QUANTIZATION OF THE DIRAC FIELD IN RIEMANNIAN SPACES

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The present paper is an attempt to generalize the canonical quantization procedure for the Dirac field to curved spaces. Thereby the metric field acts as an unquantized background field which is not influenced by the quantized Dirac field. After the investigation of a 3-dimensional eigenvalue problem the equations of motion are derived for the field operators in the Heisenberg picture. Finally we consider special Riemannian spaces admitting the Killing vectors.

1. *Introduction*

After the investigation of the electromagnetic field in an earlier paper [1] we consider the quantization of the Dirac field with regard to the study of quantum electrodynamics in Riemannian spaces. Thereby the Dirac field is a "test field" which is influenced by the metric background field, but this does not act back on the latter. We do not restrict ourselves to spaces admitting the Killing vectors. In the same way as in [1] we consider spacelike hypersurfaces S and define 3-covariant time derivatives of the geometrical objects (bispinors and bispintensors).

With the aid of these definitions we succeed in formulating the general relativistic Dirac equation and the iterated Dirac equation in a fully 3-covariant form.

As in the case of the Maxwell field the expansion of the field operators in terms of a complete orthonormal system leads to a quantum mechanical problem with a quadratic Hamiltonian. Commutation rules are postulated and we derive equations of motion for the field operators in the Heisenberg picture.

Finally we treat space-times which have certain symmetries (Killing vectors) in connection with a preceding paper [2].

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2. *Decomposition formalism*

In this section we apply the decomposition formalism explained for tensor fields in [4] to bispinors and bispintensors.

The starting point is the selection of the space-like hypersurfaces S which have the time-like unit normal vector n_i[1] ($n_i n^i = -1$). We adapt the coordinate system to the hypersurfaces described by $x^4 =$ constant as follows:

$$n_i = n\delta_i^4, \qquad n = (-g^{44})^{-1/2}. \tag{2.1}$$

In the coordinate system (2.1) the metric tensor has the form

$$(g_{ij}) = \left(\begin{array}{c|c} g_{ab} & N_a \\ \hline N_b & -n^2 + N_a N^a \end{array}\right), \qquad (g^{ij}) = \left(\begin{array}{c|c} \overset{(3)ab}{g} - \dfrac{N^a N^b}{n^2} & \dfrac{N^b}{n^2} \\ \hline \dfrac{N^a}{n^2} & -\dfrac{1}{n^2} \end{array}\right)$$

with $\overset{(3)ab}{g} \; g_{bc} = \delta^a_c$.

By the selection of the space-like hypersurfaces we have restricted the admissible coordinate transformations to

$$x^{a'} = x^{a'}(x^a, x^4), \qquad x^{4'} = x^{4'}(x^4). \tag{2.2}$$

In the following all equations have to be formulated in such a way that they are invariant under this transformation group (3-covariance). All metric operations in such equations have to be performed with the aid of g_{ab} and $\overset{(3)ab}{g}$, e.g.

$$N^a = \overset{(3)ab}{g} \; N_b.$$

In order to formulate all equations in a 3-covariant manner we have to distinguish with respect to covariant derivatives of tensors between

$$V_{i;k} = V_{i,k} - \Gamma^m_{ik} V_m \tag{2.3a}$$

and

$$V_{a\|b} = V_{a,b} - {}^{(3)}\Gamma^c_{ab} V_c, \tag{2.3b}$$

where Γ^m_{ik} and ${}^{(3)}\Gamma^c_{ab}$ are to be performed with g_{ik} and g_{ab}, respectively. The following relation is valid [4]:

$$\Gamma^a_{bc} = {}^{(3)}\Gamma^a_{bc} - \frac{N^a}{n} K_{bc}, \qquad K_{ab} = n\Gamma^4_{ab} = \tfrac{1}{2}\,\partial_4 g_{ab}.$$

It follows that $g_{ab\|c} = 0$ and $\overset{(3)ab}{g}{}_{\|c} = 0$.

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[1] $a, b, c, \ldots = 1, 2, 3$ and $i, j, k, \ldots = 1, 2, 3, 4$.

3. *Bispinors, Dirac matrices*

3.1. Algebraic part

Regarding the bispinor formalism we adopt the corresponding treatment in [3]. The bispinor components are invariant under (2.2), but contrary to the invariants of the tensor calculus they additionally transform as follows under equivalence transformations:

$$\Psi' = \mathscr{S}\Psi, \qquad \overline{\Psi}' = \overline{\Psi}\mathscr{S}^{-1},$$

where the adjoint bispinor $\overline{\Psi}$ is defined by

$$\overline{\Psi} = \Psi^{\dagger}\beta, \ \beta^2 = 1. \tag{3.1}$$

The Dirac matrices γ_k (metric bispintensors) satisfy the basic anticommutator relation

$$\{\gamma_i, \gamma_k\} = 2g_{ik}. \tag{3.2}$$

Since they transform like tensors under coordinate transformations we can take over the following two statements from [1]:

1. Covariant spatial components of the γ_k-matrices transform like 3-tensors under (2.2).
2. Contravariant 4-components of the Dirac matrices, multiplied by n, transform like invariants under (2.2):

$$\hat{\gamma}' = \hat{\gamma}, \qquad \hat{\gamma} \equiv n\gamma^4.$$

Under equivalence transformations the Dirac matrices transform according to the rule

$$\gamma_k' = \mathscr{S}\gamma_k\mathscr{S}^{-1}.$$

Table I contains important algebraic relations for the Dirac matrices in 3-covariant form.

We have to require the validity of the relations (3.4a, b) in order to derive the adjoint Dirac equation with the aid of (3.1).

TABLE I

Algebraic relations for Dirac matrices

General covariant form	3-covariant form		Remarks
$\{\gamma_i, \gamma_k\} = 2g_{ik}$	$\{\gamma_a, \gamma_b\} = 2g_{ab}$	(3.2a)	$\overset{(3)}{\gamma}{}^a \equiv \overset{(3)}{g}{}^{ab}\gamma_b$
	$\{\gamma_a, \hat{\gamma}\} = 0$	(3.2b)	
	$\{\hat{\gamma}, \hat{\gamma}\} = -2$	(3.2c)	$\hat{\gamma}^2 = -1$
$\sigma_{ik} \equiv \frac{1}{2i}[\gamma_i, \gamma_k]$	$\sigma_{ab} \equiv \frac{1}{2i}[\gamma_a, \gamma_b]$	(3.3a)	
	$\sigma_a \equiv \frac{1}{2i}[\gamma_a, \hat{\gamma}] = i\hat{\gamma}\gamma_a$	(3.3b)	$\sigma_a \equiv n\sigma_a^4$
	$\sigma \equiv 0$	(3.3c)	$\sigma \equiv n^2\sigma^{44}$
$(\gamma^i)^{\dagger}\beta = -\beta\gamma^i$	$(\overset{(3)}{\gamma}{}^a)^{\dagger}\beta = -\beta\overset{(3)}{\gamma}{}^a$	(3.4a)	
	$\hat{\gamma}^{\dagger}\beta = -\beta\hat{\gamma}$	(3.4b)	

3.2. Analytic part

Since the bispinor formalism has to be generally covariant and moreover covariant under equivalence transformations, we have to replace partial derivatives by covariant ones:

$$\Psi_{;k} = \Psi_{,k}+\Gamma_k\Psi, \qquad \overline{\Psi}_{;k} = \overline{\Psi}_{,k}-\overline{\Psi}\Gamma_k \tag{3.5}$$

with bispinor connection coefficients satisfying the transformation law

$$\Gamma'_k = \mathcal{S}\Gamma_k\mathcal{S}^{-1}-\mathcal{S}_{,k}\mathcal{S}^{-1}.$$

The bispinor connection coefficients transform like tensors under coordinate transformations and the two statements given in Sect. 3.1 for the γ_k-matrices are valid for these coefficients too.

The covariant derivatives of the γ_k-matrices are determined by

$$\gamma_{k;j} = \gamma_{k,j}-\Gamma^m_{kj}\gamma_m+[\Gamma_j, \gamma_k] = \gamma_{k\cdot j}+[\Gamma_j, \gamma_k] \tag{3.6}$$

and for β we have

$$\beta_{;i} = 0. \tag{3.7}$$

TABLE II

Analytic relations for bispinors and bispintensors

General covariant form	3-covariant form	Remarks
$\Psi_{;k} = \Psi_{,k}+\Gamma_k\Psi$	$\Psi_{\|\|a} = \Psi_{,a}+\Gamma_a\Psi$ (3.5a) $\partial_4\Psi = \frac{1}{n}(\Psi_{,4}-\Psi_{,a}N^a)-\Gamma\Psi$ (3.5b)	$\Psi_{\|\|a} = \Psi_{;a}$ since Γ_a is a 3-tensor under (2.2)
$\Psi_{k;m} = \Psi_{k,m}-\Gamma^j_{km}\Psi_j$ $+\Gamma_m\Psi_k$	$\partial_4\Psi_a = \frac{1}{n}(\Psi_{a,4}-\Psi_{a,b}N^b-\Psi_bN^b{}_{,a})-\Gamma\Psi_a$	
$\gamma_{i;k} = 0$	$\gamma_{a\|\|b} = -\hat{\gamma}K_{ab}$ (3.6a) $\partial_4\gamma_a = -\frac{n_{,a}}{n}\hat{\gamma}+\overset{(3)}{\gamma}{}^bK_{ab}$ (3.6b) $\hat{\gamma}_{\|\|a} = -\overset{(3)}{\gamma}{}^bK_{ab}$ (3.6c) $\partial_4\hat{\gamma} = -\frac{n_{,a}}{n}\overset{(3)}{\gamma}{}^a$ (3.6d)	$\gamma_{a\|\|b} = \gamma_{b\|\|a}$
$\beta_{;i} = 0$	$\beta_{\|\|a} = 0$ (3.7a) $\partial_4\beta = 0$ (3.7b)	[a]
$\Psi_{;i;k}-\Psi_{;k;i} = G_{ki}\Psi$	$\Psi_{\|\|a\|\|b}-\Psi_{\|\|b\|\|a} = G_{ba}\Psi$ $\partial_4(\Psi_{\|\|a})-(\partial_4\Psi)_{\|\|a} = \frac{n_{,a}}{n}\partial_4\Psi+G_a\Psi$	G_{ki} : Curvature bispintensor $G_a \equiv nG_a{}^4$

[a] Comparison of (3.7a, b) with (3.6c, d) shows that the relation $\beta = i\hat{\gamma}$ used in Minkowski space is not valid in general.

In Table II important analytic relations are collected for bispinors and bispintensors. Thereby we have generalized the invariant time derivatives given in [1] for tensors to bispinors and bispintensors. Comparison with the invariant derivatives given in [1] clearly points out which terms are due to coordinate transformations and which terms are connected with equivalence transformations.

Let us turn to the bispinor connection coefficients Γ_k. These are determined by the requirement $\gamma_{i;j} = 0$. If we construct the γ_k-matrices from the constant Dirac matrices $\gamma_{(j)}$ of the Minkowski space with the aid of tetrad fields $\lambda_k^{(j)}$,

$$\gamma_k = \lambda_k^{(j)}\gamma_{(j)}, \quad \gamma_{k.j} = \lambda_{k;j}^{(m)}\gamma_{(m)},$$

the bispinor connection coefficients of the Dirac field reduce to [5]:

$$\Gamma_k = \tfrac{1}{4}\,\gamma^j\gamma_{j.k}. \tag{3.8}$$

According to (2.1) we fiix the tetrad fields $\lambda_k^{(j)}$ as follows:

$$\lambda_i^{(4)} = n_i, \quad n_i\lambda^{(a)i} = 0.$$

Applying the decomposition formalism to (3.8) we get

$$\Gamma_a = \tfrac{1}{4}\,\overset{(3)}{\gamma^b}\,\lambda_b^{(c)}{}_{a}\gamma_{(c)}, \tag{3.8a}$$

$$\Gamma \equiv n\Gamma^4 = -\tfrac{1}{4}\,(\overset{(3)}{\gamma^a}\,\gamma_{(c)}\partial_4\lambda_a^{(c)} - K) \tag{3.8b}$$

with $K = \overset{(3)}{g^{ab}}\,K_{ab}$.

4. Dirac equation

In the previous sections we obtained all the elements necessary to formulate physical laws in a 3-covariant manner. Only the derivatives described by ∂_4 and $\|$ may occur in all the following equations.

After a short calculation we get for the Dirac equation

$$\gamma^k\Psi_{;k} + \kappa_0\Psi = 0 \tag{4.1}$$

the 3-covariant form

$$\overset{(3)}{\gamma^a}\,\Psi_{\|a} + \hat{\gamma}\partial_4\Psi + \kappa_0\Psi = 0. \tag{4.1a}$$

In an analogous way we determine the 3-covariant form of the iterated Dirac equation

$$\Psi_{;k}{}^{;k} + \left(\frac{R}{4} - \kappa_0^2\right)\Psi = 0 \tag{4.2}$$

and obtain the result

$$\frac{1}{n}\,(n\Psi_{\|a})^{\|a} - \partial_4(\partial_4\Psi) - K\partial_4\Psi + \left(\frac{R}{4} - \kappa_0^2\right)\Psi = 0. \tag{4.2a}$$

5. Orthonormal systems

Let us look for a differential operator which provides an orthonormal system of functions in terms of which it is possible to expand the solutions of the Dirac equation. As this operator we take

$$D \equiv in\left\{\hat{\gamma}\overset{(3)}{\gamma^a}\nabla_a + \left[\tfrac{1}{2}\left(K + \frac{n_{,a}}{n}\hat{\gamma}\overset{(3)}{\gamma^a}\right) + \kappa_0\hat{\gamma}\right]\right\} \tag{5.1}$$

with $\nabla_a u_\Sigma \equiv u_{\Sigma\|a}$. Further we consider the eigenvalue equations

$$Du_\Sigma = m_\Sigma u_\Sigma \tag{5.1a}$$

and

$$Dv_\Sigma = -m_\Sigma v_\Sigma. \tag{5.1b}$$

In general, the eigenvalues m_Σ as well as the eigenfunctions u_Σ and v_Σ are time dependent, and the interior metric of the hypersurfaces determines the modes of the fields. (Σ labels the various eigenfunctions and is not a bispinor index.)

The differential operator D is self-adjoint if certain integrals over 3-dimensional divergencies vanish. Consequently the eigenvalues m_Σ are real and after an appropriate normalization the eigenfunctions satisfy the following orthonormality relations:

$$\int \bar{u}_\Sigma\hat{\gamma}u_{\Sigma'}\sqrt{\overset{(3)}{g}}\,d^3x = -i\delta_{\Sigma\Sigma'} = \int \bar{v}_\Sigma\hat{\gamma}v_{\Sigma'}\sqrt{\overset{(3)}{g}}\,d^3x,$$

$$\int \bar{u}_\Sigma\hat{\gamma}v_{\Sigma'}\sqrt{\overset{(3)}{g}}\,d^3x = 0. \tag{5.2}$$

Further we postulate the completeness of the system of eigenfunctions

$$\sqrt{\overset{(3)}{g}}\sum\left[u_\Sigma(\bar{x})\bar{u}_\Sigma(x) + v_\Sigma(\bar{x})\bar{v}_\Sigma(x)\right]\big|_{x,\bar{x}\in S} = i\hat{\gamma}\delta(x,\bar{x}), \tag{5.3}$$

where the two-point function on the right-hand side of each hypersurface is defined by

$$u_\Sigma(\bar{x}) = \int \delta(x,\bar{x})u_\Sigma(x)d^3x. \tag{5.4}$$

We emphasize that the completeness relation (5.4) represents an additional postulate which is not immediately derivable from (5.1). In any case it is a rather difficult mathematical problem to prove the completeness of the system of eigenfunctions of a differential operator.

The completeness relation allows for the following generalized Fourier expansion of the field Ψ:

$$\Psi = \sum(b_\Sigma u_\Sigma + d_\Sigma^\dagger v_\Sigma), \tag{5.5a}$$

$$\bar{\Psi} = \sum(b_\Sigma^\dagger \bar{u}_\Sigma + d_\Sigma\bar{v}_\Sigma). \tag{5.5b}$$

Thereby the sum covers the complete set of eigensolutions of equations (5.1) and the coefficients depend only on the time coordinate.

6. *Commutation relations*

After having investigated the classical theory let us now turn to quantum theoretical considerations. From the usual quantum field theory we take over the definition of the canonical momenta

$$\pi = \frac{\partial \mathscr{L}}{\partial \dot{\Psi}}.^{2}$$

With the aid of the Lagrangian of the Dirac field

$$\mathscr{L} = -n\sqrt{\overset{(3)}{g}}\,[\overline{\Psi}\,\overset{(3)}{\gamma^a}\,\Psi_{||a} + \overline{\Psi}\hat{\gamma}\partial_4\Psi + \kappa_0\overline{\Psi}\Psi]$$

we get

$$\pi = -\sqrt{\overset{(3)}{g}}\,\overline{\Psi}\hat{\gamma}.$$

The covariant generalization of the canonical anticommutation relations reads:

$$\sqrt{\overset{(3)}{g}}\,\{\overline{\Psi}(x)\hat{\gamma},\,\Psi(\bar{x})\}_{x,\bar{x}\in S} = -i\delta(x,\bar{x}),$$

$$\{\Psi(\bar{x}),\,\Psi(x)\}_{x,\bar{x}\in S} = 0 = \{\overline{\Psi}(\bar{x}),\,\overline{\Psi}(x)\}_{x,\bar{x}\in S}. \tag{6.1}$$

Together with expansions (5.5) these relations are completely equivalent to the equal time anticommutation relations

$$\{b_\Sigma,\,b^\dagger_{\Sigma'}\} = \delta_{\Sigma\Sigma'} = \{d_\Sigma,\,d^\dagger_{\Sigma'}\}. \tag{6.2}$$

All further anticommutators vanish.

Analogous to the usual quantum field theory the operators $b^\dagger_\Sigma$ and $d^\dagger_\Sigma$ are formally considered as creation operators and b_Σ and d_Σ as annihilation operators, respectively.

7. *Equations of motion in the Heisenberg picture*

Because of the completeness relation (5.3) we can write down the following expansions:

$$n\left[\partial_4 + \tfrac{1}{2}\left(K + \frac{n_{,a}}{n}\hat{\gamma}\,\overset{(3)}{\gamma^a}\right)\right]u_\Sigma = \sum (c_{\Sigma'\Sigma}u_{\Sigma'} + a_{\Sigma'\Sigma}v_{\Sigma'}),$$

$$n\left[\partial_4 + \tfrac{1}{2}\left(K + \frac{n_{,a}}{n}\hat{\gamma}\,\overset{(3)}{\gamma^a}\right)\right]v_\Sigma = \sum (d_{\Sigma'\Sigma}v_{\Sigma'} + b_{\Sigma'\Sigma}u_{\Sigma'}).$$

With the aid of the orthonormality relations we obtain for the coefficients, e.g.

$$c_{\Sigma'\Sigma} = i\int n\bar{u}_{\Sigma'}\hat{\gamma}\left[\partial_4 + \tfrac{1}{2}\left(K + \frac{n_{,a}}{n}\hat{\gamma}\,\overset{(3)}{\gamma^a}\right)\right]u_\Sigma\sqrt{\overset{(3)}{g}}\,d^3x \tag{7.1}$$

with $c^*_{\Sigma'\Sigma} = -c_{\Sigma\Sigma'}$, $d^*_{\Sigma'\Sigma} = -d_{\Sigma\Sigma'}$ and $a_{\Sigma'\Sigma} = -b^*_{\Sigma\Sigma'}$.

[2] Point denotes partial time derivative.

Obviously the result (7.1) is 3-covariant and we can make the coefficients vanish for static gravitational fields. Since the bispinors u_Σ and v_Σ are related to each other by the orthonormality relations (5.2) there is no connection between the coefficients $c_{\Sigma'\Sigma}$ and $d_{\Sigma'\Sigma}$. This expresses the well known fact that the bispinor field Ψ describes two different particles in Minkowski's space, namely electrons and positrons.

Inserting the expansions (5.5) into the Dirac equation (4.1a) it is possible to derive the equations of motion for the time dependent operators b_Σ and d_Σ. The result is the following system of ordinary differential equations:

$$\dot{b}_\Sigma = -im_\Sigma b_\Sigma - \sum (b_{\Sigma'} c_{\Sigma\Sigma'} + d^\dagger_{\Sigma'} b_{\Sigma\Sigma'}), \tag{7.2a}$$

$$\dot{d}^\dagger_\Sigma = \quad im_\Sigma d^\dagger_\Sigma - \sum (d^\dagger_{\Sigma'} d_{\Sigma\Sigma'} + b_{\Sigma'} a_{\Sigma\Sigma'}). \tag{7.2b}$$

By means of these equations it can easily be shown that the canonical anticommutation rules (6.2) are time independent. It means that if these anticommutation rules are fulfilled on an initial hypersurface S, the same statement is valid for all other hypersurfaces.

We can give the equations of motion (7.2) the alternative form

$$\dot{b}_\Sigma = i[H, b_\Sigma], \qquad \dot{d}^\dagger_\Sigma = i[H, d^\dagger_\Sigma] \tag{7.3}$$

if we take for the quantum Hamiltonian the Hermitian operator

$$H = \sum m_\Sigma (b^\dagger_\Sigma b_\Sigma - d_\Sigma d^\dagger_\Sigma) - i \sum \sum (c_{\Sigma\Sigma'} b^\dagger_\Sigma b_{\Sigma'}$$
$$+ d_{\Sigma\Sigma'} d_\Sigma d^\dagger_{\Sigma'} + a_{\Sigma\Sigma'} d_\Sigma b_{\Sigma'} + b_{\Sigma\Sigma'} b^\dagger_\Sigma d^\dagger_{\Sigma'}). \tag{7.4}$$

The Hamiltonian (7.4) is closely connected with the Hamiltonian H' which is given by

$$H' = \int n T^4_4 \sqrt{\overset{(3)}{g}}\, d^3x,$$

and that appears in the equation of motion

$$\dot{\Psi} = i[H', \Psi].$$

But only in stationary gravitational fields H and H' are identical because of the time-independence of u_Σ and v_Σ (see Sect. 8).

Finally we remark that the coefficients (7.1) are not conformally invariant. Therefore, in general they do not vanish in conformally flat space-times. This means that massive spin-$\frac{1}{2}$ particles can be also produced from the vacuum state in conformally flat space-times. This statement agrees with corresponding results obtained by other authors for special cosmological models [6].

8. Gravitational fields with symmetries

The goal of this section is to extend the results obtained for the real scalar field and the electromagnetic field in [2] to the Dirac field on the one hand and to specialize the results of the preceding sections of this paper to metrics with certain symmetries on the other hand.

In the classical theory the following conserved quantities correspond to the symmetries expressed by the Killing vectors ξ^i_Ω in accordance with Noether's theorem

$$E_\Omega = \int_S \xi^m_\Omega T^k_m n_k \sqrt{\overset{(3)}{g}}\, d^3x.^3 \tag{8.1a}$$

Since in the case of the Dirac field the Lagrangian vanishes if the field equations are satisfied the conserved quantities E_Ω can be written in the form

$$E_\Omega = -\int_S \overline{\Psi}\hat{\gamma}\underset{\Omega}{\pounds}\Psi \sqrt{\overset{(3)}{g}}\, d^3x, \tag{8.1b}$$

with $\underset{\Omega}{\pounds}$ describing the Lie derivative with respect to the Killing vector ξ^i_Ω.

In the quantum theory the bispinors Ψ are operators, and each symmetry transformation in space-time induces a unitary transformation in the Hilbert space [2]. The Hermitian generators of these unitary transformations correspond to the conserved quantities (8.1b) of the classical theory and the equation

$$[E_\Omega, \Psi] = -i\underset{\Omega}{\pounds}\Psi \tag{8.2}$$

is valid. Further one has to expect the validity of the relation

$$[E_\Omega, E_\Gamma] = -iC^\theta_{\Omega\Gamma}E_\theta$$

also in the case of the Dirac field. Thereby, the $C^\theta_{\Omega\Gamma}$ denote the structure constants of the group of motion. After some calculations we have

$$[E_\Omega, E_\Gamma] = -iC^\theta_{\Omega\Gamma}E_\theta - i\int R^k_{\Omega\Gamma;k} \sqrt{\overset{(3)}{g}}\, d^3x \tag{8.3}$$

with

$$R^k_{\Omega\Gamma} = -2h^k_m \xi^m_{[\Omega} \overline{\Psi}\hat{\gamma}\pounds_{\Gamma]}\Psi$$

and

$$h_{km} = g_{km} + n_k n_m.$$

If certain conditions are fulfilled it can be shown by means of the Gaussian theorem that the integral over a 3-dimensional divergence vanishes.

The existence of the time-like Killing vector $\xi^m = \delta^m_4$ gives rise to a conserved quantity (the energy) which coincides with the Hamiltonian (7.4). In this special case (8.2) yields

$$[H, \Psi] = -i\dot{\Psi}.$$

[3] For a group of motion with r parameters we have $\Omega = 1, \ldots, r$.

9. *Summary*

We investigated the quantization of the free Dirac field which is influenced by a classical gravitational background field. Our method consists in the selection of 3-dimensional spacelike hypersurfaces in an arbitrary space-time.

It is possible to construct a self-adjoint first order differential operator. Thereby, we have to pay attention to the fact that the 3-covariant derivatives and the invariant time-derivatives of the Dirac matrices do not vanish. The discussion of the corresponding eigenvalue equations (5.1a, b) shows that it is possible to derive the correct orthonormality relations (5.2).

The expansion of the field operators leads to a quantum mechanical problem with the quadratic Hamiltonian. Because of the time dependence of the eigenfunctions u_Σ and v_Σ this Hamiltonian does coincide with the one constructed from the energy-momentum tensor only in stationary gravitational fields.

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