# COULOMB DISPLACEMENT ENERGY IN THE ISOBARIC ANALOG PAIRS Ti — V

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(*Received November 15, 1975*)

Coulomb displacement energy in the isobaric analog pairs Ti — V is considered. The theoretical calculations of Coulomb displacement energy for $^{49}Ca-^{49}Sc$, $^{49}Ti-^{49}V$, $^{51}Ti-^{51}V$ are performed and the results are discussed and compared with the experimental data.

## 1. *Introduction*

A manifestation of the electromagnetic interaction between the protons in the nucleus is expressed by the energy displacement of the isobaric analog states. An experimental study of the Coulomb energies began with the mirror nuclei, but after the experiments of Anderson and Wong [1] and Fox, Robson and Moore [2], concerning the discovery of analog states, systematic accurate measurements of the Coulomb energy difference $\Delta E_c$ (the Coulomb energy shift, or the Coulomb displacement energy) in the neighbouring nuclei of the isobaric pairs have been a subject of a number of recent investigations. The main purpose of these measurements is usually to understand the isobaric analog states as such, but each measurement gives a numerical value for the Coulomb displacement energy.

As Janecke [3] pointed out "the major objectives for studying Coulomb energies are: 1) to obtain an understanding of the dependence of the Coulomb energies on $Z$ and $A$ (or on $A$ and $T$) for a larger group of nuclei, or to obtain an understanding of the dependence on the detailed nuclear structure for a smaller group of nuclei or even a single isobaric pair, 2) to obtain information about the charge distribution and the charge radii of atomic nuclei, 3) to obtain information about other charge dependent effects such as charge-dependent nuclear forces. All these problems are of course related".

There is also a common interest in studying the shell effect on the Coulomb displacement energy. Most suitable for these investigations are the measurements of as many as possible the isobaric analog pairs of the same elements, especially in the vicinity of the magic numbers of nucleons. This was the reason for our investigation of the isobaric analog pairs of Ti — V.

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## 2. *Theoretical description*

The Coulomb displacement energy is defined by the equation

$$\Delta E_c = E_{as} - E_{gs}, \tag{1}$$

where $E_{gs}$ is the energy of the parent nucleus $Z_<$ in the ground state and $E_{as}$ — the energy of the daughter nucleus $Z_>$ in the excited analog state. If we take into account commutation relations between the isospin components, and relation between ground state of parent nucleus and analog state (with notation used in isospin formalism)

$$|T_0, T_0-1\rangle_{as} = \frac{1}{\sqrt{2T_0}} \hat{T}_- |T_0, T_0\rangle_{gs}, \tag{2}$$

we obtain the expression used for further calculations

$$\Delta E_c = \frac{1}{2T_0} \langle T_0, T_0| [[\hat{T}_+, H_{EM}], \hat{T}_-] |T_0, T_0\rangle_{gs}. \tag{3}$$

We have assumed that the relation

$$[\hat{\hat{T}}, H_s] = 0 \tag{4}$$

is valid, where $H_s$ is the Hamiltonian of the strong interaction.

Next, we make an assumption concerning the form of the electromagnetic Hamiltonian $H_{EM}$.

In our considerations we shall use a model constructed as follows:

1) For the nuclei belonging to the analog pairs considered below we assume a physical substantiated core, in our case the core of $^{48}Ca$ nucleus (double magic nucleus).

2) The interaction between the protons outside the core is defined by the Coulomb law.

The interaction $H_{EM}$ can be represented in the form

$$H_{EM} = H_c^{1p} + H_c^{2p}, \tag{5}$$

where $H_c^{1p}$ is the one-particle Coulomb interaction of the protons with the core

$$H_c^{1p} = \sum_{jm} V_c^j a_{jm}^+(p) a_{jm}(p), \tag{6}$$

$H_c^{2p}$ is the two-particle Coulomb interaction between protons which are outside the core

$$H_c^{2p} = \frac{1}{4} \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} \langle \alpha_3\alpha_4 | V_c^{2p} | \alpha_1\alpha_2 \rangle_{AS} a_{\alpha_3}^+(p) a_{\alpha_4}^+(p) a_{\alpha_2}(p) a_{\alpha_1}, \tag{7}$$

$$V_c^j \equiv \langle j \| V_c^{1p} \| j \rangle / \sqrt{2j+1},$$

$V_c^{1p}$ is the proton-core interaction potential, $V_c^{2p}$ — proton-proton interaction potential, $a_\alpha^+(p)$ — creation operator for proton, $\alpha = (n, l, j, m)$ — the complete set of quantum numbers describing the one-particle state in the potential with the spherical symmetry. After we put (6) and (7) into the expression for the Coulomb displacement energy, we get

$$\Delta E_c = \Delta E_c^{1p} + \Delta E_c^{2p}, \tag{8}$$

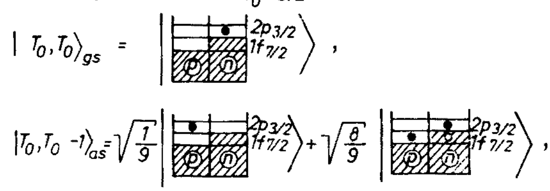
where

$$\Delta E_c^{1p} = \frac{1}{2T_0} \sum_{nljm} V_c^{nlj} \langle T_0, T_0 | \hat{N}_{nljm}(\mathrm{n}) - \hat{N}_{nljm}(\mathrm{p}) | T_0, T_0 \rangle_{gs}$$

$$= \frac{1}{2T_0} \sum_{nlj > \alpha_F} V_c^{nlj} [N_{nlj}(\mathrm{n}) - N_{nlj}(\mathrm{p})]. \tag{9}$$

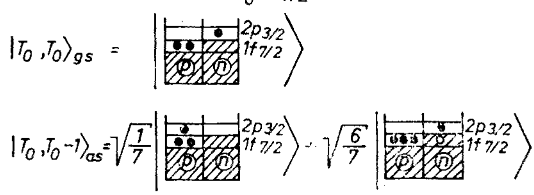
$N_{nlj}$ is the number of protons or neutrons in the shell. Below we give examples illustrating expression (9) applied to the calculations of the one-particle Coulomb displacement energy.

1. $^{49}Ca - ^{49}Sc$ $\quad T_0 = 9/2$




$$\Delta E_c = \frac{1}{9} V_c^{2p3/2} + \frac{8}{9} V_c^{1f7/2}$$

2. $^{51}Ti - ^{51}V$ $\quad T_0 = 7/2$




$$\Delta E_c^{1p} = \frac{1}{7} V_c^{2p3/2} + \frac{6}{7} V_c^{1f7/2}$$

The energy $\Delta E_c^{2p}$ in (8) represents the part of the Coulomb displacement energy which is caused by the interaction of the protons outside the core.

$$\Delta E_c^{2p} = \frac{1}{2T_0} \langle T_0, T_0 | [[\hat{T}_+, H_c^{2p}], \hat{T}_-] | T_0, T_0 \rangle_{gs}. \tag{10}$$

In many cases, for better agreement of the theoretical calculations with the experimental results it is necessary to take into account, when calculating $\Delta E_c$, also the difference of the proton and neutron magnetic moments. The interaction energy of the nucleon magnetic moment with the magnetic moment caused by the orbital motion is, according to Nolen and Schiffer (4),

$$V_{\mu l}^{(\mathrm{p,n})} = \frac{1}{2} \left( \frac{\hbar}{M_p c} \right)^2 \frac{1}{r} \frac{dV_c^{1p}}{dr} \begin{Bmatrix} (\mu_p - \frac{1}{2}) & \text{for p} \\ \mu_n & \text{for n} \end{Bmatrix} \vec{\sigma} \cdot \vec{l}. \tag{11}$$

Hence, as in the expression for $\Delta E_c^{1p}$, we have

$$\Delta E_{\mu l} = \frac{1}{2T_0} \sum_{nlj > \alpha_F(p)} (V_{\mu l}^{(p)nlj} - V_{\mu l}^{(n)nlj})\,(N_{nlj}(n) - N_{nlj}((p). \tag{12}$$

The non-reducible matrix elements $V_{\mu l}^{nlj}$ have the form

$$V_{\mu l}^{nlj} = \langle nlj \| V_{\mu l} \| nlj \rangle / \sqrt{2j+1}\,. \tag{13}$$

## *3. Numerical calculations*

Numerical calculations of the Coulomb displacement energy for the analog pairs Ti — V are performed for a few sets of parameters of the accepted model, i.e. the parameters describing the charge distribution and the shell model parameters of $^{48}$Ca. In

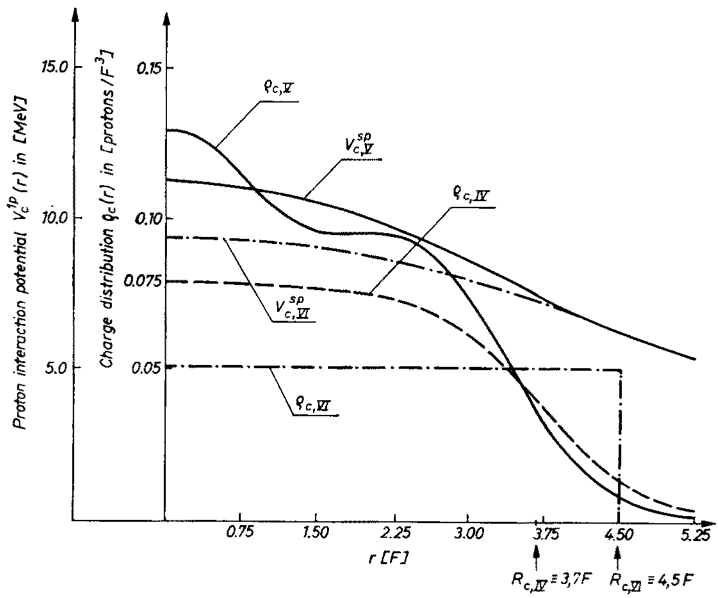



Fig. 1. Proton-core interaction potential and charge distribution. $\varrho_{c,\,IV}$ — Fermi parabolic distribution with parameters set IV. $\varrho_{c,\,V,\,VI}$ and $V_{c,\,V,\,VI}$ — charge distribution and proton-core interaction with parameters set V and VI respectively

order to calculate the one-particle contribution to the energy $\Delta E_c$ we used models with the following charge distribution:

a) Charge distribution habitually used in the shell model (Fig. 1), i.e. the spherical symmetric distribution with the charge density

$$\varrho_0 = 0.05111[\text{protons/F}^3]$$

and the radius

$$R_c = 4.5\text{F}(R_c = 1.24\text{A}^{1/3}),$$

b) Fermi parabolic distribution

$$\varrho_c(r) = \varrho_0 \frac{1+w\left(\frac{r}{c}\right)^2}{1+e^{\frac{r-c}{z}}}.$$

The parameters $w$, $z$ and $c$, taken from [5, 6] are given in Table I. These parameters are obtained (as the result of a fit) from the scattering of the high energy electrons.

TABLE I

Charge distribution in $^{48}$Ca $\varrho(r) = \varrho_0[1+wr^2/c^2]\{1+\exp[(r-c)/z]\}^{-1}$ Charge distribution parameters obtained from the electron scattering experiments

| Electron energy [MeV] | Parameters set | $c$ [F] | $z$ [F] | $w$ [F] | $r_{0.5}$ [F] | $r_{0.5}/A^{1/3}$ [F] | $t\,(90–10)$ [F] | $\langle r^2\rangle^{1/2}$ [F] |
|---|---|---|---|---|---|---|---|---|
| 250 | I | 3.7444 | 0.5255 | −0.03 | 3.7133 | 1.0218 | 2.351 | 3.4762 |
| 250 | II | 3.918 | 0.521 | −0.124 | 3.682 | 1.013 | 2.48 | 3.444 |
| 500 | III | 3.797 | 0.534 | −0.048 | 3.746 | 1.031 | 2.42 | 3.517 |
| 757.5 | IV | 3.7369 | 0.5245 | −0.03 | | | | |

c) Charge distribution calculated from the shell model

$$\varrho_c(r) = \frac{1}{4\pi} \sum_{nlj \leqslant \alpha_F(p)} (2j+1) R_{nlj}^2(r),$$

where $R_{nlj}(r)$ are the radial wave functions of the protons.

These functions were numerically calculated with the Saxon-Woods-type potential parameters given in Table II. One should remember that the parameters given in the

TABLE II

Parameters of the one-particle potential

| Nucleon | Central part of the potential | | | Spin-orbital part of the potential | | | Remarks | |
|---|---|---|---|---|---|---|---|---|
| | $r_0$ [F] | $a$ [F] | $V_c$ [MeV] | $r_{0\,so}$ [F] | $a_{so}$ [F] | $v_{so}$ | | |
| Proton | 1.24 | 0.63 | −58.63 | 1.24 | 0.63 | 31.88 | Shell model (SM) | Coulomb potential taken from electron scattering experiments |
| | 1.17 | 0.75 | −65.68 | 1.07 | 0.75 | 16.3 | Optical model (OM) | |
| Neutron | 1.24 | 0.63 | −48.22 | 1.24 | 0.63 | 31.88 | Shell model (SM) | |
| | 1.17 | 0.75 | −52.62 | 1.75 | 0.75 | 21.79 | Optical model (OM) | |

TABLE III

One-particle Coulomb energy

| Sets of charge distribution parameters | One-particle state | | $V_c^{nlj}$ [MeV] One-particle Coulomb energy | $(V_{\mu l}^{(p)nlj} - V_{\mu l}^{(n)nlj})$ [MeV] Energy caused by magnetic moments |
|---|---|---|---|---|
| | $\alpha$ | $E_{sp}$ [MeV] | | |
| I | $1f_{7/2}$ | −9.08 | 6.911 | −0.184 |
| | $2p_{3/2}$ | −5.14 | 6.771 | −0.057 |
| II | $1f_{7/2}$ | −9.08 | 6.889 | −0.179 |
| | $2p_{3/2}$ | −5.14 | 6.736 | −0.056 |
| III | $1f_{7/2}$ | −9.08 | 6.887 | −0.180 |
| | $2p_{3/2}$ | −5.14 | 6.742 | −0.056 |
| IV | $1f_{7/2}$ | −9.08 | 6.915 | −0.188 |
| | $2p_{3/2}$ | −5.14 | 6.777 | −0.057 |
| Charge distribution from SM with the parameters of OM V | $1f_{7/2}$ | −7.39 | 7.278 | −0.225 |
| | $2p_{3/2}$ | −5.14 | 6.987 | −0.066 |
| Charge distribution with $\varrho_c$ = const and OM parameters VI | $1f_{7/2}$ | −7.39 | 7.022 | −0.163 |
| | $2p_{3/2}$ | −5.14 | 6.599 | −0.047 |

second and fourth line, taken from the optical model [7], give much better agreement with the experimental values of the energy levels than the traditional parameters used in the shell model [8]. The one-particle Coulomb energies obtained for the individual models of the charge distribution and the parameters describing this distribution are given in Table III. In this table are also given the one-particle energies connected with the difference of the interaction of the proton and neutron magnetic moment with the orbital magnetic moment. The one-particle Coulomb energy gives the main contribution to the Coulomb displacement energy, what we can see in Table VI. It is characteristic that all four sets of parameters I, II, III, IV from Table III, resulting from the electron scattering experiments, give practically the same one-particle Coulomb energies. These

TABLE IV

Two-particle diagonal matrix elements $V_J^{1f^2_{7/2}, 1f^2_{7/2}}$ in MeV

| $J$ | SM | OM |
|---|---|---|
| 0 | 0.7636 | 0.7874 |
| 2 | 0.6399 | 0.6651 |
| 4 | 0.5763 | 0.6008 |
| 6 | 0.5643 | 0.5890 |

energies are obviously too small. The spherical charge distribution with a constant density (row four, Table III) gives the energy higher by about 100 keV, but still too small in comparison with the experimental value. However, the one-particle Coulomb energy calculated using the optical model parameters seems to be proper, what can be seen from the last row of the data in Table VI, where the calculated and measured Coulomb displace-

TABLE V

Two-particle non-diagonal matrix elements $V_J^{1f_{7/2}2p_{3/2},\,1f_{7/2}2p_{3/2}} = V_J^D + (-1)^J V_J^{Ex}$ in MeV

| $J$ | SM | | OM | |
|---|---|---|---|---|
| | $V_J^D$ | $V_J^{Ex}$ | $V_J^D$ | $V_J^{Ex}$ |
| 2 | 0.3212 | 0.0026 | 0.3252 | 0.0026 |
| 3 | 0.2915 | 0.0065 | 0.2957 | 0.0069 |
| 4 | 0.2796 | 0.0054 | 0.2839 | 0.0060 |
| 5 | 0.3093 | 0.0179 | 0.3134 | 0.0193 |

TABLE VI

Coulomb displacement energy

| Analog pair | | Parameters | $\Delta E_c(1p)$ [MeV] | $\Delta E_c^{\mu l}$ [MeV] | $\Delta E_c(2p)$ [MeV] | $\Delta E_c^{th}$ [MeV] | $\Delta E_c^{exp}$ [MeV] | $\delta(\Delta E_c)$ [MeV] | $\delta(\Delta E_c)/\Delta E_c$ [%] |
|---|---|---|---|---|---|---|---|---|---|
| Ti — V | $A = 51$ | I | 6.891 | −0.166 | 0.563 | 7.288 | | 0.466 | 6.0 |
| | | II | 6.867 | −0.166 | 0.563 | 7.264 | | 0.490 | 6.3 |
| | | III | 6.866 | −0.162 | 0.563 | 7.267 | 7.754 | 0.487 | 6.3 |
| | | IV | 6.895 | −0.169 | 0.563 | 7.289 | | 0.465 | 6.0 |
| | | V | 7.236 | −0.202 | 0.586 | 7.620 | | 0.134 | 1.7 |
| | | VI | 6.961 | −0.146 | 0.586 | 7.401 | | 0.353 | 4.5 |
| | $A = 49$ | I | 6.911 | −0.184 | 0.623 | 7.350 | | 0.434 | 5.6 |
| | | II | 6.889 | −0.179 | 0.623 | 7.333 | | 0.451 | 5.8 |
| | | III | 6.887 | −0.180 | 0.623 | 7.330 | 7.784 | 0.454 | 5.8 |
| | | IV | 6.915 | −0.188 | 0.623 | 7.350 | | 0.434 | 5.6 |
| | | V | 7.278 | −0.225 | 0.651 | 7.704 | | 0.080 | 1.0 |
| | | VI | 7.022 | −0.163 | 0.651 | 7.510 | | 0.274 | 3.5 |
| $^{49}Sc$—$^{49}Ca$ | | V | 7.246 | −0.207 | — | 7.039 | | 0.051 | 0.7 |

ment energy for the analog pair $^{49}$Sc — $^{49}$Ca are in agreement with the accuracy of about 0.7%. In the case of the analog pair $^{49}$Sc—$^{49}$Ca, the energy $\Delta E_c$ is defined by the one--particle Coulomb energy. To calculate the two-particle part of $\Delta E_c$ one has to know the non-reducible matrix elements $V_J^{1f^2_{7/2},1f^2_{7/2}}$ and $V_J^{1f_{7/2}2p_{3/2},1f_{7/2}2p_{3/2}}$. The radial integrals $F_L$ and $G_L$, coming into the matrix elements mentioned above, were numerically calculated. For the neutron radial function describing the states $1f_{7/2}$ and $2p_{3/2}$, we accepted Saxon–Woods potential parameters given in Table II. Tables IV and V contain the numerical values of the appropriate matrix elements.

Let us point out that the exchange terms are, on an average, about two orders of magnitude smaller than the direct terms. The final value of the Coulomb displacement energy for individual sets of parameters and individual analog pairs are given in Table VI. One can also find there the comparison with the experimental values [3] from which it is evident that all sets of the charge distribution parameters taken from the experiments of electron scattering give too small (by about 6%) the Coulomb displacement energy. The homogeneous charge distribution is only slightly better than the others. The only charge distribution which gives the values of the Coulomb displacement energy in good agreement with the experimental results was taken from the shell model with parameters deduced from the optical model [7][1].

For the analog pair $^{51}$Ti—$^{51}$V, calculated $\Delta E_c^{th}$ gives the value different by 1.7% from $\Delta E_c^{exp}$, for $^{49}$Ti—$^{49}$V by 1%, and by 0.7% in the case of the analog pair $^{49}$Sc—$^{49}$Ca. The conclusion is that the most reasonable way of calculating the Coulomb displacement energy is to use the charge distribution from the shell model with the optical model parameters. As we can see, the Coulomb displacement energy can also be a sensitive test of the optical model parameters.

The authors would like to thank dr Małgorzata Haensel-Wygonowska for help in the numerical calculations and for a valuable discussion.

Detailed calculations of the expressions used in this work are given in [9].

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[1] The potential depth has been chosen to get agreement with the known one-particle energies, i. e. the proton energy $E = -9.622$ MeV for the state $1f_{7/2}$ and the energies $E = -5.14$ MeV and $E = -7.22$ MeV for the states $2p_{3/2}$ and $1f_{7/2}$ for neutrons, respectively.