RELATIVE DYNAMICS OF THE CLASSICAL THEORY OF FIELDS*

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The paper discusses dynamical properties of the Jacobi field, i.e. the field of perturbations, in general, classical field theories. Propositions about the connection between the Jacobi equations and the principle of stationary action in field theories are formulated and then some first integrals of these equations derived. Their comparison with first integrals of the Lagrange equations suggests the necessity of a generalization of the concept of the Jacobi field. Such a generalization leads to an infinite set of fields which defines the relative dynamics of a given field theory. In concluding remarks attention is paid to the possible meaning of the relative dynamics for the quantization of nonlinear field theories, and in particular of general relativity.

1. *The Jacobi fields*

We shall here consider a general, dynamical theory of fields $\psi^A(x^\mu)$ on an n-dimensional manifold M covered with domains of a coordinate system $\{x^\mu\}$; $\mu = 1, 2, ..., n$. The manifold M may be, in particular, a four dimensional Riemannian or Minkowski space ($\mu = 0, 1, 2, 3$), as well as the one dimensional time axis ($\mu = 0$) of classical mechanics. The index A ($= 1, 2, ..., N$) is a collective index representing collections of tensorial, spinorial ect. indices of appropriate fields of the classical field theory or of the generalized coordinates of mechanics. We assume that the functions $\psi^A(x^\mu)$ admit continuous derivatives of an order sufficiently high to make the here presented constructions feasible. Denoting[1] $\psi^A_\mu := \partial\psi^A/\partial x^\mu$, we assume that in the considered theory an action functional of the form

$$W[\psi^A] = \int_\Omega L(\psi^A, \psi^A_\mu, x^\mu)dx \tag{1.1}$$

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[1] The sign := means "equal by definition".

is given, with the Lagrangian[2] L being a sufficiently regular function of its arguments. In the formula above the domain of integration Ω is an n-dimensional region of the manifold M. Its choice may depend on the particular properties of a specific theory under consideration. In spacetime, for instance, it frequently is the region bounded by two spacelike hypersurfaces, and in mechanics a finite time interval.

The stationary action principle: *The variation of the action (1.1) $\delta W = 0$ for variations $\delta\psi^A$ arbitrary in the interior of the region Ω, and vanishing on its boundary $\partial\Omega$,* leads, obviously, to the dynamical equations

$$\frac{\delta W}{\delta\psi^A} := \frac{\partial L}{\partial\psi^A} - \frac{d}{dx^\mu}\frac{\partial L}{\partial\psi^A_\mu} = 0, \tag{1.2}$$

being differential equations (partial for $n > 1$, and ordinary for $n = 1$) of the second order for fields $\psi^A(x^\mu)$ (d/dx^μ denotes here the complete derivative with respect to x^μ).

Let us suppose that there is given a one-parametric family of solutions $\psi^A = \psi^A(x^\mu, \varepsilon)$ of (1.2) which corresponds to a one-parametric family of initial conditions. We assume the functions ψ^A to be differentiable with respect to the parameter ε. Let us distinguish in the considered family the field $\tilde{\psi}^A(x^\mu) := \psi^A(x^\mu, 0)$, and define on M a new field $\varrho^A(x^\mu) := \partial\psi^A/\partial\varepsilon|_{\varepsilon=0}$. From the Eqs (1.2), which are now, after the substitution of $\psi^A = \psi^A(x^\mu, \varepsilon)$, fulfilled identically, we get an identity[3] for the just defined field $\varrho^A(x^\mu)$,

$$\frac{\delta^2 W}{\delta\psi^A\partial\psi^B}(\varrho^B) := \frac{\partial}{\partial\psi^A}\left(\frac{\partial L}{\partial\psi^B}\varrho^B + \frac{\partial L}{\partial\psi^B_\sigma}\varrho^B_\sigma\right) - \frac{d}{dx^\mu}\frac{\partial}{\partial\psi^A_\mu}\left(\frac{\partial L}{\partial\psi^B}\varrho^B + \frac{\partial L}{\partial\psi^B_\sigma}\varrho^B_\sigma\right), \tag{1.3}$$

in which all coefficients are evaluated for the distinguished solution $\tilde{\psi}^A(x^\mu)$. The differential operator $\delta^2 W/\delta\psi^A\partial\psi^B(\cdot)$ appearing in (1.3), is called here *the second variational derivative of the functional* (1.1). The details of its intrinsic definition are stated in [1]. The differentiation in (1.3) should be performed under the assumption that ϱ^A is independent of ψ^A, and does only depend on x^μ.

When a field $\varrho^A(x^\mu)$, fulfilling (1.3), is already given, we may forget about its construction by means of a family of solutions of the Lagrange equations (1.2), and accept as the starting point the relation (1.3) treated now as a set of differential equations for $\varrho^A(x^\mu)$. In more precise terms, to any fixed solution $\psi^A(x^\mu)$ of (1.2) there corresponds a field equation for $\varrho^A(x^\mu)$, the Eq. (1.3), and its coefficients evaluated for the fixed $\psi^A(x^\mu)$ are given functions of x^μ. According to the terminology of variational calculus, the Eq. (1.3) is then called *the Jacobi equation* of the variational problem for the action (1.1) and the field $\varrho^A(x^\mu)$ — *the Jacobi field* corresponding to the chosen solution $\psi^A(x^\mu)$ of the Lagrange equations. The field $\varrho^A(x^\mu)$ is, for a given $\psi^A(x^\mu)$, defined by the Eq. (1.3) with some subsidiary conditions imposed which usually amount to initial or boundary conditions. The

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[2] The term "Lagrangian" is used in this article also as a synonym of the normally used "density of the Lagrange function".

[3] The summation convention with respect to repeated indices of all kinds is accepted all over this article.

freedom of choice of such subsidiary conditions now corresponds to the possibility of defining the field $\varrho^A(x^\mu)$ by means of several one-parametric families $\psi^A(x^\mu, \varepsilon)$ (which for $\varepsilon = 0$ reduce always to the same $\psi^A(x^\mu)$) according to the first of just described ways of defining this field.

In mechanics the Jacobi field may be interpreted as a vector field along a given trajectory of the Lagrange equations. It describes, in the first approximation, the relative motion along a trajectory adjacent to the given one. The Jacobi field of geodesics in a Riemannian space is called the geodesic deviation [2]. In general relativity the geodesic deviation is a useful tool for the physical interpretation of curvature [3–8]; it is also the basic quantity in the theory of gravitational waves detectors of the type used by Weber [9, 10]. In the field theory the Jacobi field describes, in the first approximation, the perturbations of the field caused by perturbations of the imposed subsidiary conditions. In this context one should mention that *the Eq. (1.3) as a homogeneous one will, in general, admit a non-vanishing solution only for the nonhomogeneous conditions imposed.* Physically, it means, for instance, that the vanishing of the Jacobi field and its gradients at an initial instant of time implies their vanishing for ever.

From the mathematical point of view, the set of all solutions of the Lagrange equations (1.3), endowed with an appropriate topology, forms a differentiable manifold Ψ, in general infinitely many dimensional, modelled on the Banach space. The set of all linear combinations of the Jacobi fields corresponding to a solution ψ^A is then the tangent space to the manifold Ψ at ψ^A.

The Jacobi fields were usually discussed in connection with the problem of stability of solutions of (1.2) and with the formulation of sufficient conditions for the variational problem of the action functional (1.1). Here these properties of the Jacobi fields, belonging rather to global problems of mathematical analysis, will not be touched at all. This paper will, instead, concentrate on some local, but dynamical properties of these fields and of some other fields which form their natural generalization and are defined in the third section.

2. The Jacobi equations and the variational principle

One may expect that the Jacobi equation (1.3), being closely linked with the Lagrange equations, will also be related to the variational principle for the action (1.1). Such a relation forms, among others, the subject of the following three propositions.

Proposition 2.1. Let ψ^A be a solution of the Lagrange equations (1.2), and its variation $\delta\psi^A$ a solution of the Jacobi equation (1.3)

$$\frac{\delta^2 W}{\delta\psi^A \partial\psi^B}(\delta\psi^B) = 0,$$

then $\delta W + \frac{1}{2}\delta^2 W$, the variation of the action (1.1) up to the second variation, is equal to a surface integral over the boundary $\partial\Omega$ of the domain Ω of integration in (1.1) (or, differently speaking, $\delta W + \frac{1}{2}\delta^2 W$ vanishes *modulo* the "surface terms").

The proof of this proposition directly follows from the following expression for the variation of the action W up to the second variation

$$\delta W+\tfrac{1}{2}\,\delta^2 W = \int\limits_{\Omega}\left[\left(\frac{\delta W}{\delta\psi^A}+\tfrac{1}{2}\,\frac{\delta^2 W}{\delta\psi^A\partial\psi^B}(\delta\psi^B)\right)\delta\psi^A+\tfrac{1}{2}\,\frac{\delta W}{\delta\psi^A}\,\delta^2\psi^A\right]dx$$

$$+\int\limits_{\partial\Omega} dS\, n_\mu\left[\delta\psi^A\,\frac{\partial}{\partial\psi^A_\mu}(L+\tfrac{1}{2}\,\delta L)+\tfrac{1}{2}\,\delta^2\psi^A\,\frac{\partial L}{\partial\psi^A_\mu}\right], \tag{2.1}$$

where n_μ is a unit covariant vector normal to the boundary $\partial\Omega$ of the domain Ω and

$$\delta L := \frac{\partial L}{\partial\psi^B}\,\delta\psi^B+\frac{\partial L}{\partial\psi^B_\mu}\,\delta\psi^B_\mu.$$

The proposition above or statements equivalent to it were known since the beginning of variational calculus. One ought to mention that from the vanishing of the second variation *modulo* surface terms it is not possible to deduce that $\delta\psi^A$ must fulfil the Jacobi equation, since to integrals of the form $\int\delta\psi F[\delta\psi]dx$ ($F[\delta\psi]$ is a functional of $\delta\psi$) one cannot apply the Lagrange–Haar lemma: *the vanishing of the integral does not imply here the vanishing of* $F[\delta\psi]$. The Jacobi equation is, however, related to a stationary action principle with a Lagrangian being the second differential of the Lagrangian in (1.1). A more precise statement of that fact is contained in the following proposition.

Proposition 2.2. (The accessoric action principle.) Let ψ^A be a given solution of the Lagrange equations (1.2) and let

$$\mathscr{L}(\varrho^A,\varrho^A_\mu,x^\sigma) := \tfrac{1}{2}\left[\left(\frac{\partial^2 L}{\partial\psi^A\partial\psi^B}\right)_{\tilde\psi}\varrho^A\varrho^B+2\left(\frac{\partial^2 L}{\partial\psi^A\partial\psi^B_\mu}\right)_{\tilde\psi}\varrho^A\varrho^B_\mu+\left(\frac{\partial^2 L}{\partial\psi^A_\mu\partial\psi^B_\nu}\right)_{\tilde\psi}\varrho^A_\mu\varrho^B_\nu\right], \tag{2.2}$$

where the derivatives on the right hand side are evaluated for $\psi^A = \tilde\psi^A(x^\mu)$, i.e. for a given $L(\psi^A,\psi^A_\sigma,x^\sigma)$ are given functions of x^μ. Then the Euler–Lagrange equations of the variational principle

$$\delta\int\limits_{\Omega}\mathscr{L}(\varrho^A,\varrho^A_\mu,x^\sigma)dx = 0 \tag{2.3}$$

under the condition $\delta\varrho^A = 0$ for $x\in\partial\Omega$ turn out to be the Jacobi equations

$$\frac{\delta^2 W}{\delta\psi^A\partial\psi^B}(\varrho^B) = 0.$$

The proof of this proposition is based on the standard procedure used in variational calculus and on the form of (1.3). The thesis of Proposition 2.2 is ascribed to Carathéodory [11] who also introduced the term "accessoric variational problem" meant as additional to the main or original variational problem for the action (1.1). The accessoric variational problem for the geodesic line in a Riemannian space was formulated in a nonpublished work of Plebański [12]. From the point of view of certain aesthetics of the theory, the accessoric Lagrangian may appear to be constructed *ad hoc*, being not grounded before its Euler–Lagrange equations. The relation between the variational principle for the action (1.1) and the Jacobi equations seems to find a more perfect form in the next proposition.

Proposition 2.3. (The action principle for mappings of fields.) Let Ψ be a set of all sufficiently regular fields ψ^A on a region Ω of the manifold M. Let us consider an arbitrary, infinitesimal mapping of Ψ onto itself, described by a generator $\Delta\psi^A$, or in other words, a mapping $\psi^A(x^\mu) \mapsto \tilde{\psi}^A(x^\mu)$, for which there is such an $\varepsilon \in \boldsymbol{R}$, and such a function $O(x^\mu, \varepsilon)$ that

$$\tilde{\psi}^A(x^\mu) = \psi^A(x^\mu) + \Delta\psi^A(x^\mu)\varepsilon + O(x^\mu, \varepsilon) \tag{2.4}$$

and $\lim_{\varepsilon \to 0} \varepsilon^{-1} O(x^\mu, \varepsilon) = 0$ for any $x \in \Omega$, i.e. $O(x^\mu, \varepsilon)$ tends to zero, for $\varepsilon \to 0$, faster than ε.

1) The necessary and sufficient condition that the mapping (2.4), characterized by a generator $\Delta\psi^A$, transforms the fields $\psi^A(x^A)$ fulfilling the stationary action principle for W of the form (1.1), into fields $\tilde{\psi}^A(x^\mu)$ fulfilling the same principle up to terms which tend to zero, for $\varepsilon \to 0$, faster than ε, reads as

$$\delta W^{(1)}[\psi^A; \Delta\psi^A] = 0, \tag{2.5}$$

where

$$W^{(1)}[\psi^A; \Delta\psi^A] := \int_\Omega \left(\frac{\partial L}{\partial \psi^A} \Delta\psi^A + \frac{\partial L}{\partial \psi^A_\mu} \Delta\psi^A_\mu \right) dx \tag{2.6}$$

is the first Fréchet differential of the functional (1.1), and the variation in (2.5) is performed for variations being arbitrary in the interior of Ω and fulfilling on its boundary $\partial\Omega$ the condition

$$\delta\psi^A = 0 \quad \text{for} \quad x \in \partial\Omega. \tag{2.7}$$

The variables $\Delta\psi^A$ are not subjected to variations in (2.5) at all.

2) The functional (2.6), depending on two fields[4]: ψ^A and $\Delta\psi^A$, is an action functional of a new variational principle of the form (2.5), but now the variation is performed for variations $\delta\psi^A$ and $\delta\Delta\psi^A$ arbitrary in the interior of Ω and satisfying on its boundary the conditions

$$\delta\psi^A = 0, \quad \delta\Delta\psi^A = 0 \quad \text{for} \quad x \in \delta\Omega. \tag{2.7a}$$

The Euler–Lagrange equations of this action principle are:

(*i*) the original Lagrange equations (1.2), when (2.6) is being varied with respect to $\Delta\psi^A$;

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[4] It should be stressed that $\Delta\psi^A$ is here an entity by itself — a completely new field on M, wholly independent of ψ^A. The notation $\Delta\psi^A_\mu$ is used as abbreviation of $\partial\Delta\psi^A/\partial x^\mu$. The symbol Δ alone has in this paper no operational meaning and thus such symbols as e.g. $\Delta\ \partial\psi^A/\partial x^\mu$ will here not occur.

(*ii*) the Jacobi equations

$$\frac{\delta^2 W}{\delta\psi^A \partial\psi^B}(\Delta\psi^B) = 0, \tag{2.8}$$

when one varies (2.6) with respect to ψ^A.

The first part of this proposition directly follows from the definition of the differential $W^{(1)}[\psi^A; \Delta\psi^A]$ of a functional. According to this definition

$$W[\tilde{\psi}^A] = W[\psi^A] + W^{(1)}[\psi^A; \Delta\psi^A] + r(\varepsilon),$$

under the condition for the remainder: $r(\varepsilon)\varepsilon^{-1} \to 0$ for $\varepsilon \to 0$. Varying with respect to ψ^A both sides of the equation above we get, up to $\varepsilon^{-1}r(\varepsilon)$, the thesis (2.5).

The proof of the second part uses the standard procedure of variational calculus. For the variation $\delta W^{(1)}[\psi^A; \Delta\psi^A]$ with respect to $\delta\psi^A$ and $\delta\Delta\psi^A$ we get

$$\delta W^{(1)}[\psi^A; \Delta\psi^A] = \int\limits_\Omega \Bigg[\frac{\partial^2 L}{\partial\psi^A \partial\psi^B}\delta\psi^A \Delta\psi^B + \frac{\partial^2 L}{\partial\psi^A \partial\psi^B_\nu}(\delta\psi^A \Delta\psi^B_\nu + \delta\psi^B_\nu \Delta\psi^A)$$

$$+ \frac{\partial^2 L}{\partial\psi^A_\mu \partial\psi^B_\nu}\delta\psi^A_\mu \Delta\psi^B_\nu + \frac{\partial L}{\partial\psi^A}\delta\Delta\psi^A + \frac{\partial L}{\partial\psi^A_\mu}\delta\Delta\psi^A_\mu\Bigg] dx$$

$$= \int\limits_\Omega \Bigg[\frac{\delta^2 W}{\partial\psi^A \partial\psi^B}(\Delta\psi^B)\delta\psi^A + \frac{\delta W}{\delta\psi^A}\delta\Delta\psi^A\Bigg] dx$$

$$+ \int\limits_\Omega dSn_\mu \Bigg[\delta\psi^A \frac{\partial}{\partial\psi^A_\mu}\left(\frac{\partial L}{\partial\psi^B}\Delta\psi^B + \frac{\partial L}{\partial\psi^B_\nu}\Delta\psi^B_\nu\right) + \delta\Delta\psi^A \frac{\partial L}{\partial\psi^A_\mu}\Bigg]$$

and hence, because of the Haar lemma (now $\delta\psi^A$ and $\delta\Delta\psi^A$ being independent) and the condition (2.7a), the thesis follows.

One should stress that there is an essential difference between the formulations of Propositions 2.2 and 2.3. In the variational principle (2.3), in Proposition 2.2, the variables subjected to variations are the fields $\varrho^A(x^\mu)$ the Lagrange field $\tilde{\psi}^A(x^\mu)$ being fixed. In Proposition 2.3, however, the variational principle (2.5) occurs, and there the dynamical variables are the original fields as well as the Jacobi field. Moreover this variational principle is not a result of any guess-work, but due to the first part of Proposition 2.3 a logical extension of the original principle based on the action functional (1.1).

The difference of formulations of these propositions leads to two various interpretations of the Jacobi field. Both of these interpretations are of course, like both of the formulations, equivalent to each other.

Proposition 2.2 allows to treat the Jacobi field, for a fixed solution $\psi^A(x^\mu)$ of the Lagrange equations, as an independent dynamical system, the degrees of freedom of which are described by the components of the object $\varrho^A(x^\mu)$, and its Lagrangian is given by (2.2). The direct consequence of this proposition is thus the applicability of all constructions and theorems of analytical dynamics to the Jacobi field. Among others, for instance, one may pass from the accessoric Lagrange formalism, defined by (2.2), to the accessoric Hamiltonian formalism. The particular form (2.2) of the Lagrangian assures that the Hesse determinants of the original theory and of the accessoric one are equal to each other,

$$\det\left(\frac{\partial^2 L}{\partial\psi_0^A \partial\psi_0^B}\right) = \det\left(\frac{\partial^2 \mathscr{L}}{\partial\varrho_0^A \partial\varrho_0^B}\right). \tag{2.9}$$

Hence it follows that *the accessoric Hamiltonian formalism will be singular if and only if the original formalism is singular.*

Another corollary from Proposition 2.2, also connected with the particular form (2.2) of the Lagrangian, is the relation

$$\frac{d}{dx^\mu}\left(\frac{\partial\mathscr{L}}{\partial\varrho_\mu^A}\,\varrho^A\right) = 2\mathscr{L} \tag{2.10}$$

which follows from the Jacobi equations (1.3) multiplied by ϱ^A and from the definition of the Lagrangian (2.2). Besides the relations relying on the particular form (2.2) of the Lagrangian, there are also relations of quite a general nature the proof of which refers only to the existence of the accessoric Lagrangian. One of the most important relations of this kind reads as

$$\frac{d}{dx^\mu}\mathscr{T}_\nu^\mu = -\frac{\partial\mathscr{L}}{\partial x^\nu}, \tag{2.11}$$

where

$$\mathscr{T}_\nu^\mu := \frac{\partial\mathscr{L}}{\partial\varrho_\mu^A}\,\varrho_\nu^A - \mathscr{L}\delta_\nu^\mu \tag{2.12}$$

is the canonical energy momentum tensor for the Lagrangian (2.2). Since $\mathscr{L}$ is a quadratic form in the variables ϱ^A and ϱ_μ^A with coefficients depending, in general, on x^μ, the tensor $\mathscr{T}_\nu^\mu$ does not fulfil any conservation law, even if such a law is held for the canonical energy momentum tensor $\mathscr{T}_\nu^\mu$ of the original problem defined by the Lagrangian L.

In Proposition 2.2, instead, the degrees of freedom in the variational principle are defined by both the fields $\psi^A(x^\mu)$ and $\Delta\psi^A(x^\mu)$. The Jacobi equations form thus a certain additional aspect of the dynamics of the fields $\psi^A(x^\mu)$. According to such an approach the dynamics based on the Lagrange equations (1.2) should not be discussed separately from the accessoric one, defined by the Lagrangian (2.2), but a sort of union of both of them is to be considered. And indeed, taking for the starting point the set of the Lagrange (1.2) and the Jacobi (1.3) equations one can derive quite a few identities containing both Lagrangians: $L(\psi^A, \psi_\mu^A, x^\sigma)$ and $\mathscr{L}(\varrho^A, \varrho_\mu^A, x^\sigma)$. Here is one of the more important relations of this kind

$$\frac{d}{dx^\mu}\left[\varrho_\nu^A\frac{\partial L}{\partial\psi_\mu^A} + \psi_\nu^A\frac{\partial\mathscr{L}}{\partial\varrho_\mu^A} - \delta_\nu^\mu\left(\varrho^A\frac{\partial L}{\partial\psi^A} + \varrho_\sigma^A\frac{\partial L}{\partial\psi_\sigma^A}\right)\right]$$
$$= -\frac{\partial^2 L}{\partial x^\nu\partial\psi^A}\,\varrho^A - \frac{\partial^2 L}{\partial x^\nu\partial\psi_\sigma^A}\,\varrho_\sigma^A \tag{2.13}$$

which one gets after multiplying (1.3) by ψ_ν^A and performing some algebraic operations based on (1.2) and the definition (2.2); cf. Appendix A. From the relation (2.13) the next proposition immediately follows.

Proposition 2.4. If the Lagrangian $L(\psi^A, \psi_\mu^A)$ defining the action (1.1) does not explicitly depend on x^μ, i.e. if $\partial L/\partial x^\mu = 0$, then the tensor

$$\tau_\nu^\mu := \varrho_\nu^A \frac{\partial L}{\partial \psi_\mu^A} + \psi_\nu^A \frac{\partial \mathscr{L}}{\partial \varrho_\mu^A} - \delta_\nu^\mu \left(\varrho^A \frac{\partial L}{\partial \psi^A} + \varrho_\sigma^A \frac{\partial L}{\partial \psi_\sigma^A} \right) \tag{2.14}$$

fulfils, for the solution $\psi^A(x^\mu)$ of the Lagrange equations (1.2) and the solutions $\varrho^A(x^\mu)$ of the Jacobi equations (1.3) corresponding to $\psi^A(x^\mu)$, the conservation law

$$\frac{d}{dx^\mu} \tau_\nu^\mu = 0. \tag{2.15}$$

This conservation law may also be derived as a corollary of the generalized Noether identity for the variational principle in Proposition 2.3. More details about it may be found in [1].

An elementary example. The here introduced notions and relations will be followed in the case of dynamics of a single material point in a potential field of forces in the Euclidean space $\boldsymbol{R}^3$. The manifold M is then the one-dimensional time axis, i.e. $x^\mu =: t$, and $\psi^A(x^\mu) =: x^a(t)$ $(a = 1, 2, 3)$ is the function which maps M into $\boldsymbol{R}^3$. The Lagrange function is equal to

$$L(x^a, \dot{x}^a) = \tfrac{1}{2} m\dot{x}^2 - V(x), \tag{2.16}$$

where $\dot{x}^2 = \dot{x}^a\dot{x}^a$ (the summation convention does still apply) and the potential $V(x)$ is a given function of x^a. The equations of motion are then, of course, the Newton equations

$$m\ddot{x}^a = -\frac{\partial V}{\partial x^a}. \tag{2.17}$$

For the considered system $\partial L/\partial t = 0$, and the law of conservation of energy

$$E = \tfrac{1}{2} m\dot{x}^2 + V(x) = \text{const} \tag{2.18}$$

is fulfilled. One defines the Jacobi field $\varrho^A(x^\mu) =: r^a(t)$, for a fixed solution $x^a = x^a(t)$ of the equations of motion (2.17), as a vector field given along a curve γ which represents the motion $x^a = x^a(t)$ in the space $\boldsymbol{R}^3 \times M$ and is therefore "a world line" of the considered material point. The accessoric Lagrangian (2.2) has here the form

$$\mathscr{L}(r^a, \dot{r}^a, t) = \tfrac{1}{2} m\dot{r}^2 - \tfrac{1}{2}\left(\frac{\partial^2 V}{\partial x^a \partial x^b}\right)_\gamma r^a r^b, \tag{2.19}$$

where $\dot{r}^2 = \dot{r}^a\dot{r}^a$, and the Jacobi equations (1.3) amount to the equations

$$m\ddot{r}^a = -\left(\frac{\partial^2 V}{\partial x^a \partial x^b}\right)_\gamma r^b. \tag{2.20}$$

One should emphasize that the second partial derivative of V in (2.19) and (2.20), as evaluated along a given curve γ, is a given function of time. From the formal point of view, the Lagrangian (2.19) is that of an anisotropic oscillator whose "tensorial restoring force" is a given function of time. The Jacobi vector describes in a linear approximation the motion of adjacent particles, also having the mass m, with respect to a given particle with the motion described by the fixed solution $x^a = x^a(t)$. This interpretation directly follows from the first definition of the Jacobi vector quoted in Section 1.

The identity (2.11) takes for a general Lagrange function in mechanics the form

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{r}^a}\,\dot{r}^a - \mathscr{L}\right) = -\frac{\partial \mathscr{L}}{\partial t} \tag{2.21}$$

and for the accessoric Lagrangian (2.19) it yields

$$\frac{d}{dt}\left(\tfrac{1}{2}\,m\dot{r}^2 + \tfrac{1}{2}\,\frac{\partial^2 V}{\partial x^a \partial x^b}\,r^a r^b\right) = \tfrac{1}{2}\,\frac{\partial^3 V}{\partial x^c \partial x^a \partial x^b}\,r^a r^b \dot{x}^c. \tag{2.22}$$

From here then easily follows that the expression

$$G := \tfrac{1}{2}\,m\dot{r}^2 + \tfrac{1}{2}\,\frac{\partial^2 V}{\partial x^a \partial x^b}\,r^a r^b, \tag{2.23}$$

formally being the energy of the system with the Lagrangian (2.19) is not in general conserved. Thus, the quantity (2.23) cannot be the relative energy of two adjacent particles, since, as we know, the "absolute" energy of each of them is being conserved. This apparent inconsistency is due to the fact that the Jacobi vector describes the relative dynamics of two adjacent particles in the linear approximation only. Therefore, one cannot, in general, identify dynamical functions of the Lagrangian (2.19) with corresponding functions characterizing the relative motion of two adjacent particles. In the next section we shall define a collection of quantities which together with the Jacobi vector characterize the relative dynamics of neighbouring particles, and then it will turn out that the quadratic form (2.23) may be completed by some additional terms to a conserved quantity. An analogous construction for the geodesic deviation in general relativity has been done in [7] and [8].

Since in our case the original Lagrangian (2.16) does not depend on time, there exists a first integral of the form (2.14). In mechanics, for a general Lagrangian $L(x^a, \dot{x}^a)$ with a corresponding accessoric Lagrangian $\mathscr{L}(r^a, \dot{r}^a, t)$, instead of (2.14) there holds

$$\tau = \dot{x}^a\,\frac{\partial \mathscr{L}}{\partial \dot{r}^a} - r^a\,\frac{\partial L}{\partial x^a}\,. \tag{2.24}$$

Hence for the particular Lagrangians (2.16) and (2.19) we obtain

$$\tau = m\dot{r}^a\dot{x}^a + \frac{\partial V}{\partial x^a}\,r^a. \tag{2.25}$$

To get an interpretation of the conserved quantity above and similarly of the general quantity (2.14), we are introducing a one-parametric family of solutions $x^a = x^a(t, \varepsilon)$ of the equations (2.17). Then the constant of energy $E = E(\varepsilon)$ is a function of the parameter ε and applying the first of the considered definitions of the Jacobi field we see that $\tau = (dE/d\varepsilon)_{\varepsilon=0}$, i.e. $\tau\varepsilon$ is the linear approximation of the relative energy of a particle moving along $x^a = x^a(t, \varepsilon)$ with respect to a particle whose motion is described by $x^a = x^a(t, 0)$. This energy is a linear function of the quantities r^a and $\dot{r}^a$ characterizing the relative motion. Taking into account the equations of motion (2.17) one can present relations (2.25) in the form

$$\frac{\tau}{m} = \dot{r}^a\dot{x}^a - r^a\ddot{x}^a. \tag{2.26}$$

The constancy of τ/m has thus a clear kinematical meaning. Its analogy with the constancy of the scalar product $\dot{r}^a\dot{x}^a$ in the case of the geodesic deviation is also apparent.

The unified Lagrangian $L^{(1)}(x^a, r^a, \dot{x}^a, \dot{r}^a)$ corresponding to the action (2.6) is here of the form

$$L^{(1)}(x^a, r^a, \dot{x}^a, \dot{r}^a) = m\dot{x}^a\dot{r}^a - \frac{\partial V}{\partial x^a}\, r^a.$$

3. The Jacobi fields of higher order

The before considered definition of the Jacobi fields corresponding to given Lagrange fields may in turn be applied to the Jacobi fields themselves. Returning to the first of the discussed ways of defining these fields let us consider a one-parametric family of solutions $\psi^A = \psi^A(x^\mu, \varepsilon)$ of the Eq (1.2), corresponding to a one-parametric family of initial conditions. To every solution of such a family there is related a certain Jacobi equation (1.3). In other words, we have given a family of Jacobi equations in which the coefficients are functions of the parameter ε. Suppose that $\varrho^A(x^\mu, \varepsilon)$ is a solution of this family of Jacobi equations and that this solution corresponds to a family of initial conditions parametrized by ε too. Let us assume that $\varrho^A(x^\mu, \varepsilon)$ are functions at least of class C^1 with respect to ε. Now we may define on the manifold M a new field $\varrho^A_{(2)}(x^\mu) := \left.\frac{\partial \varrho^A(x^\mu, \varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}$. From the Eqs (1.3), taking into account that the coefficients in them as well as their solution depend on ε, we get as the result of differentiation with respect to ε the identity

$$\frac{\delta^2 W}{\delta\psi^A\partial\psi^B}(\varrho^B_{(2)}) = -\frac{\delta^3 W}{\delta\psi^A\partial\psi^B\partial\psi^C}(\varrho^B, \varrho^C). \tag{3.1}$$

The operator $\delta^3 W/\delta\psi^A\partial\psi^B\partial\psi^C(\cdot\,,\cdot)$ standing above is called *the third variational derivative of the action* (1.1). Its value on arbitrary, but, of course, appropriately regular fields $\varphi^B(x^\mu)$ and $\chi^C(x^\mu)$ is defined as

$$\frac{\delta^3 W}{\delta\psi^A\partial\psi^B\partial\psi^C}(\varphi^B, \chi^C) := \frac{\partial}{\partial\psi^A} L^{(2)}[\varphi^B; \chi^C] - \frac{d}{dx^\mu}\frac{\partial}{\partial\psi^A_\mu} L^{(2)}[\varphi^B; \chi^C], \tag{3.2}$$

where

$$L^{(2)}[\varphi^B;\chi^C] := \frac{\partial^2 L}{\partial\psi^B\partial\psi^C}\varphi^B\chi^C + \frac{\partial^2 L}{\partial\psi^B\partial\psi^C_\nu}(\varphi^B\chi^C_\nu+\chi^B\varphi^C_\nu) + \frac{\partial^2 L}{\partial\psi^B_\mu\partial\psi^C_\nu}\varphi^B_\mu\chi^C_\nu. \tag{3.3}$$

The formula (3.2) may, for our present purposes, be treated as the definition of the third variational derivative; its intrinsic definition is given in [1]. The derivatives of L which do appear in (3.1) through the operators of the second and third variational derivatives are evaluated for a fixed field $\psi^A(x^\mu) := \psi^A(x^\mu, \varepsilon)|_{\varepsilon=0}$ and the operator on the right-hand side of (3.1) acts on a fixed Jacobi field $\varrho^A(x^\mu) := \varrho^A(x^\mu, \varepsilon)|_{\varepsilon=0}$.

Similarly like in Section 1, when the fields $\psi^A(x^\mu)$ and $\varrho^A(x^\mu)$ fulfilling the Eqs (1.2) and (1.3) are already given, the Eq. (3.1), with coefficients evaluated for these fields, is accepted to define the field $\varrho^A_{(2)}$, whereas the primary construction of the field $\varrho^A_{(2)}$ by means of families of solutions of the Eqs (1.2) and (1.3) is left aside. The Eq. (3.1), provided ψ^A and ϱ^A are given, is a linear partial differential equation for $\varrho^A_{(2)}(x^\mu)$, and completed by appropriate subsidiary conditions will always uniquely define the unknown field, for L fulfilling the necessary regularity conditions.

This definition of the field $\varrho^A_{(2)}$, based on the differential Eq. (3.1), as well as the corresponding definition of the field ϱ^A before, does not rely on the regularity of solutions with respect to the parameter ε, but on the regularity of the Lagrangian L alone, which is less troublesome in applications.

The Eq. (3.1), in contradistinction to (1.3), is not homogeneous. Therefore, even if one imposes on the field $\varrho^A_{(2)}$ homogeneous initial conditions, i.e. conditions demanding the field and its time derivative to vanish initially, provided the first Jacobi field will not vanish, there will develop as a result of the time evolution a nonvanishing field $\varrho^A_{(2)}$.

The field $\varrho^A_{(2)}(x^\mu)$ exhibits a few properties similar to those described in the propositions of Section 2. Thus, the following propositions will take place.

Proposition 3.1. Let $\psi^A(x^\mu)$ be a solution of the Lagrange equations (1.2), its first variation $\delta\psi^A$ a solution of the Jacobi equation (1.3) and the second variation $\delta^2\psi^A$ a solution of the Eq. (3.1), then $\delta W + \frac{1}{2!}\delta^2 W + \frac{1}{3!}\delta^3 W$, the variation of the action (1.1) up to the third variation, is equal to a surface integral over the boundary $\partial\Omega$ of the domain of integration Ω in (1.1).

Proposition 3.2. (The second accessoric action principle). Let $\tilde\psi^A(x^\mu)$ be a given solution of the Lagrange equations (1.2), $\tilde\varrho^A(x^\mu)$ a solution of the Jacobi equations (1.3), and let

$$\mathscr{L}_{(2)}(\varrho^A_{(2)}, \varrho^A_{(2)\mu}, x^\sigma) := \tfrac{1}{2}\Bigg[\left(\frac{\partial^2 L}{\partial\psi^A\partial\psi^B}\right)_{\tilde\psi}\varrho^A_{(2)}\varrho^B_{(2)} + 2\left(\frac{\partial^2 L}{\partial\psi^A\partial\psi^B_\nu}\right)_{\tilde\psi}\varrho^A_{(2)}\varrho^B_{(2)\nu}$$
$$+\left(\frac{\partial^2 L}{\partial\psi^A_\mu\partial\psi^B_\nu}\right)_{\tilde\psi}\varrho^A_{(2)\mu}\varrho^A_{(2)\nu} + \varrho^A_{(2)}\Bigg]\frac{\delta^3 W}{\delta\psi^A\delta\psi^B\delta\psi^C}(\tilde\varrho^B, \tilde\varrho^C), \tag{3.4}$$

where the derivatives on the right hand side are evaluated for $\psi^A = \tilde\psi^A(x^\mu)$, i. e. for a given $L(\psi^A, \psi^A_\mu, x^\sigma)$ are fixed functions of x^σ. Then the Euler-Lagrange equations of the variational principle

$$\delta \int_\Omega \mathscr{L}_{(2)}(\varrho^A_{(2)}, \varrho^A_{(2)\mu}, x^\sigma)dx = 0$$

under the condition

$$\delta\varrho^A_{(2)} = 0 \quad \text{for} \quad x \in \partial\Omega \tag{3.5}$$

turn out to be the Eqs (3.1).

Proposition 3.3. (The action principle for mappings of fields). Let Ψ be a set of all sufficiently regular fields on a region Ω of the manifold M. Let us consider a mapping of Ψ onto itself, under which $\psi^A(x^\mu) \mapsto \tilde{\psi}^A(x^\mu)$, of the form

$$\tilde{\psi}^A(x^\mu) = \psi^A(x^\mu) + \Delta\psi^A(x^\mu)\varepsilon + \tfrac{1}{2}\Delta^2\psi^A(x^\mu)\varepsilon^2 + O(x^\mu, \varepsilon^2), \tag{3.6}$$

where ε is a real parameter, $\Delta\psi^A(x^\mu)$ and $\Delta^2\psi^A(x^\mu)$ two independent generators of the considered mapping, and $O(x^\mu, \varepsilon^2) \to 0$ faster then ε^2. We additionally assume that the generator $\Delta\psi^A(x^\mu)$ is a given solution of the Jacobi equation (1.3). Then

1) The necessary and sufficient condition that the mapping (3.6) transforms the fields $\psi^A(x^\mu)$ fulfilling the stationary action principle for W of the form (1.1) into the fields $\tilde{\psi}^A(x^\mu)$ fulfilling the same principle up to terms which tend to zero faster than ε^2 reads as

$$\delta(W^{(1)}[\psi^A; \Delta^2\psi^A] + W^{(2)}[\psi^A; \Delta\psi^A, \Delta\psi^A]) = 0, \tag{3.7}$$

where $W^{(1)}$ is the functional (2.6) and

$$W^{(2)}[\psi^A; \varphi^A, \chi^A] := \int_\Omega L^{(2)}[\varphi^A; \chi^A]dx \tag{3.8}$$

(cf (3.3)) is the second differential of the functional (1.1). The variation in (3.7) is performed for variations $\delta\psi^A$ being arbitrary in the interior of Ω and fulfilling on its boundary $\partial\Omega$ the condition

$$\delta\psi^A(x^\mu) = 0 \quad \text{for} \quad x \in \partial\Omega. \tag{3.9}$$

The variables $\Delta\psi^A(x^\mu)$ and $\Delta^2\psi^A(x^\mu)$ are not subjected to variations in (3.7) at all.

2) The functional $W^{(1)}[\psi^A, \Delta^2\psi^A] + W^{(2)}[\psi^A; \Delta\psi^A, \Delta\psi^A]$ is an action functional of a new variational principle of the form (3.7) in which the variation must now be performed with respect to all the fields: ψ^A, $\Delta\psi^A$ and $\Delta^2\psi^A$ treated as independent dynamical variables. The variations of these variables are arbitrary in the interior of Ω and on its boundary $\partial\Omega$ subjected to conditions

$$\delta\psi^A = 0, \quad \delta\Delta\psi^A = 0, \quad \delta\Delta^2\psi^A = 0 \quad \text{for} \quad x \in \partial\Omega. \tag{3.9a}$$

The Euler-Lagrange equations of this action principle are:

(*i*) the original Lagrange equations (1.2), when the variation in (3.7) is taken with respect to $\Delta^2\psi^A$;

(*ii*) the Jacobi equations (2.8), when one varies[5] with respect to $\Delta\psi^A$;

[5] Let us observe that if one artificially disregards in the new action principle ψ^A and $\Delta^2\psi^A$ as dynamical variables, leaving $\Delta\psi^A$ as the only ones, the new principle is turning into the accessoric action principle (2.3). This fact demonstrates once again the factitiousness of the last principle.

(iii) the Eqs (3.1) for $\Delta^2\psi^B$

$$\frac{\delta^2 W}{\delta\psi^A\partial\psi^B}(\Delta^2\psi^B) = -\frac{\delta^3 W}{\delta\psi^A\partial\psi^B\partial\psi^C}(\Delta\psi^B, \Delta\psi^C), \tag{3.10}$$

when the new action is being varied with respect to ψ^A.

Among the relations which may be derived from the set of Eqs (1.2), (1.3) and (3.1) the following identity being a generalization of (2.13) deserves for particular attention:

$$\frac{d}{dx^\mu}\left[\tfrac{1}{2}\varrho^A_{(2)\nu}\frac{\partial L}{\partial\psi^A_\mu}+\tfrac{1}{2}\psi^A_\nu\frac{\partial\mathscr{L}_{(2)}}{\partial\varrho^A_{(2)\mu}}-\tfrac{1}{2}\delta^\mu_\nu\left(\varrho^A_{(2)}\frac{\partial L}{\partial\psi^A}+\varrho^A_{(2)\sigma}\frac{\partial L}{\partial\psi^A_\sigma}\right)+\mathscr{T}^\mu_\nu+\psi^A_\nu\frac{\partial\mathscr{L}}{\partial\psi^A_\mu}\right]$$

$$= -\tfrac{1}{2}\frac{\partial^2 L}{\partial x^\nu\partial\psi^A}\varrho^A_{(2)}-\tfrac{1}{2}\frac{\partial^2 L}{\partial x^\nu\partial\psi^A_\sigma}\varrho^A_{(2)\sigma}-\frac{\partial^3 L}{\partial x^\nu\partial\psi^A\partial\psi^B}\varrho^A\varrho^B-2\frac{\partial^3 L}{\partial x^\nu\partial\psi^A\partial\psi^B_\sigma}\varrho^A\varrho^B_\sigma$$

$$-\frac{\partial^3 L}{\partial x^\nu\partial\psi^A_\varrho\partial\psi^B_\sigma}\varrho^A_\varrho\varrho^B_\sigma, \tag{3.11}$$

where the notation

$$\frac{\partial\mathscr{L}}{\partial\psi^A_\mu} := \frac{\partial^3 L}{\partial\psi^A_\mu\partial\psi^B\partial\psi^C}\varrho^B\varrho^C+2\frac{\partial^3 L}{\partial\psi^A_\mu\partial\psi^B\partial\psi^C_\sigma}\varrho^B\varrho^C_\sigma+\frac{\partial^3 L}{\partial\psi^A_\mu\partial\psi^B_\varrho\partial\psi^C_\sigma}\varrho^B_\varrho\varrho^C_\sigma$$

is accepted. The identity (3.11) may be derived as a result of multiplication of (3.1) by ψ^A_ν, and of an algebraic procedure given in the Appendix B, based on the Eqs (1.2) and the identity (2.11). An immediate consequence of this is the next proposition.

Proposition 3.4. If the Lagrangian $L(\psi^A, \psi^A_\mu)$ defining the action (1.1) does not depend explicitly on the variables x^μ i. e. if $\partial L/\partial x^\mu = 0$, then the tensor

$$\vartheta^\mu_\nu := \tfrac{1}{2}\varrho^A_{(2)\nu}\frac{\partial L}{\partial\psi^A_\mu}+\tfrac{1}{2}\psi^A_\nu\frac{\partial\mathscr{L}_{(2)}}{\partial\varrho^A_{(2)\mu}}$$

$$-\tfrac{1}{2}\delta^\sigma_\nu\left(\varrho^A_{(2)}\frac{\partial L}{\partial\psi^A}+\varrho^A_{(2)\sigma}\frac{\partial L}{\partial\psi^A_\sigma}\right)+\mathscr{T}^\mu_\nu+\psi^A_\nu\frac{\partial\mathscr{L}}{\partial\psi^A_\mu} \tag{3.12}$$

fulfils, for solutions $\psi^A(x^\mu)$ of the Lagrange equations (1.2), solutions $\varrho^A(x^\mu)$ of the corresponding to them Jacobi equations (1.3), and solutions $\varrho^A_{(2)}(x^\mu)$ of the Eqs (3.1), the conservation law

$$\frac{d}{dx^\mu}\vartheta^\mu_\nu = 0. \tag{3.13}$$

The last proposition, similarly to Proposition 2.4, may be derived as a corollary of the generalized Noether identity. The conservation law (3.13) could also be interpreted as another form of the relation (2.11) in which the term $\partial\mathscr{L}/\partial x^\mu$ has been expressed as the complete divergence, which was enabled by introducing a new quantity $\varrho^A_{(2)}(x^\mu)$.

Now we shall discuss a particular case of the conservation law (3.13) in the before considered elementary example.

The elementary example (continued). The second accessoric Lagrangian $\mathscr{L}_{(2)}$, corresponding to the Lagrangian (2.16), has the form

$$\mathscr{L}_{(2)}(r^a_{(2)}, \dot{r}^a_{(2)}, t) = \tfrac{1}{2}\, m\dot{r}^2_{(2)} - \tfrac{1}{2}\left(\frac{\partial^2 V}{\partial x^a \partial x^b}\right)_\gamma r^a_{(2)} r^b_{(2)} - \left(\frac{\partial^3 V}{\partial x^a \partial x^b \partial x^c}\right)_\gamma r^a_{(2)} r^b r^c \tag{3.14}$$

in which $r^a_{(2)} = r^a_{(2)}(t)$ is, besides the field $r^a = r^a(t)$ defined in Section 2, a new vector field along the curve γ representing the motion $x^a = x^a(t)$ in the space $\mathbf{R}^3 \times M$. Thus, the field $r^a_{(2)}$ now corresponds to the field $\varrho^A_{(2)}(x^\mu)$ from the general theory. The stationary action principle for the Lagrangian leads to the following equations for $r^a_{(2)}(t)$

$$m\ddot{r}^a_{(2)} = -\left(\frac{\partial^2 V}{\partial x^a \partial x^b}\right)_\gamma r^b_{(2)} - \left(\frac{\partial^3 V}{\partial x^a \partial x^b \partial^c}\right)_\gamma r^b r^c.$$

In these equations the partial derivatives of the function V are evaluated along the curve γ and $r^a = r^a(t)$ is a fixed solution of the Jacobi equation (2.20). In mechanics, for general Lagrangians L, $\mathscr{L}$, and $\mathscr{L}_{(2)}$ the tensor (3.12) will reduce to a scalar

$$\vartheta = \tfrac{1}{2}\, \dot{x}^a \frac{\partial \mathscr{L}_{(2)}}{\partial \dot{r}^a_{(2)}} - \tfrac{1}{2}\, r^a_{(2)} \frac{\partial L}{\partial x^a} + G + \dot{x}^a \frac{\partial \mathscr{L}}{\partial \dot{x}^a}, \tag{3.16}$$

where G is defined by (2.23). For the particular Lagrangians (2.16), (2.19) and (3.14) it reads as

$$\vartheta = \tfrac{1}{2}\, m\dot{r}^2 + \tfrac{1}{2}\left(\frac{\partial^2 V}{\partial x^a \partial x^b}\right)_\gamma r^a r^b + \tfrac{1}{2}\, m\dot{r}^a_{(2)}\dot{x}^a + \tfrac{1}{2}\, r^a_{(2)}\left(\frac{\partial V}{\partial x^a}\right)_\gamma. \tag{3.17}$$

Because of Proposition 3.4 the quantity above is conserved, as the Lagrangian (2.16) does not explicitly depend on time. Comparing (3.17) and (2.23) we may see that ϑ differs from G in terms linear with respect to $r^a_{(2)}$ and $\dot{r}^a_{(2)}$. Taking into account that the conservation of ϑ follows from the time independence of the Lagrangian L, and also that after one rejects the last two terms in (3.17) ϑ passes into G, we can interpret the quantity (3.17) as being proportional to the relative energy of two particles with equal masses, moving correspondingly along the trajectories $\gamma_0\colon x^a = x^a(t, 0)$ and $\gamma_\varepsilon\colon x^a = x^a(t, \varepsilon)$ of the Eqs (2.17). One can easily convince oneself also directly that such an interpretation is reasonable. Therefore one should return to the definition of fields $r^a(t)$ and $r^a_{(2)}(t)$ according to which they are correspondingly equal to the first or the second derivative with respect to the parameter of a one-parametric family of motions $x^a = x^a(t, \varepsilon)$ being solutions of (2.17). The constant of energy $E(\varepsilon)$ is then also a function of the parameter ε and instead of (2.18) we have

$$E(\varepsilon) = \tfrac{1}{2}\, m\dot{x}^2(t, \varepsilon) + V(x(t, \varepsilon)). \tag{3.18}$$

Hence making use of the Taylor formula and the definitions:

$$r^a := \left.\frac{\partial x^a}{\partial \varepsilon}\right|_{\varepsilon=0} \quad \text{and} \quad r^a_{(2)} := \left.\frac{\partial^2 x^a}{\partial \varepsilon^2}\right|_{\varepsilon=0},$$

we get

$$E(\varepsilon) = E(0)+\tau|_{\varepsilon=0}\varepsilon+\vartheta|_{\varepsilon=0}\varepsilon^2+\mathscr{R}(\varepsilon^2). \tag{3.19}$$

By an appropriate choice of initial conditions imposed on r^a and $\dot{r}^a$(e. g. by taking at the initial instant of time $\dot{r}^a$ perpendicularly to $\dot{x}^a$ and r^a to $\ddot{x}^a$), we can, according to (2.26), achieve the vanishing of τ. Then, the quantity $\vartheta\varepsilon^2$ will be, up to terms of the order ε^2, equal to the relative energy $E(\varepsilon)-E(0)$.

This discussion may be almost repeated for the energy momentum tensor in the general theory.

4. The possible physical meaning of the Jacobi fields

In mechanics the Jacobi field $r^a(t)$ describes the relative motion of particles infinitesimally close to the given one. The Jacobi equations are, as we have seen, dynamical equations characterized by the Lagrangian (2.2). The dynamical functions, however, constructed by means of this Lagrangian according to the formalism of analytical mechanics do not characterize, as it follows from the example with energy, the dynamics of the relative motion of particles. In the case of energy, i. e. of dynamical function being a quadratic form in relative positions and velocities, it has been sufficient to complete the notion of the Jacobi field by the field $r^a_{(2)}$, defined by the Eqs (3.15), to derive a dynamical function ϑ of the variables x^a, $\dot{x}^a$, r^a, $\dot{r}^a$, $r^a_{(2)}$, $\dot{r}^a_{(2)}$ which characterizes in an appropriate approximation the relative energy of a physical situation. Similarly to the procedure of defining the fields r^a and $r^a_{(2)}$, one can define further fields $r^a_{(k)}$, first as the k-th derivative with respect to the parameter and then by an appropriate equation of motion in which all previous fields: x^a, r^a, $r^a_{(2)}$, ..., $r^a_{(k-1)}$ will enter as given functions of t. These new fields are needed in cases where dynamical functions more complex than a simple quadratic form and the conservation laws combined with them are under consideration. Also in cases of dynamical functions which are quadratic forms like energy, all of these fields, up to $r^a_{(k)}$ including, will define new dynamical functions approximating the given ones with accuracy of the order ε^k. Thus, a single field $r^a_{(k)}$ alone, despite that it formally constitutes a dynamical system, does not desciibe the dynamics of the relative motion of neighbouring particles. Dynamics of such a kind is, however, determined by the infinite collection of fields $r^a_{(k)}$ ($k = 0, 1, ...$), where $r^a_{(0)} = x^a$ and $r^a_{(1)} = r^a$. The basis of thus understood relative dynamics is formed by the principle of stationary action together with the implied by it variational principles which are contained in Propositions 2.3 and 3.3 and in their appropriate generalization for all fields $r^a_{(k)}$. In most applications, however, only the knowledge of a finite sequence of consecutive fields $r^a_{(k)}$ will be essential. The number of fields in such a sequence will, in general, depend upon what are the dynamical functions which are interesting for the given purpose. Very often, the sequence in question will consists of those fields $r^a_{(k)}$ which are necessary to define the considered dynamical function in the lowest order of ε. One of the applications of the relative dynamics consists in the analysis of motion of test particles in the gravitational field in general relativity. To that question is devoted a separate paper [8]; introductory elements of such an analysis are also contained in [7].

In the general field theory one also introduces a sequence of fields $\varrho^A_{(k)}$ defining each field first as the k-th derivative with respect to the parameter and then by postulating appropriate field equations in which, in general, all the before defined fields ψ^A, ϱ^A, $\varrho^A_{(2)}$, ..., $\varrho^A_{(k-1)}$, enter as given solutions of the field equations corresponding to all values of the index smaller than k. The interpretation of the infinite collection of all fields $\varrho^A_{(k)}$ is here, of course, different than in mechanics. In case the solution of the equations is analytic with respect to a parameter introduced by initial conditions, the collection of all fields $\varrho^A_{(k)}$ defines a perturbation of the field ψ^A caused by the perturbation of initial conditions. In general, the sequence of fields $\varrho^A_{(k)}$ provides certain characteristic of the "dynamical surrounding" of a state of the field determined by ψ^A. By analogy to mechanics, also in the general case of a collection of fields $\varrho^A_{(k)}$ we shall be saying that it determines *the relative dynamics of a given field theory* characterized by either the field equations (1.2) or the action (1.1).

Quite an important question arises here: for what field theories all the equations for the fields $\varrho^A_{(k)}$ are identical to the Lagrange equations (1.2) for the field ψ^A? It easily may be verified that the following proposition is true:

Proposition 4.1. It is necessary and sufficient for the Lagrange equations (1.2) to be identical with all the equations for the fields $\varrho^A_{(k)}$ that the field theory characterized by the action (1.1) is linear.

Differently speaking, in linear field theory the "dynamical surrounding" of an arbitrary state does not depend on that state and is such as the surrounding of the state of vacuum. Thus, any linear theory is characterized by a sui generis degeneration consisting in dynamical indistinguishability of the fields ψ^A and $\varrho^A_{(k)}$. It implies, among others, that to a procedure worked out for the dynamics of the fields ψ^A in a linear theory there does not necessarily correspond an analogous procedure formulated solely for the fields ψ^A of a nonlinear theory since an adequate generalization of such a procedure might include the fields $\varrho^A_{(k)}$ as well.

This conclusion may, in particular, apply to the procedure of quantization of a field theory. The quantization of linear field theories, mainly of electrodynamics, leads to microscopic theories which are ideally confirmed by experiment. This success provides the stimulus for numerous attemps of quantization of nonlinear field theories, and particularly of the Einsteinian theory of gravitation. According to the generally accepted conviction a procedure of quantization, e. g. the so-called canonical quantization, is closely linked to the dynamics of fields. From the point of view of the general dynamical structure of only the fields ψ^A themselves, however, the difference between a linear and a nonlinear theory might not appear essential, and therefore the majority of attemps of quantization of nonlinear theories consists in possibly faithful imitation, on the level of the fields ψ^A, of the metods worked out in linear theories. One cannot, however, eliminate the possibility that in nonlinear theories the quantization procedure applies to the whole collection of fields ψ^A, ϱ^A, $\varrho^A_{(2)}$, ..., $\varrho^A_{(k)}$, ... Such a procedure would then pass, in the limit of the linear theory, due to the dynamical indistinguishability of these fields, to a usual, e. g. the canonical, quantization of dynamics of the fields ψ^A. It might be worthwile to undertake a more thorough analysis of such a possibility, and the present author intends to return to it in the future. Now it would, anyhow, go beyond the classical field theory being the subject of this paper.

Another possibility, which may here also be only mentioned, amounts to the supposition that the fields ψ^A in any classical field theory are macroscopic fields, not subjected to quantization at all, while the microscopic fields, those which should be quantized, are only the fields $\varrho^A_{(k)}$. Proposition 4.1 could then be regarded as a principle of indistinguishability of classical macro- and microscopic fields in linear fields theories.

In the general theory of relativity, for instance, this latter point of view would mean that the gravitational fields, being represented in a given reference frame by the components of a tensor $g_{\alpha\beta}$ which fulfils the Einstein equations

$$R_{\alpha\beta}-\tfrac{1}{2}\, g_{\alpha\beta}R = -8\pi T_{\alpha\beta} \tag{4.1}$$

are macroscopic fields. Besides the fields $g_{\alpha\beta}$ (corresponding here to the fields ψ^A), one can introduce for every solution of the Eqs (4.1), according to the propositions of the two preceding sections, the fields ${}_{(k)}h_{\alpha\beta}$ (corresponding to the fields $\varrho^A_{(k)}$) which are objects of the relative dynamics connected with the action leading to the Eqs (4.1). The first of these fields, the Jacobi field ${}_{(1)}h_{\alpha\beta} = : h_{\alpha\beta}$ fulfils the equations

$$\begin{aligned}\tfrac{1}{2}\,\gamma_{\alpha\beta;}{}^{\varrho}{}_{\varrho}-\tfrac{1}{2}\,\gamma_{\alpha\varrho;}{}^{\varrho}{}_{\beta}-\tfrac{1}{2}\,\gamma_{\beta\varrho;}{}^{\varrho}{}_{\alpha}+\tfrac{1}{2}\,\gamma_{\sigma\varrho;}{}^{\varrho\sigma}-R^{\varrho}{}_{\alpha\beta\sigma}\gamma^{\sigma}{}_{\varrho}+\tfrac{1}{2}\,R_{\alpha\varrho}\gamma^{\varrho}{}_{\beta}+\tfrac{1}{2}\,R_{\beta\varrho}\gamma^{\varrho}{}_{\alpha}\\ -\tfrac{1}{2}\,R\gamma_{\alpha\beta}+\tfrac{1}{2}\,g_{\alpha\beta}\gamma^{\varrho\sigma}R_{\varrho\sigma} = -8\pi\tau_{(\alpha\beta)},\end{aligned} \tag{4.2}$$

where $\gamma_{\alpha\beta} := h_{\alpha\beta}-\frac{1}{2}g_{\alpha\beta}h$ and $\tau_{(\alpha\beta)}$ is characterizing the distribution of matter. The covariant derivative denoted in (4.2) by a semicolon, the curvature tensor, and the operation of rising indices are determined by means of the metric $g_{\alpha\beta}$ being a solution of the Eqs (4.1). The Eqs (4.2), provided $g_{\alpha\beta}$ is a solution of (4.1), are invariant under a gauge group which maps $h_{\mu\nu}\mapsto h'_{\mu\nu} = h_{\mu\nu}+2\gamma_{(\mu;\nu)}$ by means of an arbitrary vector field γ_μ. The existence of such a gauge is a consequence of the generalization of the second Noether theorem stated in [1], applied to the action of general relativity and of the invariance of this action under a group determined by four arbitrary functions of x^μ. Choosing the gauge in such a way that $\gamma_{\alpha\beta;}{}^{\beta} = 0$, one can bring the Eq. (4.2) to a simpler form which in the particular case of vacuum ($R_{\alpha\beta} = 0$, $\tau_{(\alpha\beta)} = 0$) may be reduced to the equation

$$h_{\alpha\beta;\varrho}{}^{\varrho}-2R^{\varrho}{}_{\alpha\beta\sigma}h^{\sigma}{}_{\varrho} = 0. \tag{4.3}$$

The equation above was usually derived as an approximation of the Eq. (4.1) determining a small correction to the metric tensor. In the particular case for $g_{\alpha\beta} = \eta_{\alpha\beta}$, this equation was describing the waves of a weak gravitational field. According to the here formulated program, the Eqs (4.2) and (4.3) are rigorous equations for a new field, the Jacobi field, connected with the relative dynamics of the field $g_{\alpha\beta}$ of general relativity. According to the second of above mentioned possibilities, the field $h_{\alpha\beta}$ would be the first of an infinite sequence of gravitational microscopic fields subjected to quantization. The hypothesis that the field of small perturbations of the field $g_{\alpha\beta}$ is a microscopic gravitational field is not a new one. It has been, among others, discussed in a qualitative manner in [13]. This hypothesis forms also a basis of programs of quantization of the gravitational field proposed by Lichnérowicz [14, 15] and by DeWitt [16]. However, now the field $h_{\alpha\beta}$ is not an accidental, additional element to the field $g_{\alpha\beta}$ and is not "small", but is an essential element of a whole dynamical scheme. Moreover, in the limit of the linear theory this field is indistinguishable from the field $g_{\alpha\beta}$. The relative dynamics is, however, defined not only by the field $h_{\alpha\beta}$, but by the whole sequence of fields ${}_{(k)}h_{\alpha\beta}$ determined by equations which in contradistinction to (4.3) will contain source terms depending upon all fields of the lower order. One should expect that the quantization of gravitation is connected with the quantization of all fields ${}_{(k)}h_{\alpha\beta}$, although in practical applications not all of them must be necessary for the description of simple quantum processes.

A difficulty of all such programs consists in a complete lack of any indications determining which of the fields should be taken as defining the "classical background" of the theory under consideration. It seems reasonable to admit only such fields $g_{\alpha\beta}$ which would be solutions of a well stated initial value problem, that means solutions which are analytic with respect to parameters introduced by the initial data. Among several elements of thus parametrized families the physically most important would probably be manifolds with maximal symmetry.

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APPENDIX A

Multiplying the Eq. (1.3) by ψ^A_ν we get

$$\psi^A_\nu \frac{\delta^2 W}{\delta\psi^A \delta\psi^B}(\varrho^B) = \psi^A_\nu \frac{\partial}{\partial\psi^A}\left(\frac{\partial L}{\partial\psi^B}\varrho^B + \frac{\partial L}{\partial\psi^B_\sigma}\varrho^B_\sigma\right) - \psi^A_\nu \frac{d}{dx^\mu}\frac{\partial}{\partial\psi^A_\mu} \times \left(\frac{\partial L}{\partial\psi^B}\varrho^B + \frac{\partial L}{\partial\psi^B_\sigma}\varrho^B_\sigma\right) = 0. \tag{A.1}$$

Adding and substracting from the left-hand side of the equation above the term

$$\psi^A_{\nu\mu}\frac{\partial}{\partial\psi^A_\mu}\left(\frac{\partial L}{\partial\psi^B}\varrho^B + \frac{\partial L}{\partial\psi^B_\sigma}\varrho^B_\sigma\right)$$

we may write it in the form

$$\psi^A_\nu \frac{\partial}{\partial\psi^A}\left(\frac{\partial L}{\partial\psi^B}\varrho^B + \frac{\partial L}{\partial\psi^B_\sigma}\varrho^B_\sigma\right) + \psi^A_{\nu\mu}\frac{\partial}{\partial\psi^A_\mu}\left(\frac{\partial L}{\partial\psi^B}\varrho^B + \frac{\partial L}{\partial\psi^B_\sigma}\varrho^B_\sigma\right) - \frac{d}{dx^\mu}\left[\psi^A_\nu \frac{\partial}{\partial\psi^A_\mu}\left(\frac{\partial L}{\partial\psi^B}\varrho^B + \frac{\partial L}{\partial\psi^B_\sigma}\varrho^B_\sigma\right)\right] = 0. \tag{A.2}$$

Since according to (2.2)

$$\frac{\partial}{\partial \psi^A_\mu}\left(\frac{\partial L}{\partial \psi^B}\varrho^B + \frac{\partial L}{\partial \psi^B_\sigma}\varrho^B_\sigma\right) = \frac{\partial \mathscr{L}}{\partial \varrho^A_\mu}, \tag{A.3}$$

the last term in (A.2) may be expressed as

$$-\frac{d}{dx^\mu}\left(\psi^A_\nu \frac{\partial \mathscr{L}}{\partial \varrho^A_\mu}\right). \tag{A.4}$$

Taking into account that if differentiated the Lagrangian and its partial derivatives depend on x^μ through the all arguments of the function $L(\psi^A(x^\mu), \psi^A_\sigma(x^\mu), x^\mu)$, we can bring the first two terms in (A.2) to the form

$$\varrho^B \frac{d}{dx^\nu}\frac{\partial L}{\partial \psi^B} + \varrho^B_\sigma \frac{d}{dx^\nu}\frac{\partial L}{\partial \psi^B_\sigma} - \frac{\partial^2 L}{\partial x^\nu \partial \psi^B}\varrho^B - \frac{\partial^2 L}{\partial x^\nu \partial \psi^B_\sigma}\varrho^B_\sigma. \tag{A.5}$$

The first two terms in (A.5) are equal to

$$\frac{d}{dx^\nu}\left(\varrho^B \frac{\partial L}{\partial \psi^B}\right) - \varrho^B_\nu \frac{\partial L}{\partial \psi^B} + \varrho^B_\sigma \frac{d}{dx^\nu}\left(\frac{\partial L}{\partial \psi^B_\sigma}\right); \tag{A.6}$$

and the last two terms in (A.6) may be, because of the Lagrange Eqs (1.2), expressed as

$$\varrho^B_\mu \frac{d}{dx^\nu}\frac{\partial L}{\partial \psi^B_\mu} - \varrho^B_\nu \frac{d}{dx^\mu}\frac{\partial L}{\partial \psi^B_\mu} \tag{A.7}$$

which, due to the commutativity of second derivatives, $\varrho^B_{\mu\nu} = \varrho^B_{\nu\mu}$, is equal to

$$\frac{d}{dx^\nu}\left(\varrho^B_\mu \frac{\partial L}{\partial \psi^B_\mu}\right) - \frac{d}{dx^\mu}\left(\varrho^B_\nu \frac{\partial L}{\partial \psi^B_\mu}\right). \tag{A.8}$$

Thus, all the terms in (A.2), except the last two in (A.5), have been collected in the form of a complete divergence. The substitution to the Eq. (A.2) of all results of the operations performed above leads to the relation (2.13).

APPENDIX B

Multiplying the left-hand side of (3.1) by ψ^A_ν and using the results of Appendix A we get

$$\psi^A_\nu \frac{\delta^2 W}{\delta \psi^A \delta \psi^B}(\varrho^B_{(2)}) = \frac{d}{dx^\mu}\left[\varrho^A_{(2)\nu}\frac{\partial L}{\partial \psi^A_\mu} + \psi^A_\nu \frac{\partial \mathscr{L}_{(2)}}{\partial \varrho^A_{(2)\mu}} - \delta^\mu_\nu\left(\varrho^A_{(2)}\frac{\partial L}{\partial \psi^A} + \varrho^A_{(2)\sigma}\frac{\partial L}{\partial \psi^A_\sigma}\right)\right]$$
$$- \frac{\partial^2 L}{\partial x^\nu \partial \psi^A}\varrho^A_{(2)} - \frac{\partial^2 L}{\partial x^\nu \partial \psi^A_\sigma}\varrho^A_{(2)\sigma}. \tag{B.1}$$

The product of the right-hand side of (3.1) by ψ^A_ν may be written in the form

$$\psi^A_\nu \frac{\delta^3 W}{\delta \psi^A \delta \psi^B \delta \psi^C}(\varrho^B, \varrho^C) = \psi^A_\nu \frac{\partial}{\partial \psi^A}(2\mathscr{L}) - \psi^A_\nu \frac{d}{dx^\mu}\frac{\partial}{\partial \psi^A_\mu}(2\mathscr{L}), \tag{B.2}$$

where the function $\mathscr{L}$, given by the formula (2.2), is treated here as a function of all the variables: ψ^A, ψ^A_μ, ϱ^A, ϱ^A_μ, and x^μ. This function should be distinguished from $\mathscr{L}|_\psi$ which is the restriction of the complete function $\mathscr{L}$ to a given solution $\psi^A(x^\mu)$. In Sections 1–4 of this work (except for the shorthand notation $\partial\mathscr{L}/\partial\psi^A_\mu$ accepted in (3.11)) only the function $\mathscr{L}|_\psi$ was present, but it was there denoted by $\mathscr{L}$, abusing the notation. The left-hand side of (B.2) may be brought to the form

$$\psi^A_\nu \frac{\partial}{\partial\psi^A}(2\mathscr{L})+\psi^A_{\nu\mu}\frac{\partial}{\partial\psi^A_\mu}(2\mathscr{L})-2\frac{d}{dx^\mu}\left(\psi^A_\nu\frac{\partial\mathscr{L}}{\partial\psi^A_\mu}\right). \tag{B.3}$$

The derivative $\partial\mathscr{L}|_\psi/\partial x^\nu$ may, according to its definition, be expressed through the partial derivatives of the functions $\mathscr{L}$ and L:

$$\frac{\partial\mathscr{L}|_\psi}{\partial x^\nu} = \psi^A_\nu\frac{\partial\mathscr{L}}{\partial\psi^A}+\psi^A_{\nu\varrho}\frac{\partial\mathscr{L}}{\partial\psi^A_\varrho}+\tfrac{1}{2}\left[\frac{\partial^3 L}{\partial x^\nu\partial\psi^A\partial\psi^B}\varrho^A\varrho^B+2\frac{\partial^3 L}{\partial x^\nu\partial\psi^A\partial\psi^B_\sigma}\varrho^A\varrho^B_\sigma\right.$$
$$\left.+\frac{\partial^3 L}{\partial x^\nu\partial\psi^A_\varrho\partial\psi^B_\sigma}\varrho^A_\varrho\varrho^B_\sigma\right]. \tag{B.4}$$

From the other side, the formula (2.11) is in the new notation to be written as

$$\frac{\partial\mathscr{L}|_\psi}{\partial x^\nu} = -\frac{d}{dx^\mu}\mathscr{T}^\mu_\nu. \tag{B.5}$$

Taking into account (B.4) and (B.5) we can rewrite the expression (B.3) in the form

$$-\frac{d}{dx^\mu}\left[2\mathscr{T}^\mu_\nu+2\psi^A_\nu\frac{\partial\mathscr{L}}{\partial\psi^A_\mu}\right]-\frac{\partial^3 L}{\partial x^\nu\partial\psi^A\partial\psi^B}\varrho^A\varrho^B-2\frac{\partial^3 L}{\partial x^\nu\partial\psi^A\partial\psi^B_\sigma}\varrho^A\varrho^B_\sigma$$
$$-\frac{\partial^3 L}{\partial x^\nu\partial\psi^A_\varrho\partial\psi^B_\sigma}\varrho^A_\varrho\varrho^B_\sigma. \tag{B.6}$$

The expression (B.6) as being equal to the right-hand side of (3.1) multiplied by ψ^A_ν must be equal to the right-hand side of (B.1). Hence the relation (3.11) then follows.

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