LETTERS TO THE EDITOR

P-WAVE SCATTERING LENGTH SUM RULES FOR THE $\pi\pi$ SYSTEM

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Sum rules for the $I = 1$ P-wave $\pi\pi$ scattering length a_P^1 are derived for the modulus and inverse of the scattering amplitude. These non-linear dispersive sum rules depend crucially on the positivity of the amplitudes' imaginary part. Numerical examples are given which seem to prefer larger values of a_P^1. Comparison is made to the commonly used relations for a_P^1.

Because of its simple structure $\pi\pi$ scattering has long been a favorite proving ground for theoretical models of hadronic interactions. Unfortunately, experimental access to the on-shell $\pi\pi$ system is shrouded by many difficulties. Nevertheless, spurred on by the high-statistics experiments of the type $\pi N \to \pi\pi N$ [1–3] and the theoretical input of analyticity and crossing symmetry expressed via the Roy equations [4], the phenomenological analysis of $\pi\pi$ scattering has made great strides. Recently Basdevant, Froggatt and Petersen [5] presented a comprehensive analysis of the available data ($M > 600$ MeV) based on the Roy equations and found that the data are not yet sufficient to completely determine the threshold parameters. At all points along the universal curve of Morgan and Shaw [6] they were able to find solutions (characterized by the $I = 0$ S-wave scattering length in the range $-.05 < a_S^0 < .60$) consistent with the data and Roy's equations. Different solutions lead to appreciably different values of $2a_S^0 - 5a_S^2$, a quantity which arises naturally in $\pi\pi$ analysis (see, e.g., Eq. (4)). The phenomenological value of the P-wave scattering length a_P^1 is rather stable (changing from .03 to .04) but disagrees with the preliminary experimental data ($a_P^1 \sim .07$, cf. Fig. 6 Ref. [7]).

In this note we analyse some typical solutions of Ref. [5] by means of sum rules for a_P^1 which follow from non-linear dispersion relations (e.g. dispersing in the modulus or inverse of the amplitude) and show that solutions with larger values of a_P^1 are preferred.

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Consider the $s \leftrightarrow u$ crossing symmetric $\pi^{\pm}\pi^0$ scattering amplitude $F(s, t) = F^1(s, t)+F^2(s, t)$ and assume $F(4, t) < 0$. Due to the positivity of Im $F(s, t)$ for $s > 4$ and $t \geqslant 0$ the function $F(s, t)$ has no zeros and the modulus representation of $F(s, t)$ is given by [8]

$$\frac{\arg F(s, t)}{\sqrt{s-4}} = -\frac{\sqrt{s+t}}{\pi}\int_4^\infty \frac{ds'(s'-2+t/2)\,[\ln|F(s', t)|^2-\ln|F(4, t)|^2]}{\sqrt{(s'+t)\,(s'-4)}\,(s'-s)\,(s'-4+s+t)}. \tag{1}$$

Differentiating both sides with respect to t and taking the limits $t \to 0$ and $s \to 4$, we obtain

$$3\pi a_{\rm P}^1 = \int_4^\infty \frac{ds(s-6)\ln(D^2+A^2)}{s^{5/2}(s-4)^{1/2}} + 8\int_4^\infty \frac{ds(s-2)}{[s(s-4)]^{3/2}}\left[\frac{2(DD'+AA')}{D^2+A^2} - \frac{3a_{\rm P}^1}{a_{\rm S}^2}\right], \tag{2}$$

where $D = \operatorname{Re} F(s, 0)$ and $A = \operatorname{Im} F(s, 0)$. The same procedure applied to $[F(s, t)\sqrt{(s-4)(s+t)}]^{-1}$ yields

$$-3\pi\frac{a_{\rm P}^1}{a_{\rm S}^2} = \int_4^\infty \frac{ds(s-6)D}{s^{5/2}(s-4)^{1/2}(D^2+A^2)} + 8\int_4^\infty \frac{ds(s-2)}{[s(s-4)]^{3/2}}\left[\left(\frac{D}{D^2+A^2}\right)'_{t=0} + \frac{1.5a_{\rm P}^1}{(a_{\rm S}^2)^2}\right]. \tag{3}$$

These sum rules are only valid when $F(4, t) < 0$. However this restriction is easily circumvented by deriving similar relations for the $s \leftrightarrow u$ symmetric function

$$F(s, t) \to F(s, t)-F(4, 0)-\varepsilon,$$

where $\varepsilon > 0$. Varying ε emphasizes different energies between threshold and 600 MeV.

To illustrate their application, a numerical analysis of Eqs (2) and (3) (and their counterparts with $\varepsilon \neq 0$) was performed. For energies below 1100 MeV the $F^I(s, t)$ were calculated according to the prescription given in Ref. [5]. We considered their three parametrizations, SAC1, SAC2 and SAC3 (with $a_{\rm P}^1$ = .0303, .0324, .0408, respectively). We used the phase shifts of Hyams et al. [9] between 1.1 and 1.9 GeV and a standard Regge parametrization above 1.9 GeV. The contribution of energies above 1 GeV was small, never more than a few per cent. The important contributions to the integrals are the regions around threshold and the (induced) zero of Re $F(s, 0)$ which is below 700 MeV. The left- and right-hand sides of the corresponding sum rules were compared for different values of ε. Significant discrepancies in the range of 10 to 70% are observed for SAC1 and SAC2. For example, with $\varepsilon = 0.1$ the left- and right-hand sides of the modulus sum rule were 0.315 and 0.24 for the SAC1 solution; our inverse amplitude relation yields for SAC2 and SAC3 (.44, .31) and (.68, .83), respectively (note the change of sign for the difference between the left- and right-hand sides when passing to the SAC3 solution). Of course, relations like (2) and (3) are a limiting case of corresponding derivative dispersion relations evaluated at $t = 0$ and $s = 4$. In order to get some feeling concerning stability of our results we performed the principal value calculations of the relevant dispersive integrals in the energy region between 285 and 750 MeV. In all cases a smooth transition to the results obtained at $\sqrt{s} = 280$ MeV was observed; none of the discussed Saclay solutions gave any significant discrepancy between 350 and 750 MeV — only the threshold region (280–330 MeV) led to differences. Therefore it is difficult to see how our threshold relations could be significantly improved by reasonable changes of the most characteristic parameters of the model amplitudes, without destroying either the agreement with the Roy equations and/or the very subtle balance of the derivative dispersion relations for the inverse and the modulus of the scattering amplitudes in the energy range between 350 MeV and 750 MeV.

The positivity of Im $F(s, t)$ along the cut above threshold is essential in the derivation of Eqs (2) and (3). This property is not important for the validity of the Wanders sum rule [10] (which can be derived from fixed t dispersion relations [11])

$$2a_S^0 - 5a_S^2 - 18a_P^1 = \frac{96}{\pi}\int_4^\infty \frac{ds(2s-4)A^1(s,0)}{s^2(s-4)^2} + \frac{48}{\pi}\int_4^\infty \frac{dsA_t^1(s,0)}{s^2(s-4)}, \tag{4}$$

where $A_t^1(s, 0) = (2A^0(s, 0)+3A^1(s, 0)-5A^2(s, 0))/3$. Similarly, essential for deriving [5]

$$3\pi a_P^1 = 16\int_4^\infty \frac{ds(2s-4)A^1(s,0)}{s^2(s-4)^2} + 2\int_4^\infty \frac{dsA_t^1(s,0)}{s^2} \tag{5}$$

are only analyticity, $s \leftrightarrow u$ crossing and the assumption of rho-dominance of the $I_t = 1$ amplitude at high energies. Oppositely Eq. (2) does not require any detailed assumption about the asymptotic behaviour of $F(s, 0)$. With at least constant total cross sections at infinity the integrals of Eq. (3) have faster convergence properties than those of Eq. (4). In practical applications of Eqs (1) and (2) the behavior of $F(s, 0)$ below 1 GeV is determining.

The importance of positivity is not surprising. As was recently mentioned by Piguet Wanders [12] a simultaneous requirement of crossing, analyticity and positivity of the absorptive parts of the partial wave amplitudes leads to constraints on the higher partial waves in the unphysical interval $0 < s < 4$. Our sum rules contain only physical quantities which depend essentially on the S- and P-waves and therefore have a more straightforward connection to phenomenological models.

The discussion above was limited to a particular case of the $s \leftrightarrow u$ crossing symmetric linear combination of the s-channel isospin amplitudes

$$F(s, t) = \alpha F^0(s, t) + \beta F^1(s, t) + (2\alpha+\beta)F^2(s, t)$$

with $\alpha = 0$ and $\beta = 1$. By changing α and β one varies the relative strength of the three isospin amplitudes. With $\alpha = 1$ and $\beta = 0$ one can derive in similar fashion all the constraints presented in Refs [13] and [14] without invoking total $s \leftrightarrow t \leftrightarrow u$ crossing symmetry. With our choice of α and β we have replaced the (rather inconvenient) strong D-wave dependence (through the derivative) for the $\pi^0\pi^0$ case with the fairly well known P-wave.

More detailed results of our analysis will be presented elsewhere.

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REFERENCES

[1] J. P. Baton, G. Laurens, J. Reignier, *Phys. Lett.* **25B**, 419 (1967); *Nucl. Phys.* **B3**, 349 (1967); *Phys. Lett.* **33B**, 525, 528 (1970): G. Laurens, Thesis, Saclay report CEA-N-1497 (1971).
[2] S. D. Protopopescu et al., *Phys. Rev.* **D7**, 1279 (1973).
[3] G. Grayer, B. Hyams, C. Jones, P. Schlein, P. Weilhammer, W. Blum, H. Dietl, W. Koch, E. Lorenz, G. Lütjens, M. Männer, J. Meissburger, W. Ochs, U. Stierlin, *Nucl. Phys.* **B75**, 189 (1974).
[4] S. M. Roy, *Phys. Lett.* **36B**, 353 (1971).
[5] J. L. Basdevant, C. D. Froggatt, J. L. Petersen, *Nucl. Phys.* **B72**, 413 (1974).
[6] D. Morgan, G. Shaw, *Phys. Rev.* **D2**, 520 (1970).
[7] W. Männer, *New Results in $\pi\pi$ Scattering*, paper presented at the IV Int. Conference on Experimental Meson Spectroscopy, Boston, USA, April 1974.
[8] C. Cronström, *Phys. Lett.* **49B**, 283 (1964): cf. R. Wit, *Acta Phys. Pol.* **28**, 865 (1965).
[9] B. Hyams, C. Jones, P. Weilhammer, W. Blum, H. Dietl, G. Grayer, W. Koch, E. Lorenz, G. Lütjens, W. Männer, J. Meissburger, W. Ochs, U. Stierlin, F. Wagner, *Nucl. Phys.* **B64**, 313 (1974).
[10] G. Wanders, *Helv. Phys. Acta* **39**, 228 (1966).
[11] I. I. Orlov, D. V. Shirkov, *J. Nucl. Phys.* **6**, 638 (1967).
[12] O. Piguet, G. Wanders, *Nucl. Phys.* **B78**, 313 (1974).
[13] A. K. Common, M. R. Pennington, *Nuovo Cimento* **25A**, 219 (1975).
[14] D. Robertson, R. Wit, *Nuovo Cimento Lett.* **13**, 119 (1975).