METRIC TENSORS, LAGRANGIAN FORMALISM AND ABELIAN GAUGE FIELD ON THE POINCARÉ GROUP

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All metrics on the Poincaré group which are forminvariant under transformations of the relativistic symmetry as well as spatial rotations of a basis attached to the particle are found. The lagrangian formalism for classical fields on **P** is developed and applied to the abelian gauge field. It is shown that in a particular choice of metric the gauge field has a neutral, massive and 1^- component in addition to the usual electromagnetic field.

1. *Introduction*

Experimental investigations in elementary particle physics imply that the majority of particles have a very complicated internal structure, e. g. elementary particles seem to be spatially extended.

This fact suggests that when constructing a theory of elementary particles as fundamental objects we ought to consider some objects more complicated than material points with some quantum numbers or, in other words, to consider more general fields than the fields on the Minkowski space with some spin and isospin indices, say. Examples of these more realistic fields are nonlocal fields (see e. g. [1]) and fields on manifolds larger than the Minkowski space ([2–9], [13] and recently discussed string models) (we restrict ourselves to linear models).

We are concerned about theories on manifolds larger than the Minkowski space. Their peculiar feature is a simple mathematical picture of the internal structure of elementary particles. These models were discussed in papers [2–9] but in a rather arbitrary and uncomplete way. Our aim is at investigating thoroughly one of such descriptions ([2]) of the internal structure on the level of classical fields and subsequently of quantum fields. In this way we hope to put these models into a correct field theory with a clear physical interpretation and to make their very interesting results reliable.

Now we repeat briefly the main assumptions of the model ([2]) and present some ideas connected with it.

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We assume that the position of a particle in configuration space is given by four-vector $\{x^\mu\}$ referring to the standard Minkowski space M and the orientation of a four dimensional, orhonormal basis $\{e_\lambda\}$. For simplicity, the time is included into configuration space. The orientation of the basis $\{e_\lambda\}$ can be given by pointing out the Poincaré transformation which takes an orthonormal basis connected with an observer O into $\{e_\lambda\}$. Therefore, the configuration space of a particle can be identified with the manifold on the Poincaré group P. Its elements will be denoted by p, or g: $p = (x^\mu, \Lambda)$, $x^\mu \in$ M, $\Lambda \in$ SL(2, C). As usual in a relativistic field theory, instead of the proper orthochronic Lorentz group $L^\uparrow_+$ we will use its universal covering group SL(2, C).

The improper Lorentz transformations will be taken into account separately.

In such description the time depending wave functions of the particle (or adequate classical fields) will be functions (in general multicomponent) on the Poincaré group P.

If the basis connected with the observer O′ goes into the basis connected with the observer O under the Poincaré transformation $g_0 = (a, \Lambda_0)$, then the basis attached to O′ goes into the basis $\{e_\lambda\}$ under the transformation $p' = g_0 p$.

Therefore the transformations of the relativistic symmetry are the left group translations on the Poincaré group. The wave functions can be transformed by $\psi'_{a'}(p') = S^b_{a'}$ $(\Lambda_0, p)\ \psi_b(p)$. Thus, for scalar functions on P we have $\psi'(p') = \psi(p)$, so $\psi'(p = \psi(g_0^{-1}p))$ and we obtain an operation of the left regular representation of P. Our next problem is to find equations of motionf or fields on P. The procedure for writing down an equation of motion is based on the following observation: proceeding analogically as in Sec. 2 we can introduce the metric tensors on group E(3) of motions of the three dimensional Euclidean space. By the lagrangian formalism, constructed analogically as in Sec. 3, we get physical equations of motion, namely the quantum equations for the rigid body in the nonrelativistic or relativistic ([10]) case, depending on the way in which one includes the time. We expect that this procedure, when applied to the group of motions of the Minkowski space, i. e. the Poincaré group, will also yield some physical equations of motion. Other possibilities for equations of motion for fields on P are given in papers [2, 3, 13].

In general, a particle with internal structure has an internal angular momentum described in the chosen model by differential operators with respect to the variables referring to SL (2, C), and also an angular momentum (spin) described by discrete indices referring to (j_1, j_2) representation of SL(2, C) group. Papers [2, 3] try to identify the ordinary spin with the internal angular momentum and to consider only scalar functions on P. Our opinion is that multicomponent functions on P are unavoidable, but we will not discuss this point here.

It is quite probable that the states with a definite value of the internal angular momentum can be identified with some elementary particles or resonances ([2, 3, 5]). This would open the possibility of a theory on P in which the spin would be not only a kinematical feature of a particle but also could play so important dynamical role as momenta.

As it is shown in papers [7], scalar functions on P satisfying eigenvalue equations of the Casimir operators of the group P

$$P^2\psi = m^2\psi,\ W^2\psi = -m^2 s(s+1)\psi,$$

where $W^2 = W_\mu W^\mu$, W_μ is Pauli-Lubański operator, can be obtained from usual, transforming under $(s, 0)$ representation of SL(2, C), $2s+1$ component field on the Minkowski space $\Phi_\lambda(x)$ ([11]) by smearing it with some test functions $f^\lambda(p, x)$

$$\psi(p) = \sum_\lambda \int d^4x f^\lambda(p, x)\Phi_\lambda(x), \qquad x \in \mathrm{M}, \qquad p \in \mathrm{P}.$$

The test functions $f^\lambda(p, x)$ must obey some conditions given in papers [7]. Hence, our approach has a connection with some attempts to describe internal structure of particles by nonlocal fields on the Minkowski space.

Some other very interesting features of field theories on manifolds larger than the Minkowski space are pointed out in papers [6, 8, 13].

The aim of this paper is to show that theories on P can be handled by the lagrangian formalism, to find some physical lagrangians for classical fields on P, and to call the attention to interesting properties of the abelian gauge field on P. The results we have obtained are listed below.

In Section 2 we find a general form of metrics on P which are forminvariant under the transformations of relativistic symmetry as well as rotations around the time axis of the basis attached to the particle. Only some of these metrics have a definite sign of their parts referring to the internal degrees of freedom. We show that on P a metric forminvariant under both left and right group translations does not exist.

In Section 3 the usual lagrangian formalism is extended to the classical fields on *P*. In particular the presence of the Killing vectors of the left regular representation of P makes possible to derive global conservation laws, despite of the curvature of the manifold P. For a very special choice of metric we obtain, as the Lagrange-Euler equation, the equation considered in papers [3, 9] and show that the energy is not definite. This means that the equation is an unphysical one unless one assumes an additional condition. Another possibility is to take a better metric. The whole formalism is independent of a particular choice of coordinates on the manifold of the internal variables.

In Section 4 we apply the developed lagrangian formalism to the abelian gauge field on P and show that this field has two components: Usual electromagnetic field and a neutral field of spin 1, parity minus and mass different from zero.

2. *Forminvariant metrics on the Poincaré group*

We will find here forminvariant metric tensors on P. Forminvariance of metrics under the transformations of relativistic symmetry will assure identical form of the equations of motion for fields (and consequently identity of sets of their solutions) in all reference frames related by a Poincaré transformation.

Elements of P are $p = (x, \Lambda)$, $\Lambda \in \mathrm{SL}(2, \mathrm{C})$, x-translation. The group multiplication in P is $(x_0, \Lambda_0)(x, \Lambda) = (x_0 + \Lambda_0 x, \Lambda_0 \Lambda)$, where $\Lambda_0 x = L(\Lambda_0)x$ and $L(\Lambda_0) \in L_+^\uparrow$ is matrix of the Lorentz transformation corresponding to Λ_0.

As coordinates on SL(2, C) in vicinity of the unit element we use Re ω, Im ω, or equivalently ω, ω^* (ω^* is complex conjugate of ω), where ω is defined by the following relation:

$$A = \omega^0\sigma^0 + \boldsymbol{\omega\sigma} = \sum_{\mu=0}^{3} \omega^\mu \sigma^\mu, \qquad \sigma^\mu - \text{Pauli matrices.} \tag{1}$$

Since $\det A = 1$, $\omega^{02} - \boldsymbol{\omega}^2 = \omega_\mu \omega^\mu = 1$. Let $A_0 = v^0\sigma^0 + \boldsymbol{v\sigma}$. Coordinates $\boldsymbol{\omega}'$ of the product $A' = A_0^{-1}A$ are

$$\boldsymbol{\omega}' = v^0\boldsymbol{\omega} - \omega^0\boldsymbol{v} - i\boldsymbol{v}\times\boldsymbol{\omega},$$
$$\omega^{0\prime} = v^0\omega^0 - \boldsymbol{v\omega}. \tag{2}$$

As the coordinates on P we will use $(p^I) = (x^\mu, \boldsymbol{\omega}, \boldsymbol{\omega}^*)$, $I = 1, 2, \ldots, 10$.

A general metric form on P looks as follows: $ds^2 = G_{IK}\, dp^{I*}\, dp^K$, where matric G_{IK} must be hermitian to assure that ds^2 is a real number. This metric form will be formin-variant under translations of the type $p \to (a, \sigma^0)p$ only when $G_{IK}(p) = G_{IK}(\boldsymbol{\omega}, \boldsymbol{\omega}^*)$. Then ds^2 takes the form

$$ds^2 = g^{(1)}_{\mu\nu}(\boldsymbol{\omega}, \boldsymbol{\omega}^*)dx^\mu dx^\nu + g^{(2)}_{\mu i}(\boldsymbol{\omega}, \boldsymbol{\omega}^*)dx^\mu d\omega^i + g^{(3)}_{\mu i}dx^\mu d\omega^{i*}$$
$$+ g^{(2)*}_{\mu i}dx^\mu d\omega^{i*} + g^{(3)*}_{\mu i}dx^\mu d\omega^i + g^{(4)}_{ik}d\omega^i d\omega^k$$
$$+ g^{(5)}_{ik}d\omega^{i*}d\omega^k + g^{(6)}_{ik}d\omega^{i*}d\omega^{k*} + g^{(7)}_{ik}d\omega^i d\omega^{k*}, \tag{3}$$

where $\mu, \nu = 0, 1, 2, 3$, $i, k = 1, 2, 3$,

$$g^{(1)}_{\mu\nu} = g^{(1)*}_{\nu\mu}, \qquad g^{(4)*}_{ik} = g^{(6)}_{ki}, \qquad g^{(5)}_{ik} = g^{(5)*}_{ki}, \qquad g^{(7)}_{ik} = g^{(7)*}_{ki}. \tag{4}$$

Because SL(2, C) acts on the variables $\boldsymbol{\omega}$, $\boldsymbol{\omega}^*$ transitively, we can calculate functions $g^{(i)}(\boldsymbol{\omega}, \boldsymbol{\omega}^*)$ assuming their values in only one single point (we will take $\boldsymbol{\omega} = \boldsymbol{\omega}^* = 0$) and applying the transformation law for a covariant tensor. Conditions (4) will be satisfied on whole SL(2, C) if they are satisfied in one point $\boldsymbol{\omega} = \boldsymbol{\omega}^* = 0$. In this way we get the following formulae:

$$g^{(1)}_{\mu\nu}(\boldsymbol{\omega}, \boldsymbol{\omega}^*) = g^{(1)}_{\sigma\varrho}(0)L_\mu{}^\sigma(\boldsymbol{\omega}, \boldsymbol{\omega}^*)L_\nu{}^\varrho(\boldsymbol{\omega}, \boldsymbol{\omega}^*),$$
$$g^{(2)}_{\mu i}(\boldsymbol{\omega}, \boldsymbol{\omega}^*) = g^{(2)}_{\varrho\alpha}(0)L_\mu{}^\varrho(\boldsymbol{\omega}, \boldsymbol{\omega}^*)\eta^\alpha{}_i(\boldsymbol{\omega}, \boldsymbol{\omega}^*),$$
$$g^{(3)}_{\mu i} = g^{(3)}_{\varrho\alpha}(0)L^\varrho_\mu \eta^{\alpha*}_i, \qquad g^{(4)}_{ik} = g^{(4)}_{\alpha\beta}(0)\eta^\alpha{}_i\eta^\beta{}_k,$$
$$g^{(5)}_{ik}(\boldsymbol{\omega}, \boldsymbol{\omega}^*) = g^{(5)}_{\alpha\beta}(0)\eta^{\alpha*}_i\eta^\beta{}_k, \qquad g^{(7)}_{ik}(\boldsymbol{\omega}, \boldsymbol{\omega}^*) = g^{(7)}_{\alpha\beta}(0)\eta^\alpha{}_i\eta^{\beta*}_k, \tag{5}$$

where

$$L_\mu{}^\varrho(\boldsymbol{\omega}, \boldsymbol{\omega}^*) = \tfrac{1}{2}\operatorname{Tr}\left[\sigma^\varrho A^{-1}\sigma^\mu(A^+)^{-1}\right]$$

and

$$\eta^\alpha{}_i = \omega^0\delta^\alpha{}_i - \frac{\omega^\alpha\omega^i}{\omega^0} - i\varepsilon_{\alpha li}\omega^l$$

are the reciprocal Killing vectors (see e. g. [15]) for the right group translations on SL(2, C) in parametrization (1). The matrices of the constants $g^{(i)}(0)$ satisfy conditions (4). Hence, a general forminvariant metric on P has 55 arbitrary, real parameters.

Now we assume that metrics (3) have an additional forminvariance under spatial rotations of the basis attached to the particle. This assumption corresponds to the spherical symmetry of the particle. It is easy to check that such rotations are described by right group translations on P of the type $p \to p(0, u)$, $u \in SU(2)$.

Let $\Lambda'(\omega') = \Lambda(\omega)\Lambda_0(v)$. After some algebra we get the following formula:

$$\eta^{\alpha}{}_{i}(\omega')d\omega'^{i} = O^{\alpha}{}_{\beta}(v)\eta^{\beta}{}_{k}(\omega)d\omega^{k}, \tag{7}$$

where

$$O^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta}(v^{02}+\boldsymbol{v}^{2})-2v^{\alpha}v^{\beta}-2iv^{0}\varepsilon_{\alpha s\beta}v^{s}$$

form an orthogonal, 3×3, matrix. For rotations $\boldsymbol{v}^{*} = -\boldsymbol{v}$, $v^{0*} = v^{0}$ $v^{02}-\boldsymbol{v}^{2} = 1$ and reciprocal to the $O^{\alpha}{}_{\beta}$ matrix is equal to the spatial part of $L^{\varrho}{}_{\mu}$.

As transformation $p \to p(0, u)$ changes only variables ω, ω^{*} (indices μ, ν do not transform) the forminvariance of metric requires $g^{(1)}_{\mu\nu}(\omega, \omega^{*})$ to be a constant function on the left cosets of SU(2) in SL(2, C), i. e. to be a function on SL(2, C)/SU(2). In particular $g^{(1)}_{\mu\nu}$ is a constant function on SU(2), i. e. for $\boldsymbol{v}^{*} = -\boldsymbol{v}$, $v^{0*} = v^{0}$, $v^{02}-\boldsymbol{v}^{2} = 1$,

$$g^{(1)}_{\mu\nu}(\boldsymbol{v}, \boldsymbol{v}^{*}) = g^{(1)}_{\mu\nu}(0).$$

Applying the Schur Lemma for SO(3) we obtain

$$g^{(1)}_{00}(0) = a, \qquad g^{(1)}_{sr}(0) = b\delta_{sr}, \qquad g^{(1)}_{s0}(0) = g^{(1)}_{0s}(0) = 0. \tag{8}$$

Proceeding similarly we get

$$g^{(2)*}_{\mu\beta}(0) = g^{(3)}_{\mu\beta}(0) = f\delta_{\mu\beta}, \qquad g^{(4)}_{\alpha\beta}(0) = c\delta_{\alpha\beta},$$

$$g^{(5)}_{\alpha\beta}(0) = d\delta_{\alpha\beta}, \qquad g^{(7)}_{\alpha\beta}(0) = e\delta_{\alpha\beta}, \tag{9}$$

where a, b, d are real constants and c, f are complex constants. Making use of the formula $L_{\nu}{}^{0}L_{\varrho}{}^{0}-L_{\nu}{}^{i}L_{\varrho}{}^{i} = g_{\nu\varrho}$, $g_{\nu\varrho} = (1, -1, -1, -1)$, we obtain

$$ds^{2} = g_{\mu\nu}dx^{\mu}dx^{\nu}+c_{1}(L_{\nu}{}^{0}dx^{\nu})^{2}+c_{2}(\eta^{\alpha}{}_{i}d\omega^{i})^{2}+c_{3}|\eta^{\alpha}{}_{i}d\omega^{i}|^{2}+c_{2}^{*}(\eta^{\alpha*}_{i}d\omega^{i*})^{2}$$
$$+c_{4}dx^{\mu}L_{\mu}{}^{\alpha}\eta^{\alpha}{}_{i}d\omega^{i}+c_{4}^{*}dx^{\mu}L_{\mu}{}^{\alpha}\eta^{\alpha*}_{i}d\omega^{i*}, \tag{10}$$

where

$$L_{\nu}{}^{0} = g_{\nu\nu}\omega^{0}\omega^{\nu*}+g_{\nu\nu}(1-\delta_{\nu}{}^{0})\omega^{0*}\omega^{\nu}+\delta_{\nu}{}^{0}(\omega\omega^{*})+(1-\delta_{\nu}{}^{0})i\varepsilon_{s\nu k}\omega^{s}\omega^{k*}.$$

Last six terms are due to the internal structure of the particle. They will be negative definite when $c_{4} = 0$, $c_{1} < 0$, $\frac{c_{3}}{|c_{2}|} \leq -2$. Only these metrics will lead to lagrangians having physical meaning. Constant c_{1} is dimensionless, c_{4} has the dimension of length and c_{2} and c_{3} have the dimension of squared length.

Metric (10) will be forminvariant under all four dimensional rotations of the basis attached to the particle only when $c_{1} = c_{3} = c_{4} = 0$, what implies that

$$ds^{2} = g_{\mu\nu}dx^{\mu}dx^{\nu}+c_{2}(\eta^{\alpha}{}_{i}d\omega^{i})^{2}+c_{2}^{*}(\eta^{*\alpha}{}_{i}d\omega^{i*})^{2}. \tag{11}$$

For the right group translations of the type $p \to p(a, 1) = p'$ we have $x' = x + \Lambda a$ and it is easy to see that metrics (11) are not forminvariant under these transformations unles the term $g_{\mu\nu}dx^\mu dx^\nu$ is excluded. Therefore, on P a nondegenerate metric tensor forminvariant under both left and right group translations does not exist.

Parametrizing the left group translations on SL(2, C) by Θ, $\boldsymbol{n}$, $(\boldsymbol{n}^2 = 1) : \Lambda_0 = \exp\left(-\frac{i}{2}\Theta \boldsymbol{n\sigma}\right)$ (for Im $\Theta = 0$ a rotation around $\boldsymbol{n}$, for Re $\Theta = 0$ a boost in direction $\boldsymbol{n}$), we obtain the following formulae for the generators of the left regular representation of the Poincaré group

$$P_\mu = i\frac{\partial}{\partial x^\mu}, \quad \tilde{\boldsymbol{M}} = i\boldsymbol{x} \times \nabla_x + \boldsymbol{M}, \quad \tilde{\boldsymbol{N}} = -i\left(x^0\nabla_x + \boldsymbol{x}\frac{\partial}{\partial x^0}\right) + \boldsymbol{N}, \tag{12}$$

where

$$\boldsymbol{M} = \boldsymbol{J} + \boldsymbol{K}, \quad \boldsymbol{N} = i(\boldsymbol{K} - \boldsymbol{J}),$$

$$J^\alpha = -\tfrac{1}{2}\left(\omega^0 \frac{\partial}{\partial \omega^\alpha} + i\varepsilon_{\alpha is}\omega^i \frac{\partial}{\partial \omega^s}\right),$$

$$K^\alpha = \tfrac{1}{2}\left(\omega^{0*} \frac{\partial}{\partial \omega^{\alpha *}} - i\varepsilon_{\alpha is}\omega^{i*} \frac{\partial}{\partial \omega^{s*}}\right). \tag{13}$$

It is easy to check that

$$[J^\alpha, J^\beta] = i\varepsilon_{\alpha\beta\gamma}J^\gamma, \quad [K^\alpha, K^\beta] = i\varepsilon_{\alpha\beta\gamma}K^\gamma, \quad [J^\alpha, K^\beta] = 0.$$

One defines the improper Lorentz transformations P, T so as to satisfy the following relations:

$$PP_\mu P^{-1} = g^{\mu\mu}P_\mu, \quad P\tilde{\boldsymbol{M}}P^{-1} = \tilde{\boldsymbol{M}}, \quad P\tilde{\boldsymbol{N}}P^{-1} = -\tilde{\boldsymbol{N}},$$

$$TP_\mu T^{-1} = g^{\mu\mu}P_\mu, \quad T\tilde{\boldsymbol{M}}T^{-1} = -\tilde{\boldsymbol{M}}, \quad T\tilde{\boldsymbol{N}}T^{-1} = \tilde{\boldsymbol{N}}.$$

The operations defined by

$$P\colon \psi(x^0, \boldsymbol{x}, \boldsymbol{\omega}, \boldsymbol{\omega}^*) \to \eta\,\psi(x^0, -\boldsymbol{x}, -\boldsymbol{\omega}^*, -\boldsymbol{\omega}), \; \eta = 1, \tag{14}$$

$$T\colon \psi(x^0, \boldsymbol{x}, \boldsymbol{\omega}, \boldsymbol{\omega}^*) \to \chi\,\psi^*(-x^0, \boldsymbol{x}, -\boldsymbol{\omega}^*, -\boldsymbol{\omega}), \; \chi = 1$$

have these properties and can play the role of parity and time reversal operations.

It is seen that metrics (10) would be forminvariant under the operations P and T only when $c_2 = c_2^*$, $c_4 = 0$.

Having a forminvariant metric tensor one can easily construct an invariant measure on P. It is simply $d\mu(p) \equiv d^4x\, d\mu(\omega, \omega^*) = \text{const}\,\sqrt{G}\, d^4x\, d^3\omega\, d^3\omega^*$, $G = \det[G_{IK}]$ and, in particular, for metrics (11), we have

$$d\mu(p) = \text{const}\,|\omega^0|^{-2}\, d^4x\, d^3\omega\, d^3\omega^*$$

For const $= -i$ $d\mu(p)$ will be real.

3. *Lagrangian formalism for fields on P*

Keeping in mind the factor $\sqrt{G}$ in $d\mu(p)$ and making use of the formula $\left\{ {K \atop I\,K} \right\} = \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial p^I}$ for the Christoffel symbols one can check that the variational principle $\delta S = 0$ for the action $S = \int d\mu(p)\ \mathscr{L}(\psi_i, \psi_{j;I})$ leads to the following Euler-Lagrange equations for fields $\psi_i(p)$:

$$\left.\frac{\partial \mathscr{L}}{\partial(\psi_{j;I})}\right|_{;I} - \frac{\partial \mathscr{L}}{\partial \psi_j} = 0, \tag{15}$$

where $;I$ denotes the covariant derivative.

Invariance of S under an infinitesimal transformation

$$p \to (\varepsilon^\alpha, \sigma^0)p \quad \text{i.e.} \quad p^I \to p^I + \xi^I_\alpha \varepsilon^\alpha, \quad \xi^I_\alpha = \delta^{I-1}_\alpha, \quad \alpha = 0, 1, 2, 3,$$

$$\psi'_{j'}(p') \equiv \psi_j(p) \quad \text{i.e.} \quad \psi'_j(p) = \psi_j(p) + \underset{(\alpha)}{\pounds}\psi_j \varepsilon^\alpha,$$

where $\underset{(\alpha)}{\pounds}$ denotes the Lie derivative in direction ξ^I_α (see [15]), gives

$$\delta S = \varepsilon^\alpha \int_\Omega T^I_{\alpha;I} d\mu(p) = 0,$$

where

$$T^I_\alpha = \frac{\partial \mathscr{L}}{\partial(\psi_{j;I})} \underset{(\alpha)}{\pounds}\psi_j - \xi^I_\alpha \mathscr{L}. \tag{16}$$

Four ten-vectors T^I_α obey the differential conservation laws

$$T^I_{\alpha;I} = 0.$$

Integrating (17) with respect to $d^3\boldsymbol{x}\,d\mu(\omega, \omega^*)$, and assuming that the fields are vanishing sufficiently fast at $\boldsymbol{x}, \omega, \omega^* \to \infty$ we get the global conservation laws

$$\frac{d}{dx^0} \int d^3\boldsymbol{x}\,d\mu(\omega, \omega^*) T^0_\alpha = 0 \quad (T^0_\alpha \equiv T^1_\alpha).$$

The existence of the global conservation laws regardless of a curvature of the manifold P is the result of using the vectors. It is easy to check that for the second order tensor on P the differential conservation laws do not imply the global ones (see [16]).

We will regard the quantities T^0_α as the density of energy and momentum of the field.

Similarly one can get six ten-vectors $M^I_{(\alpha,\beta)}$, $(\alpha, \beta) = (0, 1), (0, 2), \ldots, (1, 2)$ which describe the total angular momentum of the field:

$$M^I_{(\alpha,\beta)} = \xi^I_{(\alpha,\beta)} \mathscr{L} - \frac{\partial \mathscr{L}}{\partial(\psi_{i;I})} (\underset{(\alpha,\beta)}{\pounds}\psi_i).$$

The vectors $\xi_{(\alpha,\beta)}$ are the Killing vectors for the group translations on P of the type $p \to (0, \Lambda)p$.

If the lagrangian $\mathscr{L}$ is invariant under gauge transformation $\psi_j(p) \to \exp(ie\chi)\psi_j(p)$, $\chi(p) = \text{const}$, there exists the conserved current

$$j^K = ie\left(\frac{\partial \mathscr{L}}{\partial(\psi_{i;K})}\psi_i - \frac{\partial \mathscr{L}}{\partial(\psi_{i;K})}\psi_i^*\right) \tag{18}$$

which satisfies the equation $j^K{}_{;K} = 0$ and, as before for energy-momentum, we get here the global conservation law for the total charge $Q = \int d^3x\, d\mu(\omega, \omega^*)\, j^1$.

We define the Poisson brackets for functionals of fields $\psi_i(p)$ and canonical conjugate momenta $\frac{\partial \mathscr{L}}{\partial(\psi_{i;0})} \equiv \pi^i$ by

$$\{F, G\} = \int d^3x\, d\mu(\omega, \omega^*)\left[\frac{\delta F}{\delta\psi_i(x, t, \omega, \omega^*)}\frac{\delta G}{\delta\pi^i(x, t, \omega, \omega^*)} - \frac{\delta G}{\delta\psi_i(x, t, \omega, \omega^*)}\frac{\delta F}{\delta\pi^i(x, t, \omega, \omega^*)}\right],$$

where the variational derivative is defined as usual with the only modification consisting in the replacement $d^3x \to d^3x\, d\mu(\omega, \omega^*)$. One can easily check that $\{\psi_j, P_\alpha\} = \underset{(\alpha)}{\pounds}\psi_j$, $\{\psi_j, \underset{(\alpha,\beta)}{M}\} = \underset{(\alpha,\beta)}{\pounds}\psi_j$, where $P_\alpha = \int d^3x\, d\mu(\omega, \omega^*)\, T^0_\alpha$, $\underset{(\alpha,\beta)}{M} = \int d^3x\, d\mu(\omega, \omega^*)\, \underset{(\alpha,\beta)}{M^0}$.

For the simple lagrangian for complex, scalar field ψ on P

$$\mathscr{L} = G^{IK}\frac{\partial}{\partial p^I}\psi^*\frac{\partial}{\partial p^K}\psi - M^2\psi\psi^* \tag{19}$$

with metric G^{IK} given by $G_{LK}G^{KI} = \delta^I_L$, from which for metric (11)

$$g^{(4)ik} = \frac{1}{\gamma^2}(\delta^{ik} + \omega^i\omega^k), \quad g^{(4)}_{ik} = \gamma^2\left(\delta_{ik} - \frac{\omega^i\omega^k}{\omega^{02}}\right), \quad (\gamma^2 \equiv c_2),$$

as an equation of motion we obtain

$$\Delta\psi = -M^2\psi, \tag{20}$$

where $\Delta\psi = \left(G^{IK}\frac{\partial}{\partial p^K}\psi\right)_{;I}$ is the Laplace-Beltrami operator on P. Equation (20) can be written in the form

$$g_{\mu\nu}\partial^\mu\partial^\nu\psi + \frac{4}{\gamma^2}(\boldsymbol{J}^2 + \boldsymbol{K}^2)\psi = -M^2\psi. \tag{21}$$

Equation (21) is invariant with respect to the left regular representation of P (relativistic symmetry) as well as transformations of the type $p \to p(0, \Lambda)$ (Lorentz rotations of the basis connected with the particle).

Equation (21) is known from papers [3, 9]. In paper [3] this equation is introduced as a quantum equation of motion for the free hyperspherical relativistic rotator.

Energy $T_0{}^0$ for scalar complex field ψ with the lagrangian (19) is

$$T_0^0 = -\sum_{\mu=0}^{3} \frac{\partial}{\partial x^\mu}\psi^* \frac{\partial}{\partial x^\mu}\psi - M^2\psi^*\psi + \frac{4}{\gamma^2}(J\psi^*)(J\psi) + \frac{4}{\gamma^2}(K\psi^*)(K\psi)$$

and does not have a definite sign.

It is easy to see that for $\gamma^2 > 0$ the condition

$$\int d^3x\, d\mu(\omega, \omega^*)\, \psi^*(\boldsymbol{M}^2 - \boldsymbol{N}^2)\, \psi \geqslant 0 \tag{22}$$

guarantees negative sign of P_0, but the presence of any additional condition always makes a theory more complicated. The problem of indefinite energy is a consequence of indefinite sign of the internal part of metric (11). It is clear that a theory with a more suitable choice of metric (e. g. any of metrics (10) with negative sign of the internal part and $c_1 = 0$) will not suffer from such additional complications. Therefore, we think that equation (21) is out of the class of equations on P which may have a physical meaning.

For metric (10) with $c_4 = c_1 = c_2 = 0$, $c_3 < 0$ the equation $\Delta\psi = -M^2\psi$ takes the form

$$g_{\mu\nu}\partial^\mu\partial^\nu\psi + \gamma'^2(\boldsymbol{M}_R^2 + \boldsymbol{N}_R^2)\psi = -M^2\psi, \qquad \gamma'^2 > 0,$$

where

$$\boldsymbol{M}_R = \boldsymbol{J}_R + \boldsymbol{K}_R, \qquad \boldsymbol{N}_R = i(\boldsymbol{K}_R - \boldsymbol{J}_R),$$

$$J_R^\alpha = \tfrac{1}{2}\left(\omega^0 \frac{\partial}{\partial\omega^\alpha} - i\varepsilon_{t\alpha s}\omega^s \frac{\partial}{\partial\omega^t}\right), \qquad K_R^\alpha = -\tfrac{1}{2}\left(\omega^{0*} \frac{\partial}{\partial\omega^{\alpha*}} + i\varepsilon_{t\alpha s}\omega^{s*} \frac{\partial}{\partial\omega^{t*}}\right)$$

are generators of the Lorentz subgroup of the right regular representation of P. The field described by this equation has $T_0{}^0$ with a definite sign.

4. *Abelian gauge field on P*

Now we will consider a gauge field on P for the gauge group U(1). General properties of this field do not depend on the form of equations of motion for sources of gauge field. However, for simplicity and in view of some interesting properties we consider here only the gauge field generated by sources described by equation (21).

In this section we use variables $\gamma\omega$ with the dimension of length, i. e. $(p^I) = (x^\mu, \gamma\omega, \gamma\omega^*)$. We shall denote the new variables by the same symbol $\boldsymbol{\omega}$ as before. In these variables $\omega^{02} - \omega^2 = \gamma^2$, $g^{(4)ik} = \delta^{ik} + \dfrac{\omega_i\omega_k}{\gamma^2}$, $d\mu(p) = -i\gamma^{-4} d^4x |\omega^0|^{-2} d^3\omega\, d^3\omega^*$ and $\boldsymbol{J}$, $\boldsymbol{K}$ are dimensionless.

The lagrangian (19) will be invariant with respect to the gauge transformation $\psi(p) \to \exp(ie\chi(p)\,\psi(p)$ ($\chi(p)$ — real function on P) if

$$\frac{\partial}{\partial p^I}\psi \to \frac{\partial}{\partial p^I}\psi - ieA_I\psi, \qquad \frac{\partial}{\partial p^I}\psi^* \to \frac{\partial}{\partial p^I}\psi^* + ieA_I\psi^*,$$

and under gauge transformation $A_I \to A_I + \frac{\partial}{\partial p_I}\chi$. Field $\{A_I\}$ is a ten-component, covariant vector field on P. A_μ is real, $A_{7+i} = A^*_{4+i}$, $i = 1, 2, 3$. Instead of A_I we shall use the following fields

$$A_\mu \equiv A_I, \quad \mu = I-1, \quad I = 1, 2, 3, 4,$$

$$Z_\alpha \equiv \xi^i_\alpha A_{4+i}, \quad Z_{\alpha *} \equiv -\xi^{i*}_\alpha A_{7+i} = -Z^*_\alpha, \quad \boldsymbol{Z}_* = (Z_{\alpha *}), \tag{23}$$

where vectors ξ^i_α with nonvanishing components $\xi^i_\alpha = \frac{1}{\gamma}(\omega^0\delta^i_\alpha + i\varepsilon_{i\alpha t}\omega^t)$ are the contravariant Killing vectors of the left regular representation of SL(2, C) group in parametrization (1). After some algebra we see that A_μ transforms as a four vector, $\boldsymbol{Z}$ as a complex vector belonging to the (1, 0) representation of SL(2, C) and $\boldsymbol{Z}_*$ as a vector belonging to the (0, 1) representation of SL(2, C). This can be deduced also from formula (7). Hence, the splitting of A_I to A_μ, $\boldsymbol{Z}$, $\boldsymbol{Z}_*$ is the Lorentz invariant. Gauge transformations for $\boldsymbol{Z}$, $\boldsymbol{Z}_*$ are $\boldsymbol{Z} \to \boldsymbol{Z} - \frac{2}{\gamma}\boldsymbol{J}\chi$, $\boldsymbol{Z}_* \to \boldsymbol{Z}_* - \frac{2}{\gamma}\boldsymbol{K}_*\chi$. ($K_{\alpha *} \equiv K^\alpha$).

The Lagrange-Euler equations for complex scalar field $\psi(p)$ acquire now the following form:

$$(\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu)\psi + \frac{4}{\gamma^2}\left(\boldsymbol{J} + \frac{i\gamma e}{2}\boldsymbol{Z}\right)^2\psi$$

$$+\frac{4}{\gamma^2}\left(\boldsymbol{K} + \frac{i\gamma e}{2}\boldsymbol{Z}_*\right)^2\psi = -M^2\psi. \tag{24}$$

The conserved current is

$$j_K = -ie\left(\psi^*\frac{\partial}{\partial p^K}\psi - \psi\frac{\partial}{\partial p^K}\psi^*\right) + 2e^2A_K\psi\psi^* \tag{25}$$

or, when passing to projections on Killing vectors (as in (23)),

$$j_\mu = -ie\left(\psi^*\frac{\partial}{\partial x^\mu}\psi - \psi\frac{\partial}{\partial x^\mu}\psi^*\right) + 2e^2A_\mu\psi\psi^*,$$

$$j_\alpha = \frac{2ie}{\gamma}(\psi^*J_\alpha\psi - \psi J_\alpha\psi^*) + 2e^2Z_\alpha\psi\psi^*,$$

$$j_{\alpha *} = \frac{2ie}{\gamma}(\psi^*K_{\alpha *}\psi - \psi K_{\alpha *}\psi^*) + 2e^2Z_{\alpha *}\psi\psi^*. \tag{26}$$

Current j_μ is identical with the current for usual scalar field described by the Klein-Gordon equation.

For the free field A_I we postulate a gauge invariant lagrangian of the form

$$\mathscr{L}_0 = \tfrac{1}{4} F_{IK} F^{IK},$$

where

$$F_{IK} = \frac{\partial}{\partial p^I} A_K - \frac{\partial}{\partial p^K} A_I. \tag{27}$$

The Euler-Lagrange equations for A_I are $F^K_{I;K} = -j_I$, where $j_I = -\dfrac{\partial \mathscr{L}}{\partial A^I}$ is given by (25). From this we get the following equations for fields A_μ and $\boldsymbol{Z}$:

$$\text{(a)} \qquad \frac{\partial}{\partial x^\nu} F_\mu{}^\nu - \frac{2}{\gamma} \frac{\partial}{\partial x^\mu} (J_\alpha Z_\alpha + K_{\alpha *} Z_{\alpha *}) - \frac{4}{\gamma^2} (\boldsymbol{J}^2 A_\mu + \boldsymbol{K}^2 A_\mu) = -j_\mu,$$

$$\text{(b)} \qquad \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} Z_\alpha + \frac{2}{\gamma} J_\alpha \left(\partial_\mu A^\mu - \frac{2}{\gamma} J_\beta Z_\beta - \frac{2}{\gamma} K_{\beta *} Z_{\beta *} \right)$$

$$+ \frac{4}{\gamma^2} (\boldsymbol{J}^2 + \boldsymbol{K}^2) Z_\alpha - \frac{4}{\gamma^2} i \varepsilon_{\varrho\alpha\beta} J_\varrho Z_\beta + \frac{4}{\gamma^2} Z_\alpha = j_\alpha. \tag{28}$$

The equation for $Z_{\alpha *}$ can be obtained from (b) by complex conjugation.

By usual methods ([14]) we can get the gauge independent density of energy and momentum. We find that the density of energy does not have a definite sign. This is a direct consequence of a non-definiteness of the adopted metric (11). A better choice of a metric will remove this unpleasant feature.

Equation of free motion (21) is forminvariant under the Lorentz rotations of the basis attached to the particle. It is a generalization of invariance of the theory of spherically symmetrical top under rotations of a basis attached to it.

Now we postulate that also the equation (24) is invariant under Lorentz rotations of the basis attached to the particle. (*)

Analogous assumption for a usual nonrelativistic rigid body would mean that the minimal interaction does not destroy the spherical symmetry of the internal world of the top.

In view of the fact that these Lorentz rotations are represented by a subgroup of the right group translations on P (transformations of the type $p \to p(0, \Lambda)$), and the right regular representation commutes with the left one, equation (24) will manifest this additional invariance only when A_μ, $\boldsymbol{Z}$ and $\boldsymbol{Z}_*$ do not depend on the variables ω, ω^*. This condition will not be in contradiction with the gauge invariance only when we restrict gauge transformations to transformations changing A_μ, Z_α, $Z_{\alpha *}$ by functions depending only on x^μ's. Hereafter we have the following conditions:

$$J_\alpha \chi(x^\mu, \omega, \omega^*) = \varphi_\alpha(x^\mu) \text{ for all } \omega, \omega^*.$$

One can easily check that these conditions imply $\varphi_\alpha(x^\mu) = 0$ and $\chi = \chi(x^\mu)$. It means that field $A_\mu(x^\varrho)$ can be shifted by $\partial_\mu \chi(x^\varrho)$ as the usual electromagnetic field, and fields $\boldsymbol{Z}$, $\boldsymbol{Z}_*$ cannot be gauged at all (assumption (*) partially breaks the gauge symmetry).

We choose the action S_0 for free fields $A_\mu(x^\varrho)$, $\boldsymbol{Z}(x^\varrho)$, $\boldsymbol{Z}_*(x^\varrho)$ in the form

$$S_0 = \int d^4x \left[\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\left(\frac{\partial}{\partial x^\mu} Z_\alpha \frac{\partial}{\partial x_\mu} Z_\alpha + \frac{\partial}{\partial x^\mu} Z_{\alpha *} \frac{\partial}{\partial x_\mu} Z_{\alpha *} \right) \right.$$
$$\left. - \frac{2}{\gamma^2}(Z_\alpha Z_\alpha + Z_{\alpha *} Z_{\alpha *}) \right]. \tag{29}$$

It can be obtained from the action $\frac{1}{4}\int d\mu(p)\, F_{IK}F^{IK}$ by omitting the infinite factor $-\dfrac{i}{\gamma 4}\displaystyle\int |\omega^0|^{-2}\, d^3\omega\, d^3\omega^*$. The action for $\psi(p)$ does not change. Equation of motion for ψ has the same form as before (24), but now $\boldsymbol{JZ} = \boldsymbol{KZ}_* = 0$. One can check that quantity $\int d^3\boldsymbol{x}\, d\mu(\boldsymbol{\omega}, \boldsymbol{\omega}^*)\, \psi^*(M^2 - N^2)\psi$ is conserved, so the minimal interaction with gauge fields, depending only on (x^ϱ), does not lead out of the space of fields which obey condition (22). The equations of motion for gauge fields are now the following:

$$\text{(a)} \quad \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} Z_\alpha + \frac{4}{\gamma^2} Z_\alpha = -\int d\mu(\omega, \omega^*) j_\alpha,$$

$$\text{(b)} \quad \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} Z_{\alpha *} + \frac{4}{\gamma^2} Z_{\alpha *} = -\int d\mu(\omega, \omega^*) j_{\alpha *},$$

$$\text{(c)} \quad \frac{\partial}{\partial x^\mu} F^{\nu\mu} = -\int d\mu(\omega, \omega^*) j^\nu. \tag{30}$$

Decomposing $\boldsymbol{Z}$ to real and imaginary parts $\boldsymbol{Z} = -\boldsymbol{\varepsilon} + i\boldsymbol{\beta}$, $\boldsymbol{Z}_* = \boldsymbol{\varepsilon} + \boldsymbol{\beta} i$, we get

$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}\boldsymbol{\varepsilon} + \frac{4}{\gamma^2}\boldsymbol{\varepsilon} = -\frac{e}{\gamma}\int d\mu(\omega, \omega^*)\,[\psi^*\boldsymbol{N}\psi - \psi\boldsymbol{N}\psi^*] - \frac{e^2}{\gamma^2}\boldsymbol{\varepsilon}\int d\mu(\omega, \omega^*)\psi\psi^*,$$

$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}\boldsymbol{\beta} + \frac{4}{\gamma^2}\boldsymbol{\beta} = -\frac{e}{\gamma}\int d\mu(\boldsymbol{\omega}, \omega^*)\,[\psi^*\boldsymbol{M}\psi - \psi\boldsymbol{M}\psi^*] - \frac{e^2}{\gamma^2}\boldsymbol{\beta}\int d\mu(\boldsymbol{\omega}, \omega^*)\psi\psi^*.$$

These equations will be invariant with respect to spatial reflection P if under P $\boldsymbol{\varepsilon}(\boldsymbol{x}, t) \to -\boldsymbol{\varepsilon}(-\boldsymbol{x}, t)$, $\boldsymbol{\beta}(\boldsymbol{x}, t) \to \boldsymbol{\beta}(-\boldsymbol{x}, t)$. These transformation laws are the same as for electric and magnetic fields, respectively. From this analogy we conclude that $\boldsymbol{Z}$ has parity "$-$".

Field $\boldsymbol{Z}$ is neutral because its lagrangian is not invariant under the gauge transformations of the first kind. $\boldsymbol{Z}$ has nonzero mass $m^2 = \dfrac{4}{\gamma^2}$. In order to have this mass real we must put $\gamma^2 > 0$.

The part of lagrangian describing the interaction of this field with scalar field ψ differs from the kinetic energy of the "rotational motion" for ψ by factor $\frac{e}{m}$. Hence, if m is large, $\boldsymbol{Z}$-field has a short range and interacts weakly with ψ (i. e. kinetic energy $\frac{4}{\gamma^2}(\boldsymbol{J}^2+\boldsymbol{K}^2)$ dominates). The interaction of ψ with A_μ is m times smaller than the interaction of ψ with $\boldsymbol{Z}$. This does not mean that for large m the usual electromagnetic interaction is negligible, because it dominates the interaction with $\boldsymbol{Z}$ on large distances and refers to different degrees of freedom.

Equation (24) does not conserve parity only if γ^2 is complex. $\boldsymbol{Z}$ has then the complex mass, but the energy of $\boldsymbol{Z}$-field will be real as before. This follows from the fact that the part of the metric form (10) referring to the internal motion is real also for complex c_2 ($c_2 \equiv \gamma^2$). The possible interpretation would be that in this case $\boldsymbol{Z}$ describes an unstable particle.

5. *Concluding remarks*

The work on the subject can be continued in the following directions:

1) The developed formalism allows us to make an attempt to quantize classical fields on P within the usual canonical approach. After this one can consider models of interactions. We think that it will be very interesting to look for other consequences of existence of a new dynamical variable, i. e. the internal angular momentum. Some general consequences were already pointed out in papers [2, 12].

2) Paper [5] gives some examples of mechanical objects whose configuration space is P. One can try to carry out quantization for such objects and to check if we obtain as a wave equation an equation of type (20) with metric (10).

3) It would be very interesting to investigate thoroughly a spatial extension of objects described by equations on P. For example, the problem arises whether constants in the metric are in any way connected with the spatial dimensions of the object. In any case in papers [4, 7] it is pointed out that objects described by equations on a manifold larger than the Minkowski space show a spatial extension.

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