ON THE SPECTRAL FORMULA FOR THE PRODUCT $\partial_{\mu_1} \ldots \partial_{\mu_n} \Delta^+(x; a) \partial_{\nu_1} \ldots \partial_{\nu_m} \Delta^+(x; b)$

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The Källén–Lehmann spectral function for the product $\partial_{\mu_1} \ldots \partial_{\mu_n} \Delta^+(x; a) \partial_{\nu_1} \ldots \partial_{\nu_m} \Delta^+(x; b)$ is calculated. It is shown that the spectral function can be obtained without explicit calculations of the integral by means of algebraic methods.

1. *General considerations*

1. When one tries to obtain a spectral representation for the product

$$\partial_{\mu_1} \ldots \partial_{\mu_n} \Delta^+(x; a) \partial_{\nu_1} \ldots \partial_{\nu_m} \Delta^+(x; b), \tag{1.1}$$

where

$$\Delta^+(x; a) = \frac{i}{(2\pi)^3} \int d^4 q e^{-iqx} \theta(q_0) \delta(q^2 - a^2),$$

one should consider the following tensor

DEF.

$$\mathcal{F}_{\mu_1 \ldots \mu_N}(p) = \int d^4 q q_{\mu_1} \ldots q_{\mu_N} \theta(q_0) \theta(p_0 - q_0) \delta(q^2 - b^2) \delta[(p-q)^2 - a^2]. \tag{1.2}$$

This integral was calculated by Thirring [1] in his book on electrodynamics for the cases of $N = 0$ and $N = 1$. For equal masses $a = b$ and $N = 2$ the integral was given by Lukierski [2]. The problem of calculation of the explicit expression for $\mathcal{F}_{\mu_1 \ldots \mu_N}(p)$ leads, for $a \neq b$ and $N > 2$, to new complications in comparison with the cases mentioned above. The tensor $\mathcal{F}_{\mu_1 \ldots \mu_N}$ depends on masses a, b and the four vector p_μ:

$$\mathcal{F}_{\mu_1 \ldots \mu_N} \equiv \mathcal{F}_{\mu_1 \ldots \mu_N}(p_\mu; a, b).$$

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We would like to write explicitly this dependence and use it further to obtain for $\partial_{\mu_1}\ldots\partial_{\mu_n}\Delta^+(x;a)\partial_{\nu_1}\ldots\partial_{\nu_m}\Delta^+(x;b)$ a formula analogous to the one given by Thirring for the product $\Delta^+(x;a)\Delta^+(x;b)$.

At first some simple results and definitions.

$$p_\mu q^\mu \equiv p\cdot q \equiv A = \frac{p^2-a^2+b^2}{2},$$

$N = 0$

$$\mathscr{F} \equiv f \equiv f(p^2;a,b)$$

$$= \frac{\pi}{2p^2}[(p^2-a^2-b^2)^2-4a^2b^2]^{1/2}\theta(p_0)\theta[p^2-(a+b)^2],$$

$N = 1$

$$\mathscr{F}_\mu = \frac{A}{p^2}f(p^2;a,b)p_\mu.$$

The tensor $\mathscr{F}_{\mu_1\ldots\mu_N}$ is a symmetric tensor with nonvanishing traces for any pair of μ's. So, in general, we can say only that it is a linear combination of all linearly independent tensors of N-th rank built out of the tensor $g^{\mu\nu}$ and the four-vector p^μ. Now we are going to construct all these linearly independent tensors and then to express $\mathscr{F}_{\mu_1\ldots\mu_N}$ with help of them.

We introduce a short-hand notation

$$C(s) = \{i_s \neq i_r, i_s \neq i_{r-1} \ldots i_s \neq i_{s+1}\},$$

$$C(s,n) = \{i_n \neq i_1 \neq j_n, i_n \neq i_2 \neq j_n \ldots i_n \neq i_s \neq j_n\}.$$

This is a set of conditions for single sum over i_s:

$$C(s) = \{i_r \neq i_s \neq j_r, i_{r-1} \neq i_s \neq j_{r-1}, \ldots, i_{s+1} \neq i_s \neq j_{s+1},$$
$$i_r \neq j_s \neq j_r, i_{r-1} \neq j_s \neq j_{r-1}, \ldots, i_{s+1} \neq j_s \neq j_{s+1}\}.$$

This is a set of conditions for double sum over i_s and j_s: $r > s \geqslant 1$, $n = 1, 2, \ldots, r$. With the help of these abbreviations we are able to construct out of $g^{\mu\nu}$-tensor and p^μ-vectors some very useful symmetric tensors.

DEF.

$$[g;\mu_1\ldots\mu_N] = \frac{1}{r!}\sum_{\substack{i_r<j_r}}^{N}\ \sum_{\substack{i_{r-1}<j_{r-1}\\ C(r-1)}}^{N}\cdots\sum_{\substack{i_1<j_1\\ C(1)}}^{N} g_{\mu_{i_r}\mu_{j_r}}g_{\mu_{i_{r-1}}\mu_{j_{r-1}}}\cdots g_{\mu_{i_1}\mu_{j_1}},$$

$$r = \frac{N}{2},\quad N - \text{even}. \tag{1.3}$$

For $N-s$ even it follows

$$[g;\mu_1\ldots\hat{\mu}_{i_1}\ldots\hat{\mu}_{is}\ldots\mu_N]$$

$$=\frac{1}{\left(\frac{N-s}{2}\right)!}\sum_{k_r<l_r}^{N}\sum_{k_{r-1}<l_{r-1}}^{N}\cdots\sum_{k_1<l_1}^{N}g_{\mu_{k_r}\mu_{l_r}}g_{\mu_{k_{r-1}}\mu_{l_{r-1}}}\cdots g_{\mu_{k_1}\mu_{l_1}},$$

$$C(r-1)\ldots C(1)$$

$$C(r,s)C(r-1,s)\ldots C(1,s)$$

where $r=(N-s)/2$ and the sign $\wedge$ over μ_i means "no μ_i".

DEF.

$$[g;p;\mu_1\ldots\mu_N;s]$$

$$=\frac{1}{s!}\sum_{i_s}^{N}\sum_{i_{s-1}}^{N}\cdots\sum_{i_1}^{N}p_{\mu_{i_1}}p_{\mu_{i_2}}\cdots p_{\mu_{i_s}}[g;\mu_1\ldots\hat{\mu}_{i_1}\ldots\hat{\mu}_{i_s}\ldots\mu_N]. \tag{1.4}$$

Let us notice that

$$[g;p;\mu_1\ldots\mu_N;0]=[g;\mu_1\ldots\mu_N],$$

$$[g;p;\mu_1\ldots\mu_N;N]=p_{\mu_1}p_{\mu_2}\cdots p_{\mu_N}.$$

2. With the help of tensors $[g;p;\mu_1\ldots\mu_N;s]$ we can express the tensor $\mathscr{F}_{\mu_1\ldots\mu_N}$ as follows

N — even

$$\mathscr{F}_{\mu_1\ldots\mu_N}=\sum_{n=0}^{N/2}[g;p;\mu_1\ldots\mu_N;2n]fC_{2n}^{(N)}. \tag{1.5a}$$

N — odd

$$\mathscr{F}_{\mu_1\ldots\mu_N}=\sum_{n=0}^{(N-1)/2}[g;p;\mu_1\ldots\mu_N;2n+1]fC_{2n+1}^{(N)}. \tag{1.5b}$$

Our next task is to determine the coefficients $C_k^{(N)}$. We can always do it for any N because $\mathscr{F}_{\mu_1\ldots\mu_N}$ fulfils the following equations:

N — even

$$p_{\mu_1}\cdots p_{\mu_N}\mathscr{F}_{\mu_1\ldots\mu_N}=A^Nf,$$

$$g_{\mu_1\mu_2}p_{\mu_3}\cdots p_{\mu_N}\mathscr{F}_{\mu_1\ldots\mu_N}=A^{N-2}b^2f,$$

$$\vdots$$

$$g_{\mu_1\mu_2}g_{\mu_3\mu_4}\cdots g_{\mu_{N-1}\mu_N}\mathscr{F}_{\mu_1\ldots\mu_N}=b^Nf. \tag{1.6a}$$

N — odd

$$p_{\mu_1} \cdots p_{\mu_N} \mathscr{F}_{\mu_1 \dots \mu_N} = A^N f,$$
$$g_{\mu_1 \mu_2} p_{\mu_3} \cdots p_{\mu_N} \mathscr{F}_{\mu_1 \dots \mu_N} = N^{N-2} b^2 f,$$
$$\vdots$$
$$g_{\mu_1 \mu_2} \cdots g_{\mu_{N-2} \mu_{N-1}} p_{\mu_N} \mathscr{F}_{\mu_1 \dots \mu_N} = A b^{N-1} f. \tag{1.6b}$$

These are linear equations for $C_k^{(N)}$. It can be easily seen that the first one can be written in the form

N — even

$$\sum_{n=0}^{N/2} \binom{N}{2n} (N-2n)!!\, p^{N+2n} C_{2n}^{(N)} = A^N, \tag{1.7a}$$

N — odd

$$\sum_{n=0}^{(N-1)/2} \binom{N}{2n+1} (N-2n-1)!!\, p^{N+2n+1} C_{2n+1}^{(N)} = A^N, \tag{1.7b}$$

where

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}.$$

This form of equations follows from the relation

$$p_{\mu_1} \cdots p_{\mu_N} [g;\, p;\, \mu_1 \dots \mu_N;\, s] = (N-s)!! \binom{N}{s} p^{N+s}.$$

To obtain other equations in explicit form we must know how to perform the contractions written below

$$g_{\mu_1 \mu_2} \cdots g_{\mu_k \mu_{k+1}} p_{\mu_{k+2}} \cdots p_{\mu_N} [g;\, p;\, \mu_1 \dots \mu_N;\, l] \tag{1.8}$$

where

$$k, l = 1, 2, \dots, N-1.$$

In Section 2 we perform the required calculations. Here we write only the result, i.e. all the linear equations satisfied by the coefficients $C_k^{(N)}$.

N — even

$$\sum_{n=0}^{N/2} \sum_{\substack{i=-M \\ \text{step } 2}} E_i^{(M)}(2n) D_{2n+i}(N-M) C_{2n}^{(N)} = b^{N-M} A^M, \tag{1.9a}$$

step $2 \equiv \{i = -M, -M+2, \dots, M-2, M\}$.

N — odd

$$\sum_{n=0}^{(N-1)/2} \sum_{\substack{i=-M \\ \text{step } 2}}^{M} E_i^{(M)}(2n+1) D_{2n+1+i}(N-M) C_{2n+1}^{(N)} = b^{N-M} A^M, \tag{1.9b}$$

where

$$D_s(N) = \frac{N!!(N+2)!!}{s!!(s+2)!(N-s)!!} p^s \tag{1.10}$$

and $E_i^{(M)}$ (s) are given by recurrence formulas in Section 2.

3. It is easy to notice that $\partial_{\mu_1} \ldots \partial_{\mu_n} \Delta^+(x; a) \partial_{\nu_1} \ldots \partial_{\mu_m} \Delta^+(x; b)$ can be expressed in terms of $\mathscr{F}_{\mu_1 \ldots \mu_N}(p)$ tensors as follows:

$$\partial_{\mu_1} \ldots \partial_{\mu_n} \Delta^+(x; a) \partial_{\nu_1} \ldots \partial_{\nu_m} \Delta^+(x; b)$$

$$= -\frac{1}{(2\pi)^6} \sum_{l=0}^{n} \frac{(-i)^{n-l}}{l!} \sum_{i_1 \ldots i_l}^{n} i^l \partial_{\mu_{i_1}} \ldots \partial_{\mu_{i_l}} \int d^4 p e^{-ipx} \mathscr{F}_{\mu_1 \ldots \hat{\mu}_{i_1}, \hat{\mu}_{i_2} \ldots \mu_{i_l} \ldots \mu_n, \nu_1 \ldots \nu_m}. \tag{1.11}$$

We must distinguish two cases of N odd and N even, and include both in the sum $\sum_{l=0}^{n}$. For this purpose we introduce odd and even deltas

$$\delta^{\mathrm{E}}(N) = \begin{cases} 0; & N - \text{odd} \\ 1; & N - \text{even}, \end{cases}$$

$$\delta^{O}(N) = 1 - \delta^{\mathrm{E}}(N).$$

The final expression for (1.1) looks as follows:

$$\partial_{\mu_1} \ldots \partial_{\mu_n} \Delta^+(x; a) \partial_{\nu_1} \ldots \partial_{\nu_m} \Delta^+(x; b)$$

$$= \frac{i}{(2\pi)^3} \sum_{l=0}^{n} \frac{(-i)^{n-l}}{l!} \sum_{i_1, \ldots, i_l = 1}^{n} \partial_{\mu_{i_1}} \ldots \partial_{\mu_{i_l}}$$

$$\times \{\delta^{\mathrm{E}}(m+n-l) \sum_{k=0}^{(m+n-l)/2} [g; -i\partial; \mu_1 \ldots \hat{\mu}_{i_1} \ldots \hat{\mu}_{i_l} \ldots \mu_n, \nu_1 \ldots \nu_m; 2k]$$

$$\times \int_0^\infty ds C_{2k}^{(m+n-l)} f(s) \Delta^+(x; s)$$

$$+ \delta^{O}(m+n-l) \sum_{k=0}^{(m+n-l-1)/2} [g; -i\partial; \mu_1 \ldots \hat{\mu}_{i_1} \ldots \hat{\mu}_{i_l} \ldots \mu_n, \nu_1 \ldots \nu_m; 2k+1]$$

$$\times \int_0^\infty ds C_{2k+1}^{(m+n-l-1)} f(s) \Delta^+(x; s)\}. \tag{1.12}$$

2. *Calculation of $D_s(N)$ and $E_i^{(M)}(n)$ coefficients*

For even N we define

$$D_s(N) = g_{\mu_1\mu_2} \dots g_{\mu_{N-1}\mu_N}[g;\, p;\, \mu_1 \dots \mu_N;\, s]. \tag{2.1}$$

It follows from (2.1) that

$$D_N(N) = p^N, \qquad D_0(N) = \frac{(N+2)!!}{2}.$$

Using the following property of $[g; p; \mu_1 \dots \mu_N; s]$

$$[g;\, p;\, \mu_1 \dots \mu_N;\, s] = \frac{2}{N-s}\{g_{\mu_1\mu_2}[g;\, p;\, \mu_3 \dots \mu_N;\, s] \dots$$

$$+ g_{\mu_1\mu_N}[g;\, p;\, \mu_2 \dots \mu_{N-1};\, s] + g_{\mu_2\mu_3}[g;\, p;\, \mu_1\mu_4 \dots \mu_N;\, s] \dots$$

$$+ g_{\mu_2\mu_N}[g;\, p;\, \mu_1, \mu_3 \dots \mu_{N-1};\, s] + \dots + g_{\mu_{N-1}\mu_N}[g;\, p;\, \mu_1 \dots \mu_{N-2};\, s]\} \tag{2.2}$$

one obtains

$$D_s(N) = \frac{2}{N-s}\left\{\frac{N(N-1)}{2} + \frac{3N}{2}\right\} D_s(N-2). \tag{2.3}$$

From (2.3) we get

$$D_s(N) = !!\prod_{n=s+2}^{N} \frac{(n+2)n}{n-s}\, p^s, \tag{2.4}$$

where

$$!!\prod_{n=i}^{N} a_n = a_i a_{i+2} \dots a_N, \qquad i-\text{even}.$$

With the help of induction one can prove another useful property of $[g; p; \mu_1 \dots \mu_N; s]$ tensors. Namely

$$p_{\mu_1}[g;\, p;\, \mu_1 \dots \mu_N;\, s]$$

$$= p^2[g;\, p;\, \mu_2 \dots \mu_N;\, s-1] + \frac{1}{s-1}[g;\, p;\, \mu_2 \dots \mu_N;\, s+1]. \tag{2.5}$$

We can now represent the contraction $p_{\mu_1} \dots p_{\mu_n}[g; p; \mu_1 \dots \mu_N; s]$ for $M < N$ in the form

$$p_{\mu_1} p_{\mu_2} \dots p_{\mu_M}[g;\, p;\, \mu_1 \dots \mu_N;\, s]$$

$$= \sum_{\substack{i=-M \\ \text{step } 2}}^{M} E_i^{(M)}(s)\,[g;\, p;\, \mu_{M+1} \dots \mu_N;\, s+i], \tag{2.6}$$

where "step 2" means that $i = -M, \ -M+2, \ldots, M-2, M$ and $E_i^{(M)}(s) = 0$ for all $i+s < 0$ and $i+s > N-M$. Using (2.5) ans (2.6) one can show that

$$E_l^{(k)}(s) = p^2 E_{l+1}^{(k-1)}(s) + (l+s) E_{l-1}^{(k-1)}(s),$$

$$E_{-l}^{(l)}(s) = p^{2l}\theta(s-l)\theta(N-s),$$

$$E_l^{(l)}(s) = \frac{(s+l)!}{s!}\theta(N-2l-s)\theta(l+s),$$

$$l = -k+2, \ -k+4, \ldots, k-2, \tag{2.7}$$

where $\theta(k) = 0$ for $k < 0$ and $\theta(k) = 1$ for $k \geqslant 0$. Unfortunately we are unable to solve (2.7).

We now perform the contraction

$$g_{\mu_N \mu_{N-1}} \ldots g_{\mu_{M+2}\mu_{M+1}} p_{\mu_M} \ldots p_{\mu_1}[g; p; \mu_1 \ldots \mu_N; s] = \sum_{\substack{i=-M \\ \text{step } 2}} E_i^{(M)}(s) D_{s+i}(N-M). \tag{2.8}$$

Equations (1.9a) and (1.9b) follow from (2.8).

3. *Final remarks*

The scheme presented above could be applied further to the calculations of expression (1.1) for arbitrary n and m.

Our formula (1.12) is useful in the calculation of self-energy diagrams for particles with higher spins. This application will be considered in our next paper.

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