QUANTIZED SCALAR FIELD IN CURVED SPACE-TIME

By D. Kramer

Wissenschaftsbereich Relativistische Physik, Sektion Physik, Friedrich-Schiller-Universität Jena*

(Received June 18, 1975)

The influence of a time-dependent gravitational field (Robertson–Walker metric with spherical 3-space) on a quantized scalar field conformally coupled to the geometry is studied for two different cases: prescribed background field and reaction on the metric. In the latter model, the expectation values of the stress-energy tensor are the only source terms in the Einstein equations. The equation of state contains the limiting regimes of matter-dominated ($P = 0$) and radiation-dominated ($P = \frac{1}{3}\varepsilon$) dynamics.

1. Introduction

In the Heisenberg picture the quantization of the scalar field in cosmological models has been studied by several authors [1–4 and references cited therein]. The pioneering work of Parker and Zeldovich suggests the relevance of curvature-induced creation of particle-antiparticle pairs for anisotropy damping near the cosmic singularity.

Our treatment starts with the Schroedinger equation for the probability amplitude [7]. The following viewpoints are adopted:

1. The *prescribed* gravitational field varies in a special way between the remote past and the remote future where it is static (Einstein universe), therefore a definition of the vacuum state should be possible in these asymptotic regions. The considered model admits an exact solution; we find the expression (24) for the particle-creation rate of the vacuum state.

2. We investigate the problem of the *reaction of the quantum field on the classical metric* and vice versa. Apart from the expectation values of the stress-energy tensor of the quantum field no other sources are included in Einstein's equations. We choose an appropriate state and subtract the vacuum terms. The simultaneous system of the Schroedinger and Einstein equations reduces to the equations (27). They have been numerically integrated over that range of the evolution in which the results significantly deviate from the WKB computation. Near the singularity, which cannot be avoided in the model considered here, the equation of state changes continuously from $P = 0$ to $P = \frac{1}{3}\varepsilon$.

---

* Address: Sektion Physik, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, DDR 69 Jena.

2. Classical field

The real scalar field U conformally coupled to the geometry satisfies the classical equation[1]

$$U_{,i}{}^{;i}+\left(\frac{R}{6}-m^2\right)U = 0. \tag{1}$$

From the associated Lagrangian the stress-energy tensor

$$T_{ij} = \Theta_{ij}+\frac{\hbar c}{6m}(R_{ij}U^2+U^2{}_{,i;j}-g_{ij}U^2{}_{,k}{}^{;k}),$$

$$\Theta_{ij} \equiv \frac{\hbar c}{m}\left[-U_{,i}U_{,j}+\tfrac{1}{2}\,g_{ij}\left(U_{,k}U^{,k}+m^2U^2-\frac{R}{6}\,U^2\right)\right]$$

(Θ_{ij} = canonical energy-momentum tensor) can be derived in the usual way. The tensor T_{ij} is symmetric, divergence-free, and its trace vanishes for $m = 0$,

$$T_{ij} = T_{ji}, \quad T^{ij}{}_{;j} = 0, \quad T_i^i = m\hbar c U^2. \tag{2}$$

The gravitational field is described by the metric

$$ds^2 = K^2(t)\,(-dt^2+d\sigma^2), \tag{3}$$

where K is the radius of the universe and the unit 3-sphere is endowed with the line element

$$d\sigma^2 = \gamma_{ab}dx^a dx^b, \quad a,b = 1-3, \quad \gamma \equiv \det(\gamma_{ab}).$$

By a prime we denote differentiation with respect to the dimensionless time coordinate $x^0 = t$. The independent components of the Ricci tensor are

$$R_0{}^0 = -3K^{-4}(KK''-K'^2), \quad R = -6K^{-3}(K+K'').$$

We specialize the field equation (1) and the essential component of T_{ij} to the metric (3):

$$-(K^2U')'+K^2U_{,a}{}^{;a}+K^4\left(\frac{R}{6}-m^2\right)U = 0, \tag{4}$$

$$T_0{}^0 = \frac{\hbar c}{m}\left[\tfrac{1}{2}K^{-2}U'^2+\tfrac{1}{6}K^{-2}U_{,a}U^{,a}-\tfrac{1}{3}K^{-2}U_{,a}{}^{;a}U+\tfrac{1}{6}\left(R_0{}^0-\frac{R}{2}\right)U^2 + \frac{m^2}{2}U^2+K'K^{-3}UU'\right]. \tag{5}$$

Finally, we mention the definition of the canonical momentum conjugate to the field U,

$$\Pi = \frac{\hbar}{m}\sqrt{\gamma}\,K^2U'.$$

---

[1] Indices $i, j, \ldots = 0-3$, R = scalar curvature, $m \equiv m_0c/\hbar$ (m_0 = rest mass).

3. *Schroedinger equation and expectation values*

With the aid of the well-known equal-time commutation relations and the Hamiltonian

$$H = \frac{1}{c} \int \Theta_0{}^0 K^4 \sqrt{g}\, d^3x \tag{6}$$

one easily verifies that the equations of motion

$$U' = \frac{i}{\hbar} [H, U], \quad \Pi' = \frac{i}{\hbar} [H, \Pi]$$

are completely equivalent to the operator field equation (4). We prefer the coordinate representation of the commutation relations [7]

$$U = \sum q_{nl} S_{nl}, \quad \Pi = \frac{\hbar}{i} \sqrt{\gamma} \sum S_{nl} \frac{\partial}{\partial q_{nl}}, \tag{7}$$

where U and Π are expanded in terms of spherical harmonics S_{nl}, summation runs over all modes (indices n, l). The partial derivatives with respect to q_{nl} act on the probability amplitude Ψ, which satisfies the *Schroedinger equation*

$$\hbar i \frac{\partial \Psi}{\partial t} = H\Psi,$$

$$H = \frac{\hbar}{2m} \left[-m^2 K^{-2} \sum \frac{\partial^2}{\partial q_{nl}^2} + K^2 \sum n(n+2) q_{nl}^2 + K^4 \left(m^2 - \frac{R}{6} \right) \sum q_{nl}^2 \right]. \tag{8}$$

This equation can be solved by separation. Usually we shall consider only one mode and omit the indices denoting the various modes. The formulas hold for each mode separately. For states having the proper symmetry of the space-time a summation over the index l is necessary, this explains the factor $(n+1)^2$ (=number of independent modes for fixed n) in the equations (17).

We consider two specifications of the probability amplitude:
the "*coherent state*"

$$\overset{(C)}{\Psi} = v^{-1/2} \exp\left[-\tfrac{1}{2} \lambda (q-\mu)^2\right] \tag{9}$$

and the "*N-particle state*"[2]

$$\overset{(N)}{\Psi} = u^{-1/2} \mathscr{H}_N(\nu q) \exp\left(-\tfrac{1}{2} \lambda q^2\right). \tag{10}$$

In the ordinary quantum mechanics as well as in the quantum field theory in Minkowski space-time these wave functions denote a coherent state (Glauber state) and a proper N-particle state, respectively, and the quantities λ, u, v, μ, ν are constants. Both

---

[2] $\mathscr{H}_N$ = Hermite polynomial of order N.

expressions (9) and (10) are special solutions of the Schroedinger equation (8), provided that these quantities satisfy a system of ordinary differential equations: If λ is known from

$$i\lambda' = m\lambda^2 K^{-2} - A, \qquad m \cdot A \equiv K^2(n+1)^2 + K^4 m^2 + KK'', \tag{11}$$

then the remaining functions of time can be obtained from

$$-i\frac{v'}{v} = m\lambda K^{-2} + A\mu^2, \qquad i\lambda\mu' = A\mu,$$

$$-i\frac{u'}{u} = mK^{-2}(\lambda + 2N), \qquad i\frac{v'}{v} = mK^{-2}\left(\frac{\lambda}{2} - v^2\right).$$

Introducing in place of λ a new complex variable y,

$$m\lambda = -iK^2\left(\ln\frac{y}{K}\right)', \tag{12}$$

we get the differential equation of the harmonic oscillator with time-dependent frequency,

$$y + W^2(t)y = 0, \qquad W^2 \equiv (n+1)^2 + m^2K^2 \tag{13}$$

(parametric resonance). Quantization in the Heisenberg picture also gives rise to this equation. It is convenient to use the normalization

$$y'y^* - y'^*y = i \tag{14}$$

(Wronskian condition). The *expectation value* of some dynamical quantity f is defined by

$$\bar{f} \equiv \int_{-\infty}^{\infty} \Psi^* f \Psi dq, \qquad f = f(q, p, t), \qquad p \equiv \frac{\hbar}{i}\frac{\partial}{\partial q}. \tag{15}$$

Especially, for the states mentioned above we evaluate the following expectation values:

	$\overset{(C)}{\Psi}$, Eq. (9)	$\overset{(N)}{\Psi}$, Eq. (10)
vacuum state:	$\mu = 0$	$N = 0$
$\bar{q}$	$\dfrac{\lambda^*\mu^* + \lambda\mu}{\lambda + \lambda^*}$	0
$\bar{p}$	$-hi\dfrac{\lambda\lambda^*(\mu - \mu^*)}{\lambda + \lambda^*}$	0
$\overline{q^2}$	$\dfrac{1}{\lambda + \lambda^*} + \bar{q}^2$	$\dfrac{2N+1}{\lambda + \lambda^*}$
$\overline{p^2}$	$\dfrac{\hbar^2\lambda\lambda^*}{\lambda + \lambda^*} + \bar{p}^2$	$(2N+1)\dfrac{\hbar^2\lambda\lambda^*}{\lambda + \lambda^*}$

We are interested in the expectation values of the energy-momentum tensor,

$$\overline{T}_i^j \equiv \begin{pmatrix} -P & & & \\ & -P & & \\ & & -P & \\ & & & \varepsilon \end{pmatrix} \tag{16}$$

(P — pressure, ε — energy density). Using the expressions (5), (6), (7), the definitions (12), (15), (16) and the normalization (14) we obtain for the "N-particle state" (10) the result

$$\varepsilon = \frac{\hbar c(n+1)^2}{2\pi^2 K^4} (N+\tfrac{1}{2}) (y'y'^* + W^2 yy^*),$$

$$3P = \frac{\hbar c(n+1)^2}{2\pi^2 K^4} (N+\tfrac{1}{2}) (y'y'^* - W^2 yy^* + 2(n+1)^2 yy^*). \tag{17}$$

The computation of P is facilitated by employing the trace equation in (2). The divergence relation

$$\overline{T}_{i\;;j}^{j} = 0 \rightarrow (\varepsilon K^3)' = -3PK^2K' \tag{18}$$

is automatically fulfilled. Apart from the vacuum terms the expectation values (16) formed with the coherent state (9) coincide with the corresponding classical expressions.

4. Prescribed geometry

We want to avoid approximation methods (WKB) which are usually applied to solve equation (13). In order to get an exactly solvable model, we assume for the radius of the universe the simple time dependence [9]

$$K^2 = K_0^2 + \alpha^2 \operatorname{ch}^{-2}(\beta t) \tag{19}$$

(Fig. 1). The real parameters α and β can be chosen arbitrarily. Thus, we restrict ourselves to a Robertson–Walker metric which is static before and after the non-stationary period. In the asymptotic regions $t \rightarrow \pm\infty$ the usual particle number interpretation and, in partic-

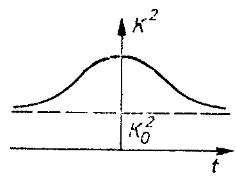

Fig. 1

ular, the definition of the vacuum state should be possible. We start with the vacuum state at $t \rightarrow -\infty$, vary the background field according to (19), and compute the change of the state in question. We have to solve the key equation (13). In general, this state taken at $t \rightarrow +\infty$ differs from the vacuum state in future time infinity. That is, the non-stationary gravitational field gives rise to particle creation [1]. The state $\overset{(0)}{\Psi}$ corresponds to the vacuum in the limit $t \to -\infty$, if we realize the asymptotic behaviour

$$t \to -\infty: \quad y \sim e^{i\omega t}; \quad \omega^2 \equiv (n+1)^2 + m^2 K_0{}^2.$$

By making the substitutions

$$y(t) = (1-\xi^2)^{i\frac{d}{2}} w(\xi), \quad z = \tfrac{1}{2}(1-\xi), \quad \xi = \text{th}\,(\beta t)$$

we transform the equation (13) into the hypergeometric differential equation

$$z(1-z)\frac{d^2 w}{dz^2} + [c-(a+b+1)z]\frac{dw}{dz} - abw = 0$$

with the abbreviations

$$a = id - s, \quad b = id + s + 1, \quad c = id + 1, \quad d = \frac{\omega}{\beta},$$

$$s = \tfrac{1}{2}\left[-1 + \left(1 + \frac{4m^2\alpha^2}{\beta^2}\right)^{1/2}\right].$$

Up to a constant factor the regular solution y with the correct behaviour for $t \to -\infty$ is

$$y = (1-\xi^2)^{\frac{id}{2}} F(a, b, a+b-c+1; 1-z) \tag{20}$$

(F — hypergeometric function). By means of the formula [9]

$$F(a, b, a+b-c+1; 1-z) = CF(a, b, c; z) + DF(a-c+1, b-c+1, 2-c; z),$$

$$C = \frac{\Gamma(a+b-c+1)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}, \quad D = \frac{\Gamma(a+b-c+1)\Gamma(-c+1)}{\Gamma(b-c+1)\Gamma(a-c+1)}$$

(Γ — gamma function) the asymptotic expression for $t \to +\infty$ ($z \to 0$) can be read off from the equation (20),

$$t \to +\infty: \quad y \sim Ce^{-i\omega t} + De^{+i\omega t},$$

where the coefficients C and D are given by

$$CC^* = \frac{\sin^2 \pi s}{\text{sh}^2(\pi d)}, \quad DD^* = 1 + CC^*. \tag{21}$$

Subtracting the vacuum terms we have, in the limit $t \to +\infty$, for the energy density and for the pressure

$$\varepsilon = \frac{\hbar c \cdot (n+1)^2}{2\pi^2 K_0{}^4}\,\omega CC^*$$

and

$$3P = \frac{\hbar c(n+1)^2}{2\pi^2 K_0{}^4 \omega} \left[(n+1)^2 CC^* - \frac{m^2 K_0{}^2}{2} (CD^* e^{-2i\omega t} + DC^* e^{+2i\omega t}) \right], \tag{22}$$

respectively. The energy density is positive definite and independent of time, while the pressure contains an oscillating part. The vacuum state in past time infinity is not energy eigenstate in the static region after the variation of the background field has been performed. The contribution of one single mode to the total energy is

$$E = \frac{\hbar c}{K_0} \omega CC^*. \tag{23}$$

The comparison of this formula with the energy eigenvalues of the scalar field in the Einstein universe [8] leads one to interpret CC^* as the mean *number of particles created* in this mode,

$$\bar{N} = CC^* = \frac{\sin^2 \pi s}{\mathrm{sh}^2 (\pi d)}. \tag{24}$$

There are certain values of the parameters, for which no particle creation occurs; in general, however, $\bar{N}$ does not vanish after the influence of the external gravitational field. For sufficiently rapid and violent variations (α and β very large) we obtain values (24) of any desired magnitude. The sum of the energies (23) over all possible modes is convergent; all zero-point energies have been eliminated. Hawking proved the conservation theorem [5]: If the stress-energy tensor obeys the *dominant energy condition* [6]

$$-\varepsilon \leqslant P \leqslant \varepsilon, \ \varepsilon \geqslant 0 \tag{25}$$

and is zero on an initial spacelike hypersurface, then it is zero at later times. Consequently, T_i^j does not obey the dominant energy condition during the process of particle creation. We state, that even in the asymptotic region $t \to +\infty$ the pressure can exceed the energy density. Choosing the parameters so that $\sin^2 \pi s = 1$, we obtain from (21), (22) for the maximum pressure

$$P_{\max} = \left[(n+1)^2 + m^2 K_0{}^2 \,\mathrm{ch} \left(\frac{\pi \omega}{\beta} \right) \right] \frac{\varepsilon}{3\omega^2}.$$

Therefore, if the inequality

$$m^2 K_0{}^2 \left(\mathrm{ch} \frac{\pi \omega}{\beta} - 1 \right) \geqslant 2\omega^2$$

holds, then the energy dominance is violated. Irrespective of the mode, this inequality can be satisfied by

$$\beta^2 < \tfrac{1}{2} \pi^2 m^2 K_0^2.$$

The model considered here does not account for the reaction of the expectation values P and ε on the geometry.

5. *Reaction on the metric*

In the previous section we considered the scalar field as a test field. Now, by contrast, the source terms of Einstein's equations are the expectation values (17). At the instant of maximum expansion of the universe ($K = K_{max}$) the state $\overset{(N)}{\Psi}$ approximately describes N independent particles with fixed energy and undetermined location. We give no account of other modes than the lowest one, $n = 0$. Because of the very large number of particles the vacuum terms can be neglected [2]. Thus, we bypass the renormalization procedure. Under the conditions

$$N \gg 1, \qquad x \equiv mK \gg 1$$

the expressions (17) are reduced to[3]

$$\begin{aligned} \varepsilon &= 3Bx^{-4}(y'y'^* + x^2yy^*), \\ P &= Bx^{-4}(y'y'^* - x^2yy^*), \end{aligned} \qquad B \text{ — const.} \tag{26}$$

The strong energy condition $3P+\varepsilon \geq 0$ is fulfilled, so that quantum effects of the scalar field in the state chosen here cannot prevent the cosmic singularity. The avoidance of the singularity by the quantum field in another state has been demonstrated in [2]; the model universe has a minimum radius in the order of the Compton wavelength of the π meson ($x \approx 1$). With the stress-energy tensor (26) the simultaneous system of the Schroedinger and Einstein equations has the remarkably simple form:

$$\left.\begin{aligned} \hbar i \frac{\partial \Psi}{\partial t} &= H\Psi \\ R_i^{\,j} - \tfrac{1}{2} g_i^{\,j} R &= \kappa \overline{T}_i^{\,j} \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} y'' + x^2 y &= 0 \\ x'' + x &= x_0 x y y^* \end{aligned}} \tag{27}$$

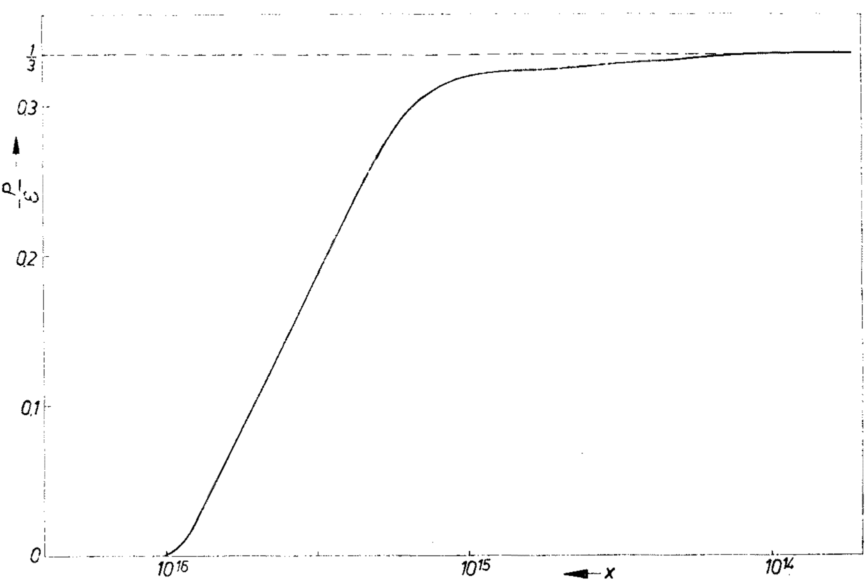

Fig. 2

[3] $B \equiv m^4 N\hbar c/6\pi^2$, numerical values: $m = 7.2 \cdot 10^{12}\ \text{cm}^{-1}$ (π mesons), $K_{max} = 5.5 \cdot 10^{27}$ cm

($x_0 \equiv mK_{max}$; $x(t)$ real, $y(t)$ complex). The first integrals of motion, (14), (18), and

$$x'^2+x^2 = x_0(y'y'^*+x^2yy^*),$$

can be used to check the numerical computations. The function x varies slowly, whereas y oscillates rapidly. The condition

$$|x'x^{-2}| \ll 1$$

(WKB regime) holds in a large range, in which the pressure is extremely small compared to the energy density. Non-interacting particles fill the universe. Therefore we approximate the model by the standard dust universe until x reaches values of about 10^{16}. Then modifications of the equation of state $P = 0$ occur. The numerical computations starting with appropriate initial values at $x = 10^{16}$ show that in the course of time the equation of state aproaches $P = \frac{1}{3}\varepsilon$. The ratio of pressure to energy density plotted against x in Fig. 2 becomes nearly constant at $x = 10^{14}$. The numerical integration has been extended up to $x \approx 10$, and no deviations from the $P = \frac{1}{3}\varepsilon$ — law have been observed. Near the singularity, the pressure and the energy density increase very rapidly — like in a universe filled with radiation.

Conclusion: The description of the matter by the expectation values $\overline{T}_i^j$ formed with the state $\overset{(N)}{\Psi}$ implies a remarkable feature: *The equation of state changes continuously from* $P = 0$ *to* $P = \frac{1}{3}$.

I want to thank Dr. W. Littke for the numerical integration.

REFERENCES

[1] Ya. B. Zeldovich, A. A. Starobinsky, *Zh. Eksp. Teor. Fiz.* **61**, 2161 (1971).
[2] L. Parker, S. A. Fulling, *Phys. Rev.* **D7**, 2357 (1973).
[3] S. A. Fulling, L. Parker, *Phys. Rev.* **D10**, 3905 (1974).
[4] A. A. Grib, S. G. Mamayev, V. M. Mostepanenko, *Izv. VUZ Fiz.* **12**, 79 (1974).
[5] S. W. Hawking, *Commun. Math. Phys.* **18**, 301 (1970).
[6] S. W. Hawking, G. F. R. Ellis, *The large scale structre of space-time*, Univ. Press, Cambridge 1973, pp. 88–96.
[7] D. Kramer, *Acta Phys. Pol.*, **B7** 227 (1976).
[8] D. Kramer, *Acta Phys. Pol.* **B4**, 11 (1973).
[9] C. D. Landau, E. M. Lifschitz, *Lehrbuch der Theor. Physik*, Quantenmechanik, Akademie-Verlag, Berlin 1965, pp. 85, 629.