singularity cannot be removed by the spin terms, this might be only a kind of cigar singularity with the exponent $\alpha < -1$; but we do not know of solutions with such "inevitable" singularities,

Though we are unable in the most cases to integrate explicitly the equations (4.3) and (4.4), or Eq. (6.3) to obtain singularity-free generalization of known solutions of general relativity, we can find which of these solutions can be thus generalized to non-singular Einstein-Cartan models. All such solutions for which the metric function $X(t)$ while approaching singularity behaves as t^α, with $\alpha > -1$, allow for a generalization to non-singular solutions. Actually, all "closed" models of Kantowski (1966) and Kantowski and Sachs (1966) satisfy this condition, and can be "regularized" by spin-spin interaction to non-singular models. Let us give two examples:

I. Solution for dust matter (Kantowski and Sachs 1966)

$$X = 1+(\eta+b)\,\mathrm{tg}\,\eta,\ Y = a\cos^2\eta,\ t = a(\eta+\tfrac{1}{2}\sin 2\eta),\ -\frac{\pi}{2} < b \leqslant 0.$$

II. Solution for a radiative universe (Kantowski 1966)

$$X = a\left[\left(\frac{a}{t}\right)^{2/3} - 1\right]^{1/2},\ Y = a\left[1-\left(\frac{t}{a}\right)^{2/3}\right]^{1/2}\left(\frac{t}{a}\right)^{2/3},\ a \neq 0.$$

Both cases correspond to an initial cigar singularity with $\alpha = 1/3$ which may be removed by the spin-spin interaction. In case II the final singularity (for $t = a$) is of a point type, but also may be removed. While in general relativity both these models start and end in a singular state, they may be generalized in the Einstein-Cartan theory to authentically pulsating universes which never go across singular states. The expressions given above have to be modified only for rather short periods of transition through the state of highest (but not infinite!) compression.

The proof for $\varepsilon = -1$ may go along similar lines, but it is not necessary to make it, as the possibility of non-singular open models of axial symmetry follows from Tafel's result (1973).

Now, in our indicating the existence of non-singular models with spin and torsion we relied on the two expressions (6.2) for which the contracted Bianchi identities are identically fulfilled. These relations are, however, no necessary condition in our theory, and it is possible to resign e.g. from the spin conservation relation as we have done it in our earlier studies on spherical symmetry (Kuchowicz 1975 b). Since we are then not allowed to use the expressions (6.2), we have to apply our equation of state of the form (6.1) directly to the expressions which are given by Eq. (4.3) and (4.4). We find easily a first integral of the resulting second-order differential equation for $R(t)$; this is

$$\dot{R}^2R^{3\gamma-2} + \int R^{3\gamma-7}\left[\frac{2-\gamma}{3}\left(\int X dt\right)^2 + \varepsilon(\gamma-\tfrac{2}{3})XR^3 + \frac{\gamma-2}{4}H^2R^6\right]dR = C, \qquad (6.5)$$

where we may consider $X = X(t)$ as some involved function of R, while R depends on t. Now it is possible to prove under what conditions our Eq. (6.5) has a non-singular solution $R = R(t)$. It is sufficient to assume that its solution R is singular, i.e. it approaches zero with $t \to 0$ (it is always possible to shift the initial point of the time scale so that this is true), and to arrive at some contradiction with the constancy of C. Now, near the singularity the following behaviour of X, Y, and R is possible

$$X \simeq X_0 t^\alpha, \quad Y \simeq Y_0 t^\beta, \quad R \simeq R_0 t^{1/3(\alpha+2\beta)}, \quad \alpha+2\beta > 0. \qquad (6.6)$$

Let us assume the following kind of asymptotic behaviour of H: $H = H_0/R^\delta$ (which is a generalization of the inverse R^3 dependence for the spin conservation case). We find that the last term in the square bracket in Eq. (6.5) dominates for t approaching zero provided the positive δ fulfils the inequality

$$\delta > \tfrac{3}{2} + \frac{3\beta}{\alpha+2\beta} \tag{6.7}$$

and that the whole integral term in Eq. (6.5) goes to infinity for $t \to 0$ provided a second inequality holds

$$\delta > \tfrac{3}{2}\gamma + \frac{3}{2(\alpha+2\beta)}. \tag{6.8}$$

Since the first term in Eq. (6.5) is always non-negative, we see that our non-singular solution implies an infinite C which is impossible. Thus we can conclude that when the exponent δ in the expression for spin density exceeds a certain minimum value (i.e. it fulfils simultaneously the two above inequalities), the spin-spin interaction removes the singularity in the Einstein–Cartan theory. As the second term on the right-hand side of the inequality (6.7) may be larger than 3/2, we find that to remove in general a singularity from the cosmological solution it may be sometimes advisable to consider a higher value of δ than the value of three which is used when spin conservation is assumed. Of course, all we give here are only the sufficient conditions for the removal of a singularity, and it is already evident from our preceding analysis of Eq. (6.3) that even with $\delta = 3$ we are able to remove the singularity, at least in some cases.

Thus for all types of non-rotating models of axial symmetry it is possible to prevent a singularity. This statement is essentially new with respect to the case S which constitutes the first non-singular, and hence essentially pulsating type of models. These are at the same time those spatially homogeneous models which do not permit a simply transitive group of motions. It is sufficient to apply the classical description of spin to remove the singularity. This contrasts sharply with the case of closed models with the same 4-parametric group, but permitting a simply transitive 3-parametric subgroup (Kuchowicz, in preparation).

7. Some exact solutions

The non-singular solutions for flat models were given in earlier studies (Kuchowicz 1975 d, e). They are solutions of the equation of state $p = (\gamma-1)\varrho$, which yields after a first integration an equation analogous to Eq. (6.3)

$$\dot{R}^2 R^{3\gamma-2} + \frac{\tilde{A}}{3} R^{3(\gamma-2)} = D, \tag{7.1}$$

where D is an integration constant, and the quantity A is defined in terms of the spin density constant H_0 and shear constant σ_0 (from Eq. (5.1)): $\tilde{A} \stackrel{\text{def}}{=} H_0^2/4-\sigma_0^2$. To admit non-singular quasi-Euclidean solutions, the effect of shear should be overwhelmed by that of the spin and torsion, i.e. $\tilde{A} > 0$. Eq. (7.1) may be interpreted as an energy conservation equation, with the term involving $\tilde{A}$ being interpreted as a "centrifugal potential energy term" resulting from the spin-spin interaction. For $\tilde{A} = 0$ we get non-shearing, for $\tilde{A} < 0$ — shearing models of Bianchi type I, both of the same form as in general relativity, i.e. with a singularity.

Eq. (7.1) admits non-singular solutions for $\tilde{A} > 0$ and $1 \leqslant \gamma < 2$, i.e. for all equations of state of the type (6.1) except of the limiting stiff case, when the velocity of sound in matter is equal to the velocity of light. This situation occurred for spherically symmetric models (Kuchowicz 1975 b). For $\gamma = 2$ the spin terms $(H^2/4)$ and shear terms on both sides of the equation of state $p = \varrho$ cancel, and there remains $-\dot{\Theta} = \Theta^2$, which yields the singular solution $R = R_0 t^{1/3}$ where we have used the second integration constant to have minimum R at the origin of time. Though Eq. (7.2) may be solved in terms of elementary functions only for the following values of γ: $\gamma = \dfrac{2(n+1)}{n+2}$, with $n = 0, 1, 2, \ldots$, and the solutions which are given elsewhere (Kuchowicz 1975 d) are non-singular, the prevention of singularity is no longer essential for sufficiently high values of R, when the models behave asymptotically like general relativistic models

$$R_{\text{as}} \simeq (\tfrac{3}{2}\gamma\sqrt{D}\,t)^{2/3\gamma}. \tag{7.2}$$

We see that for higher γ, corresponding to a more "stiff" equation of state, a lower expansion rate is obtained asymptotically. The same asymptotic behaviour follows for all various kinds of the dependence of spin on R, as the spin terms are essential only for small values of t and of R. While Eq. (7.1) and its exact solutions (Kuchowicz 1975 d) correspond to the "conservation of spin" ($H = H_0/R^3$), we get for the general behaviour $H = H_0/R^{3\gamma}$ (when the spin density decreases faster) and zero shear $\sigma_0 = 0$ the following solution

$$\sqrt{DR^{3\gamma} - \frac{2-\gamma}{12\gamma}H_0^2} = \tfrac{3}{2}\gamma Dt. \tag{7.3}$$

The corresponding formula in the previous study (Kuchowicz 1975 d) was misprinted.

When we have to do with non-shearing models, we are allowed to assume also lower increase of the spin density toward the most compressed stage of evolution of the universe, and still we have no singularity. Let us illustrate this by a dust model ($\gamma = 1$) with $H = H_0/R^2$

$$\sqrt{DR - \tfrac{1}{4}H_0^2}\,(DR + \tfrac{1}{2}H_0^2) = \tfrac{3}{2}D^2 t. \tag{7.4}$$

It is possible to compare two dust universes with $H = H_0/R^m$. We find that the amount of cosmic time in which the given minimum volume is doubled, is larger for a smaller value of the exponent m. This is the consequence of the higher importance of the "centrifugal potential energy" in the case of higher m. After the initial volume has increased by a factor of 10^3, the difference between various values of the exponent m does not play any role any more.

So far we spoke of quasi-Euclidean models with a "classical description" of spin. These models may be generalized by an inclusion of the $B = -D$ terms along the lines given in Sect. 4. A semiclosed model expanding only in the "longitudinal" direction is characterized by the following scale functions

$$X = X_0 \operatorname{ch} \frac{t-t_0}{Y_0}, \qquad Y = Y_0. \tag{7.5}$$

This model is filled by a pressureless dust of energy density ϱ and spin density $s^4{}_{23}$. Spin conservation is not valid. The physical quantities are given by

$$8\pi\tilde{G}\varrho = \frac{2}{Y_0^2}, \qquad 8\pi\tilde{G}s^4{}_{23} = H = \frac{2}{Y_0}. \tag{7.6}$$

The constant values of the physical densities in the overall expanding universe make it unrealistic; nevertheless, it is the simplest case of a strongly shearing ($\sigma^2 = \frac{1}{3}\Theta^2$), non--singular universe. Just the constancy of Y together with the strong shearing makes this models expand forever; it is found (Kuchowicz 1975 g) that only a sufficiently strong magnetic field can make the model oscillate.

We are unable to present any other simple, non-Euclidean non-singular model. Numerical computations, in which the expressions of Kantowski (1966) would be the asymptotic formulae to represent the solutions of our theory very far from the initial stage of minimum radius, should give us the behaviour of such non-singular solutions.

8. *Final remarks*

We put more emphasis on the general problem of preventing the singularity, than on giving sets of specific solutions. For the first time, a possibility of having really pulsating models has been demonstrated. Numerical studies on such models should be the next stage.

It is a pleasant duty to acknowledge the correspondence with Prof. F. W. Hehl from the Technical University in Clausthal as a constant stimulus to the studies on the Einstein--Cartan theory. Thanks are also due to the participants of the cosmological seminar at the Astronomical Observatory of the Jagellonian University, to whom these results were presented first.

APPENDIX I

Killing vectors, and general form of the torsion tensor

We give below the sets of generators of the groups of motion corresponding to the three cases under study:

1) case S: "closed" models with the Lie algebra given by Eq. (2.1);
2) case F: "Euclidean" models with the Lie algebra given by Eq. (2.2);
3) case H: "open" models with the Lie algebra given by Eq. (2.3).

$$X_1 = \frac{\partial}{\partial\varphi}, \qquad X_2 = \sin\varphi\frac{\partial}{\partial\theta} + \operatorname{cotg}\theta\cos\varphi\frac{\partial}{\partial\varphi},$$

$$X_3 = \cos\varphi \frac{\partial}{\partial\theta} - \operatorname{cotg}\theta \sin\varphi \frac{\partial}{\partial\varphi}, \qquad X_4 = \frac{\partial}{\partial x},$$

$$Y_1 = -\cos\varphi \frac{\partial}{\partial\theta} + (\coth\theta \sin\varphi - 1) \frac{\partial}{\partial\varphi},$$

$$Y_2 = \sin\varphi \frac{\partial}{\partial\theta} + (\coth\theta \cos\varphi) \frac{\partial}{\partial\varphi},$$

$$Y_3 = \cos\varphi \frac{\partial}{\partial\theta} - (\coth\theta \sin\varphi + 1) \frac{\partial}{\partial\varphi}, \qquad Y_4 = \frac{\partial}{\partial x},$$

$$Z_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad Z_2 = \frac{\partial}{\partial y}, \qquad Z_3 = \frac{\partial}{\partial z}, \qquad Z_4 = \frac{\partial}{\partial x}.$$

The generators are denoted by the symbols X_i (case S), Z_i (case F) and Y_i (case H): For the three metrics, the fourth generator (X_4, Y_4 or Z_4) generates spatial translations parallel to the axis of symmetry. The three other generators of isometry are acting transitively, in each case, on the two-dimensional subspaces of constant curvature which are locally either the closed sphere, or the plane or the pseudosphere. For $X = Y$ in Eq. (2.4), the three metrics reduce to the known closed, flat and open Friedmann metrics, respectively.

In order to obtain the most general form of the torsion tensor $Q^i{}_{jk}$ (or any other object antisymmetric in the two lower indices) we have to equate to zero the Lie derivative of this quantity with respect to any of the four Killing vector fields in each case. This may be done in a similar way as in Appendix I of the preceding part (Kuchowicz 1975c). As it turns incidentally, that the first three generators X_i are the same now as in the paper above where they applied to the study of spherical symmetry, all the previous information concerning the number of the independent non-vanishing components of $Q^i{}_{jk}$ is valid for our case of axial symmetry in a "closed" universe. In addition, these components have to be independent of r (this results from the occurrence of the fourth Killing vector).

In the cases F and H we have to write the set of equations from the beginning. We give below only the set corresponding to case F

$$-Q^i{}_{2k}\delta^3_j + Q^i{}_{3k}\delta^2_j + Q^i{}_{2j}\delta^3_k - Q^i{}_{3j}\delta^2_k - Q^2{}_{jk}\delta^3_i + Q^3{}_{jk}\delta^2_i = 0,$$

$$\frac{\partial Q^i{}_{jk}}{\partial x} = \frac{\partial Q^i{}_{jk}}{\partial y} = \frac{\partial Q^i{}_{jk}}{\partial z} = 0.$$

The set for the case H is much longer. But the final result of the study of all three cases is the same: There may exist no more than 8 independent non-vanishing components of the torsion tensor, and these are given by Eq. (2.5). When we treat them as components with respect to the 1-forms (and not to coordinates!), they have all to be functions of the cosmic time r only.

APPENDIX II

The connection 1-forms $\omega^i{}_k$

Below we express the connection 1-forms for the case S with respect to the basis 1-forms θ^i:

$$\omega^1{}_2 = -\omega^2{}_1 = C\theta^2 - \tfrac{1}{2} B\theta^3,$$

$$\omega^1{}_3 = -\omega^3{}_1 = \tfrac{1}{2} B\theta^2 + C\theta^3,$$

$$\omega^2{}_3 = -\omega^3{}_2 = (D + \tfrac{1}{2} B)\theta^1 - \frac{\operatorname{cotg}\theta}{Y}\theta^3 - (F + \tfrac{1}{2} H)\theta^4,$$

$$\omega^1{}_4 = \omega^4{}_1 = \left(A + \frac{\dot{X}}{X}\right)\theta^1 - G\theta^4,$$

$$\omega^2{}_4 = \omega^4{}_2 = \left(E + \frac{\dot{Y}}{Y}\right)\theta^2 - \tfrac{1}{2} H\theta^3,$$

$$\omega^3{}_4 = \omega^4{}_3 = \tfrac{1}{2} H\theta^2 + \left(E + \frac{\dot{Y}}{Y}\right)\theta^3.$$

Formulae for the case H differ from the set above by replacing $\operatorname{cotg}\theta$ in the expression for $\omega^2{}_3$ by $\coth\theta$. In the formulae for case F the term $\operatorname{cotg}\theta/Y\theta^3$ is absent, and this is the only difference for that case.

The set for case S may be obtained in a straightforward way from the corresponding set for spherical symmetry (as given in Appendix II of the previous part of this paper).

APPENDIX III

The curvature 2-forms $\Omega^i{}_j$

The only difference between the expressions for the three cases under study appears in the formula for $\Omega^2{}_3$ where we have

$$\varepsilon = \begin{cases} +1 \text{ for case S} \\ \ \ 0 \text{ for case F} \\ -1 \text{ for case H.} \end{cases}$$

We list the formulae:

$$\Omega^1{}_2 = -\Omega^2{}_1 = \left[\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} + A\frac{\dot{Y}}{Y} + E\frac{\dot{X}}{X} + AE + \frac{B^2}{4} + \frac{BD}{2}\right]\theta^1 \wedge \theta^2$$

$$+ \left[-\frac{H}{2}\frac{\dot{X}}{X} - \frac{AH}{2} + \frac{BC}{2} + CD\right]\theta^1 \wedge \theta^3$$

$$+ \left[(G - C)\frac{\dot{Y}}{Y} + \frac{BF}{2} + \frac{BH}{4} - \dot{C} + EG\right]\theta^2 \wedge \theta^4$$

$$+\left[\tfrac{1}{2}\left(\dot{B}+B\frac{\dot{Y}}{Y}\right)+CF+\tfrac{1}{2}CH-\tfrac{1}{2}GH\right]\theta^3\wedge\theta^4,$$

$$\Omega^1{}_3=-\Omega^3{}_1=\left[\frac{H}{2}\frac{\dot{X}}{X}+\frac{AH}{2}-\frac{BC}{2}-CD\right]\theta^1\wedge\theta^2$$

$$+\left[\frac{\dot{X}\dot{Y}}{XY}+A\frac{\dot{Y}}{Y}+E\frac{\dot{X}}{X}+AE+\frac{B^2}{4}+\frac{BD}{2}\right]\theta^1\wedge\theta^3$$

$$-\left[\tfrac{1}{2}\left(\dot{B}+B\frac{\dot{Y}}{Y}\right)+CF+\tfrac{1}{2}CH-\tfrac{1}{2}GH\right]\theta^2\wedge\theta^4$$

$$+\left[(G-C)\frac{\dot{Y}}{Y}+\frac{BF}{2}+\frac{BH}{2}-\dot{C}+EG\right]\theta^3\wedge\theta^4,$$

$$\Omega^2{}_3=-\Omega^3{}_2=-\left[\left(\frac{B}{2}+D\right)\frac{\dot{X}}{X}+\frac{\dot{B}}{2}+\dot{D}\right]\theta^1\wedge\theta^4$$

$$+\left[\frac{\varepsilon}{Y^2}+\frac{\dot{Y}}{Y}\left(\frac{\dot{Y}}{Y}+2E\right)-C^2-\frac{B^2}{4}+E^2+\frac{H^2}{4}\right]\theta^2\wedge\theta^3,$$

$$\Omega^1{}_4=\Omega^4{}_1=-\left[\frac{\ddot{X}}{X}+A\frac{\dot{X}}{X}+\dot{A}\right]\theta^1\wedge\theta^4+\left[B\frac{\dot{Y}}{Y}+BE-CH\right]\theta^2\wedge\theta^3,$$

$$\Omega^2{}_4=\Omega^4{}_2=\left[C\frac{\dot{X}}{X}+AC+\tfrac{1}{2}BD+\tfrac{1}{4}BH\right]\theta^1\wedge\theta^2$$

$$+\left[\tfrac{1}{2}B\left(\frac{\dot{Y}}{Y}-\frac{\dot{X}}{X}\right)+D\frac{\dot{Y}}{Y}-\tfrac{1}{2}AB+\tfrac{1}{2}BE+DE\right]\theta^1\wedge\theta^3$$

$$+\left[-\frac{\ddot{Y}}{Y}-E\frac{\dot{Y}}{Y}+CG-\dot{E}+\tfrac{1}{2}FH+\tfrac{1}{4}H^2\right]\theta^2\wedge\theta^4$$

$$+\left[(F+H)\frac{\dot{Y}}{Y}-\tfrac{1}{2}BG+EF+\tfrac{1}{2}EH+\tfrac{1}{2}\dot{H}\right]\theta^3\wedge\theta^4,$$

$$\Omega^3{}_4=\Omega^4{}_3=\left[\tfrac{1}{2}B\left(\frac{\dot{X}}{X}-\frac{\dot{Y}}{Y}\right)-D\frac{\dot{Y}}{Y}+\tfrac{1}{2}AB-\tfrac{1}{2}BE-DE\right]\theta^1\wedge\theta^2$$

$$+\left[C\frac{\dot{X}}{X}+AC+\tfrac{1}{2}BD+\tfrac{1}{4}BH\right]\theta^1\wedge\theta^3$$

$$-\left[(F+H)\frac{\dot{Y}}{Y}-\tfrac{1}{2}BG+EF+\tfrac{1}{2}EH+\tfrac{1}{2}\dot{H}\right]\theta^2\wedge\theta^4$$

$$+\left[-\frac{\ddot{Y}}{Y}-E\frac{\dot{Y}}{Y}+CG-\dot{E}+\tfrac{1}{2}FH+\tfrac{1}{4}H^2\right]\theta^3\wedge\theta^4.$$

REFERENCES

Doroshkevich, A. G., *Astrofizyka*, **1**, 255 (1965).

Ellis, G. F. R., *Relativistic Cosmology*, in *Proc. International School of Physics "Enrico Fermi", Course XLVII, General Relativity and Cosmology*, Academic Press, New York–London 1971, p. 104.

Heckmann, O., Schücking, E., *Relativistic cosmology*, Chapter 11 in *Gravitation: an Introduction to Current Research*, L. Witten, ed., Wiley, New York 1962.

Hehl, F. W., *Gen. Relativ. and Gravitation* **4**, 333 (1973).

Hehl, F. W., *Gen. Relativ. and Gravitation* **5**, 491 (1974).

Hehl, F. W., v. d. Heyde, P., Kerlick, G. D., *Phys. Rev.* **D10**, 1066 (1974).

Hehl, F. W., *On the energy tensor of spinning massive matter in classical field theory and general relativity*, in print (1975).

v. d. Heyde, P., *Gravitation in microphysics*, preprint (1975).

Jacobs, K. C., *Astrophys. J.* **153**, 661 (1968).

Kantowski, R., *Some relativistic cosmological models*, Dissertation, University of Texas, Austin 1966.

Kantowski, R., Sachs, R. K., *J. Math. Phys.* **7**, 443 (1966).

Kopczyński, W., *Phys. Lett.* **39A**, 219 (1972).

Kopczyński, W., *Phys. Lett.* **43A**, 63 (1973).

Kuchowicz, B., *Acta Cosmologica* **3**, 109 (1975a).

Kuchowicz, B., *Acta Phys. Pol.* **B6**, 173 (1975b).

Kuchowicz, B., *Acta Phys. Pol.* **B6**, 555 (1975c).

Kuchowicz, B., *J. Phys. A: Math. and Gen.* **8**, L 29 (1975d).

Kuchowicz, B., *Astrophys. Space Sci.* **38**, (1975e) in press.

Kuchowicz, B., *Cosmology without singularity*, Essay awarded "Honorable Mention" at the Gravity Research Foundation Annual Essay Competition, 1975f.

Kuchowicz, B., *Astrophys. Space Sci.* (1975g) in press.

Misner, C. W., Thorne, K. S., Wheeler, J. A., *Gravitation*, W. H. Freeman and Co., San Francisco 1973.

Petrov, A. Z., *New Methods in General Relativity*, "Nauka" Publ. House, Moscow 1966, (in Russian).

Shikin, I. S., *Dokl. Akad. Nauk SSSR* **171**, 73 (1966).

Tafel, J., *Phys. Lett.* **45A**, 341 (1973).

Thorne, K. S., *Astrophys. J.* **148**, 51 (1967).

Trautman, A., *Istit. Nazionale Alta Matematica, Symposia Matematica* **12**, 139 (1973a).

Trautman, A., *Nature (Phys. Sci.)* **242**, 7 (1973b).

Vajk, J. P., Eltgroth, P. G., *J. Math. Phys.* **11**, 2212 (1970).